Automatic Generation of Generating Functions for the Number of Spanning Trees for Grid Graphs (and Much More General Creatures) of Fixed (but arbitrary!) Width

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Very Important: This article comments on the Maple package `http://www.math.rutgers.edu/~zeilberg/tokhniot/KamaEtzim`. Some sample input and output can be gotten from the "front" of this article: `http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/etzim.html` .

This work extends the beautiful work of Paul Raff[1], that in turn, extended pioneering work of Frans Faase[2] and others (see the references of [1]) about counting spanning trees in grid graphs and more general creatures. However, *our* approach is very naive (in a good way!) .

For a graph G, let $T(G)$ be its number of spanning trees. Also let P_n be the path-graph of length n: $1 - 2 - \ldots - n$. It is fairly easy to see that for a fixed m, the generating function

$$F_m(z) = \sum_{n=0}^{\infty} T(\mathcal{P}_n \times \mathcal{P}_m) z^n$$

is a *rational function* of z. The Maple package `KamaEtzim` automatically computes $F_m(z)$ for *any* inputted numeric m. (See procedure `GFrect(m,z)` of `KamaEtzim`). In fact, we do something much more general. For *any* graph G, `KamaEtzim` can (explicitly!) compute

$$F_G(z) = \sum_{n=0}^{\infty} T(\mathcal{P}_n \times G) z^n \quad .$$

(See procedure `GFG(G,m,z,K)` in `KamaEtzim`) .

In fact we do something *even* more general! For *any* graph G, on m vertices, and for *any* bipartite (m, m) graph C, let $M_n(G, C)$ be the graph on mn vertices where the edges among

$$1 + im\,,\ 2 + im\,,\ \ldots,\ m + im$$

mimic the graph G (for $i = 0, \ldots, n-1$), and in addition the edges between

$$1+im\,,\,2+im\,,\,...\,,\,m+im$$

and

$$1+(i+1)m\,,\,2+(i+1)m\,,\,...\,,\,m+(i+1)m$$

$(0 \leq i < n-1)$ mimic the edges of C, given as a set of (up to m^2) ordered pairs $\{[\alpha,\beta]\}$. $[\alpha,\beta] \in C$ means that there is an edge between vertex $\alpha+im$ and vertex $\beta+(i+1)m$ for $0 \leq i < n-1$. Note that when C is the monogamy bipartite graph $\{[1,1],\ldots,[m,m]\}$, where Mr i is connected to Mrs i (but no cheating!), then $M_n(G,C)$ reduces to the Cartesian product $G \times \mathcal{P}_n$.

`KamaEtzim` can (explicitly!) compute the rational function (of z):

$$F_{G,C}(z) = \sum_{n=0} T(M_n(G,C))z^n \quad .$$

(See procedure `FGt(G,C,m,z,K)` in `KamaEtzim`.)

The output: For the (already known [1][2]) generating functions for the number of spanning trees on grid graphs $P_m \times P_n$ (resp. cylinder graphs $C_m \times P_n$) for $2 \leq m \leq 6$ see:
`http://www.math.rutgers.edu/~zeilberg/tokhniot/oKamaEtzim1`
(resp. `http://www.math.rutgers.edu/~zeilberg/tokhniot/oKamaEtzim2`) .

For the *brand-new* generating function, for the number of spanning trees on grid graphs $P_7 \times P_n$ see: `http://www.math.rutgers.edu/~zeilberg/tokhniot/oKamaEtzim3` .

The Method

The computer first constructs the graphs $M_n(G,C)$ for sufficiently many n, then finds the adjacency matrices, and then uses the matrix tree theorem (see wikipedia or any combinatorics textbook) to compute the sequence, and finally guesses a rational function describing this sequence. *Voilà tout!*

References

1. Paul Raff, *Spanning Trees in Grid Graphs*, `arXiv:0809.2551v1` , 15 Sept. 2008. (Note that some of the entries in the table of p. 9 are in error (they correctly represent something else, namely the cylinder graphs).

2. Frans Faase, *On the number of specific spanning subgraphs* $g \times p_n$, Ars Combinatorica **49** (1998), 129-154.

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