The (Symbolic and Numeric) Computational Challenges of Counting 0-1 balanced matrices

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Dedicated to our hero, Neil J. A. Sloane (b. Oct. 10, 1939), on his 85th birthday.

Abstract

A chessboard has the property that every row and every column has as many white squares as black squares. In this mostly methodological note, we address the problem of counting such rectangular arrays with a fixed (numeric) number of rows, but an arbitrary (symbolic) number of columns. We first address the "vanilla" problem where there are no restrictions, and then go on to discuss the still-more-challenging problem of counting such binary arrays that are not permitted to contain a specified (finite) set of horizontal patterns, and a specified set of vertical patterns. While we can rigorously prove that each such sequence satisfies some linear recurrence equation with polynomial coefficients, actually finding these recurrences poses major *symbolic*-computational challenges, that we can only meet in some small cases. In fact, just generating as many as possible terms of these sequences is a big *numeric*-computational challenge. This was tackled by computer whiz Ron H. Hardin, who contributed several such sequences, and computed quite a few terms of each. We extend Hardin's sequences quite considerably. We also talk about the much easier problem of counting such restricted arrays without balance conditions.

Preface: How it all started

A few weeks ago, the New York Times magazine started publishing a new kind of logic puzzle that they call *Not Alone*, created by Presanna Seshadri. You are given a 6×6 (or 8×8) array of boxes with most of them empty, but a few of them are filled with either a solid circle, that we will denote by 1, or an empty circle, that we will denote by 0. The solver has to, presumably using logic and human cleverness, fill-in the empty boxes such that the following conditions are met:

- Every row and every column must have as many zeroes as ones (i.e., they each must contain 3 zeroes and 3 ones in the 6×6 case).
- It is forbidden that on any row, and on any column, a **single** zero will be 'all alone' between two ones and that a **single** one will be all alone between two zeroes. In other words the patterns 010 and 101 are forbidden both horizontally and vertically.

Three Natural Enumeration Problems Being *enumerators*, the following three questions immediately came to mind.

- For a *fixed*, 'numeric', positive integer k, but an *arbitrary*, 'symbolic' n, how many $2k \times 2n$ 0-1 balanced matrices are there? In other words how many $2k \times 2n$ 0-1 matrices are there where every row has n zeroes and n ones and every column has k zeroes and k ones?

  Looking up some numbers from this problem leads to a family of sequences submitted by Ron H. Hardin [H], who has made a number of interesting submissions to the OEIS. Some of these were detected by an automated search of the OEIS for recurrences by Kauers and Koutschan [KK], and later proven to satisfy those recurrences by Dougherty-Bliss and Kauers [DK]. We will tell a similar story here.

  We will show that the family of sequence $B(n,k)$, the number of $2n \times 2k$ balanced matrices, is "D-finite" in for every fixed k. That is, it satisfies a linear recurrence relation with polynomial coefficients. The approach will be to use *'holonomic nonsense'* [AZ; K1; Z] which guarantees the existence of such a recurrence and also outlines a method to construct it. Using the very efficient implementations of [K2; K3; K4] we—or rather our beloved computers—were able to compute these recurrences for $k = 2$ and $k = 3$.

  Alas, it seems too difficult at present to determine rigorously *or* experimentally what recurrence $B(n,4)$, $B(n,5)$, and so on might satisfy in n. The deterministic algorithms take too long to run, and we do not have enough data to guess recurrences. It would be good to generate enough data to make conjectures.

- For an arbitrary (finite) alphabet (not just $\{0,1\}$) and arbitrary finite sets of forbidden horizontal and vertical patterns, H and V, how many $k \times n$ matrices are there avoiding the patterns of H in the rows and the patterns of V in the columns (with no balance conditions)? We will show that these sequences are much easier, both conceptually and computationally, since they always satisfy linear recurrence equations with **constant** coefficients, or equivalently, their generating function is a *rational function*.

