HOW JOE GILLIS DISCOVERED COMBINATORIAL SPECIAL FUNCTION THEORY

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Girsa d'yankuta [la mishtakcha][1] *(Talmud, Tractate Shabbat 22b)*

Nov. 19, 1993: When I checked my mail this morning, I was shocked to learn that Joe Gillis died last night in his sleep. He was eighty two years old.

Only last May, during my last visit to Israel, he discussed with me his research plans. He was hoping to generalize results on Hausdorff dimension and what are now called fractal sets, that he had found back in the mid-thirties.

Joe had a great influence on my mathematical development, as he did on many generations of Israeli mathematicians, who got their first taste of advanced mathematics through *Gilyonot Lematematika*, an outstanding mathematical magazine, in Hebrew, " for high-school students and amateurs". This magazine, which Joe edited for many years, had an extensive problem section. The problems periodically challenged the best and brightest among us . Joe also initiated the Israeli Math Olympiad, and was the coach of the Israeli I.M.O. team for a very long time.

However, Joe's influence to mathematics at large was far greater than that, in particular, with respect to my own two specialties: Combinatorics and Special Functions. Gillis, in his seminal paper with Even[2], initiated the extremely fruitful marriage of these hitherto unrelated subjects, from which was soon born the flourishing new field of combinatorial special functions (e.g.[3][5]). I feel that the story of how this unison came to be, narrated to me, years ago, by Joe himself, must be recorded for posterity, since it testifies not only to Joe's genius, but to the genius of the human spirit.

1928: Joe spent his last year in high-school preparing for the competition for the coveted scholarship to Trinity College, Cambridge, that he subsequently won. The textbook he studied was Chrystal's famous *Algebra*[1]. One of the problems discussed there particularly appealed to Joe. It was the classical *derangements* problem: *In how many ways can one stuff n different letters, in the corresponding n envelopes, in such a way that no letter gets sent to the right address?*. The well-known answer, given in Chrystal's text, is that this number, $D(n)$, equals $n!(1/0! - 1/1! + 1/2! - 1/3! + \ldots + (-1)^n/n!) = [n!/e]$.

Joe, being the bright youngster that he was, started to wonder what happens if there are *multiple* letters addressed to each address. In other words, what can one say about $D(n_1, \ldots, n_k)$, the number of ways of stuffing n_1 letters addressed to 1, ..., n_k letters addressed to k, into the corresponding $n_1 + \ldots + n_k$ envelopes, in such a way that no letter gets to the right destination? Of course, he realized that

$$D(n_1, n_2) = \delta_{n_1,n_2} \quad , \tag{1}$$

but was unable to find a "closed form" expression, even for the case $k = 3$. Failing this, he went on to establish *recurrence relations*, that enabled him to compile a table for $D(n_1, n_2, n_3)$, for small (and not so small) values of the arguments, starting from the obvious "initial conditions" (1). Having

[1] Aramaic: The lesson of infancy is not forgotten.

accomplished this, and realizing that there probably is no reasonable formula for $D(n_1, n_2, n_3)$, he went on to "bigger and better"[2] things, or so it seemed then.

1960: About one-third-century later, and long after he "changed fields" to "applied" mathematics (spurred initially by his desire to contribute to the welfare of the then young state of Israel, to where he immigrated in the late forties), he encountered a "practical" problem. In the course of trying to solve a certain differential equation, he needed to compute the following "linearization coefficients" for the Laguerre polynomials:

$$E(n_1, n_2, n_3) = (-1)^{(n_1+n_2+n_3)} \int L_{n_1}(x) L_{n_2}(x) L_{n_3}(x) e^{-x} dx \quad . \tag{2}$$

Once again, he was unable to find a "closed form" expression. However, he and George Weiss[4] obtained recurrence relations for the $E(n_1, n_2, n_3)$ which enable one to compile a table of these for any specified range of the arguments, obviating the need to integrate every time anew.

1975: A decade and a half later, as he was browsing through his old paper[4], Joe made a *connection.* He had seen these recurrence relations for (2), established 15 years earlier, *much much* earlier! They were identical (up to some trivial change of notation) to the recurrences he established for $D(n_1, n_2, n_3)$, almost half-a-century before, during his last year of high school! Matching the obvious initial conditions at $n_1 = 0, n_2 = 0, n_3 = 0$, for which E coincides with (1)(due to the orthogonality of the Laguerre polynomials), it followed that ([2])

$$D(n_1, n_2, n_3) = E(n_1, n_2, n_3) \quad . \tag{3}$$

Thus began the beautiful field of combinatorial special function theory (e.g. [3][5].)[3]

References

1. G. Chrystal, "Algebra", vol. **2**, reprinted by Chelsea, N.Y., N.Y.

2. S. Even and J. Gillis, *Derangements and Laguerre polynomials*, Proc. Cambridge Phil. Soc. **79**(1976), 135-143.

3. D. Foata, *Combinatoire de identitiés sur le polynômes orthogonaux*, Proc. Inter. Congress of Math.[Warsaw. Aug. 16-24, 1983], Warsaw, 1983.

4. J. Gillis and G. Weiss, *Products of Laguerre polynomials*, M.T.A.C. (now Math. Comp.), **14**(1960), 60-63.

5. J. Zeng, *Weighted derangements and the linearization coefficients of orthogonal Sheffer polynomials*, Proc. London Math. Soc. (3) **65**(1992), 1-22.

[2] After completing his undergraduate studies with distinction, he went on to write a brilliant dissertation under Besicovitch; was one of the first collaborators of Erdös; was stationed in Bletchley Park; made important contributions to fluid dynamics; and so on..., but this is a different story.

[3] The referee pointed out that the connection between combinatorics and function theory goes back to Euler, Gauss, and Jacobi. However the connection between combinatorics and the classical special functions of mathematical physics, was first made, as far as I am aware of, by Joe Gillis. I wish to thank the referee for many valuable comments.

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