The (Symbolic and Numeric) Computational Challenges of Counting 0-1 balanced matrices

Robert Dougherty-Bliss   Christoph Koutschan   Natalya Ter-Saakov
Doron Zeilberger

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Dedicated to our hero, Neil J. A. Sloane (b. Oct. 10, 1939), on his 85th birthday.

Abstract

A chessboard has the property that every row and every column has as many white squares as black squares. In this mostly methodological note, we address the problem of counting such rectangular arrays with a fixed (numeric) number of rows, but an arbitrary (symbolic) number of columns. We first address the "vanilla" problem where there are no restrictions, and then go on to discuss the still-more-challenging problem of counting such binary arrays that are not permitted to contain a specified (finite) set of horizontal patterns, and a specified set of vertical patterns. While we can rigorously prove that each such sequence satisfies some linear recurrence equation with polynomial coefficients, actually finding these recurrences poses major *symbolic*-computational challenges, that we can only meet in some small cases. In fact, just generating as many as possible terms of these sequences is a big *numeric*-computational challenge. This was tackled by computer whiz Ron H. Hardin, who contributed several such sequences, and computed quite a few terms of each. We extend Hardin's sequences quite considerably. We also talk about the much easier problem of counting such restricted arrays without balance conditions.

Preface: How it all started

A few weeks ago, the New York Times magazine started publishing a new kind of logic puzzle that they call *Not Alone*, created by Presanna Seshadri. You are given a 6×6 (or 8×8) array of boxes with most of them empty, but a few of them are filled with either a solid circle, that we will denote by 1, or an empty circle, that we will denote by 0. The solver has to, presumably using logic and human cleverness, fill-in the empty boxes such that the following conditions are met:

- Every row and every column must have as many zeroes as ones (i.e., they each must contain 3 zeroes and 3 ones in the 6×6 case).
- It is forbidden that on any row, and on any column, a **single** zero will be 'all alone' between two ones and that a **single** one will be all alone between two zeroes. In other words the patterns 010 and 101 are forbidden both horizontally and vertically.