- Going back to the alphabet $\{0,1\}$, for a specific k, how many $(2k) \times (2n)$ balanced 0-1 matrices are there that also avoid a prescribed horizontal set of patterns H and (another or the same) prescribed vertical set of patterns V. Once again, we will show that for each such scenario, the enumerating sequence satisfies *some* linear recurrence equation with polynomial coefficients (in n). Alas finding it is yet harder than the 'vanilla' case above. Once again this leads to numeric challenges. In particular, it turns out that for the original New York Times puzzles where $k = n = 3$ and $H = V = \{010, 101\}$ that number is **exactly** 368. It is relatively easy to actually construct the set of all such legal matrices, *once and for all*. It follows that, surprisingly, a pure *brute-force* algorithm for solving these original puzzles is more efficient than using logic, as a human solver would. Just try out all 368 possible answers and see which one agrees with the given clues. Alas for the 8×8 puzzles that started to appear shortly after, the computer has to do it the human way of using logic.

The Maple package `NotAlone.txt`, available from https://sites.math.rutgers.edu/~zeilberg/tokhniot/NotAlone.txt solves and creates such puzzles. Procedure `Ptor` implements the brute-force approach that is optimal for the 6×6 case. Procedure `SolveN` does it in a way a human would tackle it.

Theorems

In this section, that is purely *theoretical*, we will prove that the first and third kind of sequences above are P-recursive (aka *holonomic*), in other words are guaranteed to satisfy *some* linear recurrence equation with **polynomial coefficients** (see [KP] chapter 7), while any sequence that comes from the second kind of enumeration problems belongs to the simpler class of C-finite sequences ([KP], chapter 4), i.e., satisfies *some* linear equation with **constant coefficients**.

Theorem 1. *Let k be a specific positive integer, and let n be a general positive integer. Let $b_k(n)$ be the number of balanced $2k \times 2n$ 0-1 matrices, i.e., binary matrices with $2k$ rows and $2n$ columns where every row has exactly n ones (and hence exactly n zeroes), and every column has exactly k ones (and hence exactly k zeroes). Then the sequence $\{b_k(n)\}_{n=1}^{\infty}$ is holonomic. In other words there exists a positive integer L (the order) and polynomials in n, $p_i(n), 0 \leq i \leq L$, with $p_L(n) \neq 0$ such that*

$$\sum_{i=0}^{L} p_i(n) b_k(n+i) \quad = 0 \quad .$$

Proof. Let $e_k(x_1, \dots, x_n)$ be the **elementary symmetric function** of degree k:

$$e_k(x_1, \dots, x_n) \; = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k} \quad .$$

It is readily seen that $b_k(n)$ is the coefficient of $x_1^n \cdots x_{2k}^n$ in $e_k(x_1, \dots, x_{2k})^{2n}$. Indeed, each monomial of $e_k(x_1, \dots, x_{2k})$ corresponds to a way of placing k ones (and k zeroes) in any particular column, making each column balanced. $e_k(x_1, \dots, x_{2k})^{2n}$ then is the weight enumerator of all column-balanced $2k$ by $2n$ 0-1 matrices. The coefficient of $x_1^n \cdots x_{2k}^n$ collects those that are also row-balanced. Hence

$$b_k(n) = \mathrm{Coeff}_{x_1^0 \cdots x_{2k}^0} \left(\frac{e_k(x_1, \dots, x_{2k})^2}{x_1 \cdots x_{2k}} \right)^n = \left(\frac{1}{2\pi i} \right)^{2k} \int \left(\frac{e_k(x_1, \dots, x_{2k})^2}{x_1 \cdots x_{2k}} \right)^n \frac{dx_1 \cdots dx_{2k}}{x_1 \cdots x_{2k}} \quad ,$$

where the integration is over the multi-circle $|x_1| = 1, \dots, |x_{2k}| = 1$. Since the integrand is holonomic in the $2k$ continuous variables $x_1, \dots, x_{2k}$ and the one discrete variable n, it follows from *algorithmic proof theory* [AZ; K4; Z] that integrating away the $2k$ continuous variables leaves $b_k(n)$ holonomic in the surviving discrete variable n. □

So far our alphabet was $\{0, 1\}$. In the next theorem (answering the second question above) we will be more general, but we need to introduce some definitions.

