This is a short solution of Question 1 posed in the article *Fractional Counting of Partitions* by Doron and Noam Zeilberger

See `http://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/fcp.html` and

`https://arxiv.org/abs/1810.12701` . The first solution was given by Will Sawin, and is included as an appendix in the current version of their paper.

$$\textit{If} \quad C := \lim \frac{b(n)}{n} \quad \textit{exists then} \quad C = e^{-\gamma}.$$

Given $\epsilon > 0$, $|b(n) - Cn| < \epsilon n$ for n large enough. Hence there exists $\delta < 1$ such that if $B(z)$ is the generating function of $b(n)$ and $\phi(z) = z/(1-z)^2$ (the generating function of n), we have $|B(x) - C\phi(x)| < \epsilon\phi(x)$ for $1 - \delta < x < 1$. Multiplying by $(1-x)^2$, $C = \lim_{x\to 1^-}(1-x)^2 B(x)$. Taking logarithms

$$\log C = \lim_{x\to 1^-}\Big(\log(1-x) - \sum_{k=2}^{\infty}\log\big(1 - \frac{x^k}{k}\big)\Big) = -\lim_{x\to 1^-}\Big(x + \sum_{k=2}^{\infty}\Big(\log\big(1 - \frac{x^k}{k}\big) + \frac{x^k}{k}\Big)\Big).$$

Hence

$$\log C = -1 - \sum_{k=2}^{\infty}\Big(\frac{1}{k} - \log\frac{k}{k-1}\Big) = -\lim_{N\to\infty}\sum_{k=1}^{N}\Big(\frac{1}{k} - \log N\Big) = -\gamma.$$

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