A WZ PROOF OF A "CURIOUS" IDENTITY

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Abstract: We give a short WZ-proof of an identity that appeared recently in Integers.

In [1] and [2], the identity,

$$\sum_{i=0}^{m}(x+m+1)(-1)^i\binom{x+y+i}{m-i}\binom{y+2i}{i}-\sum_{i=0}^{m}\binom{x+i}{m-i}(-4)^i=(x-m)\binom{x}{m} \text{...........}(*)$$

was proved using generating functions and double recursions respectively.

Here we cleverly construct the function

$$g(i)=\frac{-\binom{x}{i}(x^2-2ix-x+i+i^2)}{2(1+i)(x+i+2)(x+i+1)(-2)^i}$$

with the motives that

$$\sum_{i=0}^{m}\binom{x+i}{m-i}(-4)^i=(-2)^m(x+m+1)\left(\frac{1}{x+1}+\sum_{i=0}^{m-1}g(i)\right) \text{..........}(**)$$

The first sum, call it $T(m)$, on the LHS of $(*)$ divided by $(x+m+1)$ satisfies the following recurrence equation as found by EKHAD.

$a_0(m)T(m)+a_1(m)T(m+1)+a_2(m)T(m+2)+a_3(m)T(m+3)=0$, where

$a_0(m)=2(x-m-1)(x-m-2); a_1(m)=-(x-m-2)(2y-x+5m+11)$,

$a_2(m)=\left(-yx+3ym-2xm+4m^2+8y-5x+21m+28\right); a_3(m)=(m+3)(y+m+3)$

Now, it is routine(use maple!) to check that the sum of the RHS's of $(*)$ and $(**)$ divided by $(x+m+1)$ satisfies the recurrence equation. Moreover, both sides agree for $m=0,1,2$. Q.E.D.

REFERENCES

[1] Alois Panholzer, and Helmut Prodinger. A generating functions proof of a curious identity. Integers, pages A6, 3 pp. (electronic), 2002.

[2] Zhi-Wei Sun. A curious identity involving binomial coefficients. Integers, pages A4, 8 pp. (electronic), 2002

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