Definition 1. Fix a finite alphabet A once and for all. A word $w_1 \dots w_n$ in the alphabet A **contains** the word $p_1 \dots p_k$ (called a *'pattern'*) if there is a location i such that $w_i = p_1, \dots, w_{i+k-1} = p_k$. For example, with the Latin alphabet, *robert* contains the words *rob*, *obe*, *t*, and many others. A word w *avoids* the pattern p if it does not contain it. For example 101010001 avoids 11.

Theorem 2. *Let A be an arbitrary (finite) alphabet, and H and V be arbitrary finite sets of words in A. Let k be a fixed (numeric) positive integer. Let $m_k(n) = m_{A,H,V,k}(n)$ be the number of $n \times k$ matrices with entries in A such that every row avoids the patterns in H, and every column avoids the patterns in V, then the sequence $\{m_k(n)\}_{n=1}^{\infty}$ satisfies a linear recurrence equation with*

constant *coefficients. In other words there exists a positive integer* L *and numbers* $c_0, c_1, \ldots, c_L$ *such that*

$$\sum_{i=0}^{L} c_i \, m_k(n+i) \,=\, 0 \quad .$$

Equivalently, there exist polynomials $P(t)$ *and* $Q(t)$ *(where* $Q(t)$ *has degree* L*) such that*

$$\sum_{n=0}^{\infty} m_k(n) \, t^n \,=\, \frac{P(t)}{Q(t)} \quad .$$

Proof. Let B be the set of words in the alphabet A with k letters that avoid the patterns in V. This is a finite set. We will view the $k \times n$ matrix as a one-dimensional word in this meta-alphabet. Then the restrictions that the rows avoid the patterns in H translate to many conditions about pattern avoiding in this meta-alphabet. This gives rise to a so-called type-3 grammar, or finite automata, whose enumerating generating functions are famously rational functions. In order to actually find them one can use the *positive* approach, using the transfer-matrix method ([S], ch. 4), or the *negative* approach, using the powerful Goulden-Jackson method, nicely exposited in [NZ]. $\square$

Comment For the motivating example (the *Not Alone* puzzles), $A = \{0, 1\}$ and $H = V = \{010, 101\}$.

The next theorem states that if one counts $2k \times 2n$ balanced 0-1 matrices and imposes *arbitrary* horizontal and vertical conditions, the resulting sequences are still holonomic.

Theorem 3. *Let* k *be a specific positive integer, and let* n *be a general positive integer. Let* H *and* V *be finite sets of words ('patterns') in* $\{0, 1\}$*. Let* $b_{H,V,k}(n) = b_k(n)$ *be the number of balanced* $2k \times 2n$ *0-1 matrices, that avoid the patterns of* H *in every row and the patterns of* V *in every column, then there exists a positive integer* L *and polynomials* $p_i(n)$*,* $0 \le i \le L$*, with* $p_L(n) \neq 0$ *such that*

$$\sum_{i=0}^{L} p_i(n) b_k(n+i) \quad = 0 \quad .$$

Proof. Instead of *naive counting* where the weight of a $2k \times 2n$ matrix was simply t^{2n}, we now introduce $2k$ formal variables $x_1, \ldots, x_{2k}$ and assign a *weight* of a matrix $A = (a_{ij}, 1 \le i \le 2k, 1 \le j \le n)$ to be

$$t^n \, x_1^{a_1} \cdots x_{2k}^{a_{2k}} \quad ,$$

where a_i is the number of ones in the i-th row.

