ZEILBERGERS FIRST QUESTION ON FRACTIONAL COUNTING OF PARTITIONS

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1. The question

Doron Zeilberger and Noam Zeilberger [1] defined

$$b(n) := \sum_{\substack{p_1+\cdots+p_k=n \\ p_1\geq p_2\geq\cdots\geq p_k>0}} \frac{1}{p_1 p_2 \cdots p_k}\,.$$

They believe that $b(n)/n$ tends to a finite limit when n goes to infinity, and asked for an identification of the limit. In particular, they wonder whether $b(n) \sim ne^{-\gamma}$, where γ is Euler's constant.

2. The generating function

Let us put $B(q) := \sum_{n\geq 0} b(n)q^n$. The summands p_j may be arranged according to their value, i. e. there are m_i times the value i. We find

$$B(q) = \sum_{m_i\geq 0, i\geq 1} \frac{1}{\prod_{i\geq 1} i^{m_i}} q^{\sum_{i\geq 1} i m_i} = \sum_{m_i\geq 0, i\geq 1} \prod_{i\geq 1} \frac{q^{im_i}}{i^{m_i}} = \prod_{i\geq 1} \sum_{m\geq 0} \frac{q^{im}}{i^m} = \prod_{i\geq 1} \frac{1}{1-q^i/i}\,.$$

We thus obtain

$$B(q) = \frac{1}{\prod_{i\geq 1}(1-q^i/i)}\,. \tag{1}$$

3. An infinite product

Let us put $P = \prod_{i\geq 2}(1-\frac{1}{i})e^{\frac{1}{i}}$. The Weierstrass product for the Γ function is given by

$$\Gamma(z+1) = e^{-\gamma z} \prod_{n\geq 1} \left(1+\frac{z}{n}\right)^{-1} e^{\frac{z}{n}}$$

for $z \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$. We thus find

$$\prod_{n\geq 2} \left(1+\frac{z}{n}\right) e^{-\frac{z}{n}} = \frac{e^{(1-\gamma)z}}{(1+z)\Gamma(z+1)} = \frac{e^{(1-\gamma)z}}{\Gamma(z+2)}\,.$$

Let z go to -1. We obtain

$$P = e^{\gamma-1}\,. \tag{2}$$

4. Asymptotics

From (1) we deduce

$$B(q) = e^{\sum_{i\geq 1}\frac{q^i}{i}} \prod_{i\geq 1}\left(1-\frac{q^i}{i}\right)^{-1} e^{-\frac{q^i}{i}} = e^{-\log(1-q)}(1-q)^{-1}e^{-q}\prod_{i\geq 2}\left(1-\frac{q^i}{i}\right)^{-1} e^{-\frac{q^i}{i}}$$
$$= \frac{e^{-q}}{(1-q)^2}\prod_{i\geq 2}\left(1-\frac{q^i}{i}\right)^{-1} e^{-\frac{q^i}{i}}\,.$$

We thus find

$$B(q) \sim \frac{e^{-1}}{(1-q)^2}P^{-1}$$

when q goes to 1^-. By (2), we obtain

$$B(q) \sim \frac{e^{-\gamma}}{(1-q)^2}\,. \tag{3}$$

If $b(n) \sim Cn$, we would have $B(q) \sim C(1-q)^{-2}$. Property (3) then shows that $C = e^{-\gamma}$ and this answers the question.

A suitable abelian theorem could even apply to deduce $b(n) \sim e^{-\gamma}n$ from (3).

References

[1] Doron Zeilberger and Noam Zeilberger, Two questions about the fractional counting of partitions, arXiv:1810.12701.v1

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