## A Fibonacci-Counting Proof Begged by Benjamin and Quinn

*Doron ZEILBERGER* *

Arthur Benjamin and Jennifer Quinn, in their delightful "*Proofs that Really Count: The Art of Combinatorial Proof*" (MAA, 2003), asked for a bijective proof of ($f_n := F_{n+1}$)

$$\sum_i \binom{2n}{i} f_{2i-\epsilon} = 5^n f_{2n-\epsilon} \quad (\epsilon \in \{0,1\}) \quad .$$

Here goes. For any vector of integers $u$, let $|u|$ denote the sum of its entries. Let, for $\epsilon \in \{0,1\}$ ,

$$A_\epsilon(n) := \{(w,u) : w \in \{0,1\}^{2n} \quad , \quad u \in \{1,2\}^* \quad , \quad 2|w| - |u| = \epsilon \quad \} \quad ,$$

$$B_\epsilon(n) := \{(w,u) : w \in \{1,2,3,4,5\}^{n} \quad , \quad u \in \{1,2\}^* \quad , \quad |u| = 2n - \epsilon \quad \} \quad .$$

The left side *counts* $A_\epsilon(n)$ and the right side counts $B_\epsilon(n)$. Both sets can be bijectively mapped to the set of $n$-step walks from $A_\epsilon$ to $A_0$ in a digraph whose vertices are $A_0$ and $A_1$, and there are ten edges from $A_0$ to $A_0$ and five edges each from $A_0$ to $A_1$, $A_1$ to $A_0$, $A_1$ to $A_1$.

For $(w,u) \in A_\epsilon(n)$, write $w = w'w''$, where $length(w') = 2$, and $u = u'u''$ where $u'$ is the **shortest** head of $u$ such that $|u'| = 2|w'| - \epsilon$ or $|u'| = 2|w'| - \epsilon + 1$, which entails $(w'', u'') \in A_0(n-1)$, $(w'', u'') \in A_1(n-1)$, respectively. There are ten [1] ways of getting from $A_0(n)$ to $A_0(n-1)$, five[2] ways of getting from $A_0(n)$ to $A_1(n-1)$, five[3] ways of getting from $A_1(n)$ to $A_0(n-1)$, and five[4] ways of getting from $A_1(n)$ to $A_1(n-1)$. To get the walk, repeat this passage from $(w,u)$ to $(w'', u'')$, $n$ times, recording the transitions, and the 'states'.

For $(w,u) \in B_\epsilon(n)$ , write $w = w'w''$, where $length(w') = 1$, and $u = u'u''$ where $u'$ is the **shortest** head of $u$ such that $|u'| = 2-\epsilon$ or $|u'| = 3-\epsilon$ which entails $(w'', u'') \in B_0(n-1)$, $(w'', u'') \in B_1(n-1)$, respectively. There are ten [1'] ways of getting from $B_0(n)$ to $B_0(n-1)$, five[2'] ways of getting from $B_0(n)$ to $B_1(n-1)$, five[3'] ways of getting from $B_1(n)$ to $B_0(n-1)$, and five[4'] ways of getting from $B_1(n)$ to $B_1(n-1)$. To get the walk, repeat this passage from $(w,u)$ to $(w'', u'')$, $n$ times, recording the transitions, and the 'states'.

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* Department of Mathematics, Rutgers University (New Brunswick), Hill Center-Busch Campus, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019, USA. `zeilberg@math.rutgers.edu` , `http://www.math.rutgers.edu/~zeilberg/`.
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[1] $(w', u') = (00, \phi), (01, 11), (01, 2), (10, 11), (10, 2), (11, 1111), (11, 112), (11, 121), (11, 211), (11, 22)$.

[2] $(w', u') = (01, 12), (10, 12), (11, 1112), (11, 122), (11, 212)$.

[3] $(w', u') = (01, 1), (10, 1), (11, 111), (11, 12), (11, 21)$.

[4] $(w', u') = (00, \phi), (01, 2), (10, 2), (11, 112), (11, 22)$.

[1'] $(w', u') = (1, 11), (2, 11), (3, 11), (4, 11), (5, 11), (1, 2), (2, 2), (3, 2), (4, 2), (5, 2)$.

[2'] $(w', u') = (1, 12), (2, 12), (3, 12), (4, 12), (5, 12)$.

[3'] $(w', u') = (1, 1), (2, 1), (3, 1), (4, 1), (5, 1)$.

[4'] $(w', u') = (1, 2), (2, 2), (3, 2), (4, 2), (5, 2)$.