Once again we can use the transfer matrix method, or the Goulden-Jackson method, to find the *weight-enumerator* of the set of all matrices avoiding H horizontally and V vertically, with the above weight. This is a very complicated rational function in the $2k+1$ variables, t and $x_1, \ldots, x_{2k}$ In order to count balanced such matrices with $2n$ columns, we have to extract the coefficient of

$$t^{2n} x_1^n \cdots x_{2k}^n \quad .$$

Let's call this giant, but explicitly computable, rational function $R(x_1, \ldots, x_{2k}; t)$ then

$$b_k(n) = \mathrm{Coeff}_{t^{2n} x_1^n \cdots x_{2k}^n} R(x_1, \ldots, x_{2k}; t)$$

$$= \left(\frac{1}{2\pi i}\right)^{2k+1} \int \frac{R(x_1, \dots, x_{2k}; t)}{(x_1 \cdots x_{2k})^n t^{2n}} \frac{dx_1 \cdots dx_{2k} dt}{x_1 \cdots x_{2k} t} \quad .$$

The integrand is holonomic in the $2k+1$ continuous variables $x_1, \dots, x_{2k}, t$ and the one discrete variable n, and once again, integrating with respect to the $2k+1$ continuous variables leaves us, by algorithmic proof theory [AZ; K4; Z], with a holonomic discrete function in n. □

Comment Theorem 1 is the special case of Theorem 3 where the sets of forbidden patterns H and V are empty. Nevertheless the simple explicit form of the integrand is useful, as we will see below.

Symbol Crunching

The main Maple package accompanying this article is `Hardin.txt` available from https://sites.math.rutgers.edu/~zeilberg/tokhniot/Hardin.txt.

Let us take a tour of the main features.

- `SeqB(k,N)` uses the formula in the proof of Theorem 1 to crank-out the first N terms of the sequence enumerating balanced 0-1 $2k \times 2n$ matrices for $n = 1$ to $n = N$. This is useful for checking with the OEIS.

  For example `SeqB(2,10);` gives:

  $$6, 90, 1860, 44730, 1172556, 32496156, 936369720, 27770358330,$$
  $$842090474940, 25989269017140, \dots$$

  This is a very famous sequence, listed at https://oeis.org/A002896 as the "number of walks with $2n$ steps on the cubic lattice Z^3 beginning and ending at $(0,0,0)$.

  Can you see why these two sequences are the same?

- `SeqB(3,10);` gives the first 10 terms of OEIS sequence A172556 `https://oeis.org/A172556`, given there with the same description as ours, created by Ron Hardin, who computed 49 terms. With our Maple package we were able to compute 55 terms. In fact already 49 terms suffice to *conjecture* a linear recurrence. See the output file https://sites.math.rutgers.edu/~zeilberg/tokhniot/oHardin2.txt. Later on we will see how to derive it rigorously, without guessing.

  `SeqB(4,10);` gives the first 10 terms of OEIS sequence A172555 `https://oeis.org/A172555`, also due to Hardin, who computed 33 terms.

  `SeqB(5,10);` gives the first 10 terms of OEIS sequence A172557 `https://oeis.org/A172557`, also due to Hardin, who computed 24 terms.

  While we know from Theorem 1 that these sequences do satisfy linear recurrences with polynomial coefficients, we are unable at present to find them. We need bigger and faster computers!

- `GF1t(A,H,V,n,t)`: inputs an alphabet `A`, sets of horizontal and vertical forbidden patterns `H` and `V` respectively, a positive integer `n`, and a variable `t`. It outputs the rational function whose coefficient of t^m is the number of $m \times n$ matrices avoiding the patterns of `H` in rows and the patterns of `V` in columns, whose existence is guaranteed by Theorem 2.

For example to get the rational function whose coefficient of t^n is the number of $3 \times n$ 0-1 matrices avoiding 010 and 101 both vertically and horizontally enter:

`GF1t({0,1}, {[0,1,0],[1,0,1]}, {[0,1,0],[1,0,1]},3,t);` ,

getting right away :

$$-\frac{5t^4 - 19t^2 - 4t - 1}{t^4 - 5t^2 - 2t + 1} \quad .$$

The first few terms are

$$6, 36, 102, 378, 1260, 4374, 14946, 51384, 176238, 605022, 2076288, \ldots$$

Surprise! These are in the OEIS https://oeis.org/A060521 for a different reason. They are the numbers of $3 \times n$ 0-1 matrices avoiding, both vertically and horizontally, the patterns 000 and 111.

And indeed this is confirmed by our Maple package. Typing:

`GF1t({0,1 }, {[1,1,1],[0,0,0]}, {[1,1,1],[0,0,0]},3,t);`

gives the same output. Here is an explicit bijection between these two sets of 0-1 $3 \times n$ matrices. Define the bijective map that maps the matrix entry $m_{i,j}$ to $m_{i,j} + i + j \pmod 2$, for all $1 \le i \le 3$ and $1 \le j \le n$. In other words, use as "mask" a 0-1 matrix with chessboard pattern and add it to the input matrix (in binary arithmetic). Clearly, every occurrence of 000 or 111 (either vertically or horizontally) will be mapped to 101 or 010, and vice versa. Thus, this map transforms each $\{010, 101\}$-avoiding matrix into a $\{000, 111\}$-avoiding one, and vice versa.

- `GF2t(H,V,k,x,t)`: inputs sets of horizontal and vertical forbidden patterns `H` and `V` respectively, variable names `x` and `t`, and outputs the rational function in t and $x_1, \ldots, x_{2k}$, whose coefficient of $t^n x_1^{a_1} \cdots x_{2k}^{a_{2k}}$ gives the number of $2k \times 2n$ 0-1 matrices avoiding the horizontal patterns H and vertical patterns V and having a_i ones in row i, for all $1 \le i \le 2k$.

  For example if $H = V = \{010, 101\}$ (as in the Not-Alone puzzles), the rational function for $4 \times 2n$ matrices is given in the output file https://sites.math.rutgers.edu/~zeilberg/tokhniot/oHardin5.txt.

  This is already big! But once we have it, we can Taylor expand it in t, extract the coefficient of t^{2n} followed by extracting the coefficient of $x_1^n x_2^n x_3^n x_4^n$ to get many terms, see the output file https://sites.math.rutgers.edu/~zeilberg/tokhniot/oHardin5a.txt.

  The generating function for $6 \times 2n$ 0-1 matrices avoiding $010, 101$ both horizontally and vertically is much bigger! See the output file https://sites.math.rutgers.edu/~zeilberg/tokhniot/oHardin6.txt.

  This enabled us to find the first 30 terms, via *symbolic computation*. See: https://sites.math.rutgers.edu/~zeilberg/tokhniot/oHardin6a.txt. They start with

  $$8, 64, 368, 2776, 25880, 251704, 2629080, 28964248, 331032312, 3907675376, \ldots$$

  In particular the third term, 368 is the exact number, mentioned above, of solutions to a 6×6 Not-Alone puzzle. See the next section for 70(!) terms using numeric computations.

Rigorous Recurrences for the number of balanced $4 \times 2n$ and $6 \times 2n$ matrices

Using the Maple package https://sites.math.rutgers.edu/~zeilberg/tokhniot/SMAZ.txt that accompanies [AZ], one very quickly gets the following theorem.

Theorem 4. *Let $a(n)$ be the number of 4 by $2n$ balanced matrices. Then:*

$$36(2n+3)(2n+1)(n+1)a(n) - 2(2n+3)\left(10n^2+30n+23\right)a(n+1) + (n+2)^3 a(n+2) = 0.$$

But `SMAZ.txt` was unable, with our computers, to find a recurrence for the sequence enumerating 6 by $2n$ balanced matrices. Amazingly, the second author's Mathematica package https://risc.jku.at/sw/holonomicfunctions/ did it! We have the following fully rigorously-proved recurrence.

Theorem 5. *Let $a(n)$ be the number of 6 by $2n$ balanced matrices. Then:*

$$\begin{aligned}
&51200(2n+7)(2n+5)(2n+3)(2n+1)(n+2)(n+1)\left(33n^2+242n+445\right)a(n)\\
&-128(2n+7)(2n+5)(2n+3)(n+2)\left(7491n^4+84898n^3+351364n^2\right.\\
&\qquad\left.+628997n+414370\right)a(n+1)\\
&+16(2n+5)(2n+7)\left(2772n^6+48048n^5+344379n^4+1307394n^3\right.\\
&\qquad\left.+2775099n^2+3125336n+1460132\right)a(n+2)\\
&+2(2n+7)(n+3)\left(3201n^6+61886n^5+497179n^4+2124170n^3+5089654n^2\right.\\
&\qquad\left.+6484024n+3431096\right)a(n+3)\\
&-(n+3)(n+4)^5\left(33n^2+176n+236\right)a(n+4)\;=0 \quad .
\end{aligned}$$

Number Crunching

Since it is unrealistic to try and find recurrences for enumerating $2k \times 2n$ balanced matrices for $k \geq 4$, it would be nice to extend, as far as our computers would allow, Hardin's already impressive computational feats. Note that a brute force approach is doomed.

To that purpose we have a C program available from https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/hardinC/balmat4p.c that extended Hardin's sequences quite a bit. The program computes, for $n = 1, 2, \ldots, 2N$, the coefficients of the polynomial $e_k(x_1, \ldots, x_{2k})^n$, and whenever n is even, outputs the coefficient of $(x_1 \cdots x_{2k})^n$. For $\mathbf{a} = (a_1, \ldots, a_{2k})$ let

$$c_n(\mathbf{a}) := \operatorname{Coeff}_{x_1^{a_1} \cdots x_{2k}^{a_{2k}}} e_k(x_1, \ldots, x_{2k})^n \quad .$$

The trivial identity $e_k(x_1, \ldots, x_{2k})^n = e_k(x_1, \ldots, x_{2k}) \cdot e_k(x_1, \ldots, x_{2k})^{n-1}$ immediately yields a recursive definition of these coefficients. Let $\mathcal{S} := \left\{(s_1, \ldots, s_{2k}) \in \{0,1\}^{2k} \;\middle|\; s_1 + \cdots + s_{2k} = k\right\}$ denote the support of $e_k(x_1, \ldots, x_{2k})$, then $c_n(\mathbf{a}) = \sum_{\mathbf{s} \in \mathcal{S}} c_{n-1}(\mathbf{a} - \mathbf{s})$. In this formula, one has to apply the boundary conditions $c_{n-1}(\mathbf{a} - \mathbf{s}) = 0$ whenever $\mathbf{a} - \mathbf{s}$ has a negative component, or one that is larger than $n-1$. Thanks to the symmetry in the variables $x_1, \ldots, x_{2k}$, and thanks to the fact that $e_k(x_1, \ldots, x_{2k})^n$ is a homogeneous polynomial of degree kn, it suffices to store $c_n(\mathbf{a})$ for $n \geq a_1 \geq \cdots \geq a_{2k} \geq 0$ and $a_1 + \cdots + a_{2k} = kn$. Moreover, if we fix the number N of desired terms from the very beginning, we can impose the additional condition $a_i \leq N$. Since these vectors **a** do not any more form a rectangular (multi-dimensional) array, we flatten it to a one-dimensional array, in order to handle it more easily in the C language. Conversion between these two data structures can be done by a suitable rank and unrank function. Finally the whole computation is done modulo prime numbers, using 64-bit integers. A sufficient number of primes can be determined by the trivial upper bound $\binom{2k}{k}^{2n} \geq c_{2n}(n,\ldots,n)$, the latter being the n-th term of the sequence.

- If you want to see 150 terms of the sequence enumerating $2n$ by 8 0-1 matrices with row sums 4 and column sums n, in other words OEIS sequence A172555 (Hardin only had 33 terms) see the output file `https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/hardinC/data4.txt`.

- If you want to see 50 terms of the sequence enumerating $2n$ by 10 0-1 arrays with row sums 5 and column sums n, in other words OEIS sequence A172557 (Hardin only had 24 terms) see the file `https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/hardinC/data5.txt`.

- If you want to see 39 terms of the sequence enumerating $2n$ by 12 0-1 matrices with row sums 6 and column sums n, in other words OEIS sequence A172558 (Hardin only had 19 terms) see the file: `https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/hardinC/data6.txt`.

- If you want to see 30 terms of the sequence enumerating $2n$ by 14 0-1 matrices with row sums 7 and column sums n, in other words OEIS sequence A172559 (Hardin only had 17 terms) see the file: `https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/hardinC/data7.txt`.

- If you want to see 25 terms of the sequence enumerating $2n$ by 16 0-1 matrices with row sums 8 and column sums n, in other words OEIS sequence A172560 (Hardin only had 14 terms) see the file: `https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/hardinC/data8.txt`.

- If you want to see 22 terms of the sequence enumerating $2n$ by 18 0-1 matrices with row sums 9 and column sums n, in other words OEIS sequence A172554 (Hardin only had 12 terms) see the file: `https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/hardinC/data9.txt`.

- So far for the 'vanilla case'. Above, using the Maple package `Hardin.txt` we were able to find 30 terms of the motivating sequence of this paper, i.e., the number of balanced 6 by $2n$ 0-1 matrices avoiding the patterns 010 and 101 both vertically and horizontally. Using the C program mentioned above we now have 70 terms. See the output file: `https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/hardinC/dataNA3.txt`.

Conclusion: Humankind, and even computerkind, will most probably **never** know the exact number of 100×100 0-1 matrices with row- and columns- sums all equal to 50, but it is fun to try and see how far we can go. The OEIS created, by our hero Neil Sloane, is an ideal platform for publishing these hard-to-compute numbers.

Happy 85$^{\text{th}}$ birthday, Neil. May you live to see the OEIS with 1,200,000 sequences!

References

[AZ] Moa Apagodu and Doron Zeilberger. "Multi-Variable Zeilberger and Almkvist-Zeilberger Algorithms and the Sharpening of Wilf-Zeilberger Theory". In: *Advances in Applied Mathematics* 37 (2006). Special issue in honor of Amitai Regev, pp. 139–152. URL: https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/multiZ.html.

[DK] Robert Dougherty-Bliss and Manuel Kauers. "Hardinian Arrays". In: *The Electronic Journal of Combinatorics* (2024), P2–9.

[H] Ron H. Hardin. *OEIS sequences A172555, A172557, A172558*. URL: https://oeis.org/A172555,%20https://oeis.org/A17257,%20https://oeis.org/A17258.

[K1] Manuel Kauers. *D-Finite Functions*. Springer, 2023.

[K2] Christoph Koutschan. "Advanced applications of the holonomic systems approach". 2009.

[K3] Christoph Koutschan. "A fast approach to creative telescoping". In: *Mathematics in Computer Science* 4.2 (2010), pp. 259–266.

[K4] Christoph Koutschan. *Holonomic Functions (a Mathematica package)*. URL: https://risc.jku.at/sw/holonomicfunctions/.

[KK] Manuel Kauers and Christoph Koutschan. "Some D-Finite and Some Possibly D-Finite Sequences in the OEIS". In: *Journal of Integer Sequences* 26.2 (2023), p. 3.

[KP] Manuel Kauers and Peter Paule. *The Concrete Tetrahedron*. Springer, 2011.

[NZ] John Noonan and Doron Zeilberger. "The Goulden-Jackson cluster method: extensions, applications, and implementations". In: *Journal of Difference Equations and Applications* 5 (1999), pp. 355–377. URL: https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/gj.html.

[S] Richard Stanley. *Enumerative Combinatorics, Volume 1*. Wadsworth and Brooks/Cole, 1986.

[Z] Doron Zeilberger. "A Holonomic Systems Approach To Special Functions". In: *Journal of Computational and Applied Mathematics* 32 (1990), pp. 321–368. URL: http://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimPDF/holonomic.pdf.

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