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An elegant Multi-Integral that implies an even more elegant determinant identity of Dougherty and McCammond

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Abstract We state and prove an explicit evaluation of a certain multi-variate integral and use it to furnish a new, and shorter, proof of an elegant determinant identity of Michael Dougherty and Jon McCammond that came up in their study of critical values of a complex polynomial.

1 Introduction

Beardon-Carne-Ng [1] investigated invertibility of the Jacobian of the polynomials

$$p(z) = \int_0^z (w - z_1) \cdots (w - z_n)\, dw$$

given by the n-by-n matrix

$$\boldsymbol{J}(z) = \left(\frac{\partial}{\partial z_i} p(z_j) \right)_{i,j}^{1,n}.$$

Using Topological arguments and techniques from Several Complex variables, the authors [1] show that every n-tuple of complex numbers arises as the critical *values* of some polynomial by proving that the determinant $\det(\boldsymbol{J}) \neq 0$, as along as $z_1, \ldots, z_n$ are non-zero and distinct. More recently, Dougherty and McCammond [2] computed $\det(\boldsymbol{J})$, explicitly, and reproved the above-mentioned result of [1] with a different method.

Some nomenclature adopted in the sequel: bold-face lower case letters are reserved for vectors, such as $\boldsymbol{z} = (z_1, \ldots, z_n)$ and $\hat{\boldsymbol{z}}^i = (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)$. Given $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$, denote $\bar{\boldsymbol{a}} = \sum_{i=1}^{\lceil n/2 \rceil} a_{2i-1}$ where $\lceil \cdot \rceil$ is the *ceiling function* while $\lfloor \cdot \rfloor$ is the *floor function*. Let $V_n(\boldsymbol{x}) = \prod_{1 \leq i < j \leq n} (x_j - x_i)$ be the *Vandermonde determinant* of the *Vandermonde matrix* $\boldsymbol{M} = (x_j^{i-1})_{i,j}^{1,n}$. We write $\boldsymbol{dx} = dx_1 \cdots dx_n$, the dimension n being suppressed when it is clear from the context.

2 The Main Theorem

The first purpose of this note is to prove the following elegant identity.

Theorem 2.1. *Let $z_1, \ldots z_n$ be commuting indeterminates, let n be a positive integer, and let $a_1, \ldots, a_n$, and b be non-negative integers. Then*

$$\int_0^{z_n} \cdots \int_0^{z_1} \prod_{i=1}^n x_i^b \prod_{1 \leq j,k \leq n} (x_j - z_k)^{a_k} \prod_{1 \leq i < j \leq n} (x_j - x_i)\, dx_1 \cdots dx_n$$

$$= (-1)^{\bar{\boldsymbol{a}}} \prod_{1 \leq i < j \leq n} (z_j - z_i)^{a_i + a_j + 1} \cdot \prod_{i=1}^n z_i^{a_i + b + 1} \cdot \frac{b! \prod_{i=1}^n a_i!}{(n + b + \sum_{i=1}^n a_i)!}. \tag{1}$$

3 An Example

We offer an illustrative example for Theorem 2.1. Choose $n = 2, b = 2022, a_1 = 1, a_2 = 2$. Then Eq. (1) reads

$$\int_0^{z_2}\int_0^{z_1}(x_1x_2)^{2022}(x_1-z_1)(x_2-z_1)(x_1-z_2)^2(x_2-z_2)^2(x_2-x_1)\,dx_1dx_2 = \frac{-\left(z_2-z_1\right)^4 z_1^{2024} z_2^{2025}}{17025293698597800}.$$

4 An interesting consequence

The second purpose our work here is to deduce from Theorem 2.1 (and thereby give a shorter proof) of the following even more elegant identity, discovered, and first proved in [2].

Theorem 4.1. *(***Dougherty and McCammond***) Let*

$$p(Z) := \int_0^Z \prod_{i=1}^n (w-z_i)^{a_i}\,dw,$$

and let $\mathbf{J}(z_1,\ldots,z_n)$ *be the* $n\times n$ *matrix whose* (i,j)*-entry is* $\mathbf{J}(z_1,\ldots,z_n)_{i,j} := \frac{\partial}{\partial z_i}p(z_j)$*, then*

$$\det \mathbf{J}(z_1,\ldots,z_n) = \frac{\prod_{i=1}^n a_i!}{(\sum_{i=1}^n a_i)!}\cdot\prod_{i=1}^n(-z_j)^{a_j}\cdot\prod_{\substack{1\le i,j\le n\\ i\neq j}}(z_i-z_j)^{a_j}. \qquad (2)$$

Proof that Theorem 2.1 $\Longrightarrow$ **Theorem 4.1:** Let's rewrite the determinant and apply *Cauchy's alternant formula* so that

$$\begin{aligned}
\det(\mathbf{J}) &= \det\left(-a_i\int_0^{z_j}\prod_{k=1}^n(w-z_k)^{a_k}\frac{dw}{w-z_i}\right) = \det\left(-a_i\int_0^{z_j}\prod_{k=1}^n(x_j-z_k)^{a_k}\frac{dx_j}{x_j-z_i}\right)\\
&= \prod_{i=1}^n(-a_i)\int_0^{z_n}\cdots\int_0^{z_1}\prod_{1\le j,k\le n}(x_j-z_k)^{a_k}\cdot\det\left(\frac{1}{x_j-z_i}\right)_{i,j}^{1,n}\boldsymbol{dx}\\
&= \prod_{i=1}^n(-a_i)\int_0^{z_n}\cdots\int_0^{z_1}\prod_{1\le j,k\le n}(x_j-z_k)^{a_k}\cdot\left[\frac{\prod_{1\le i<j\le n}(z_i-z_j)(x_j-x_i)}{\prod_{1\le i,j\le n}(x_j-z_i)}\right]\boldsymbol{dx}\\
&= \prod_{i=1}^n(-a_i)\cdot\prod_{1\le i<j\le n}(z_i-z_j)\cdot\int_0^{z_n}\cdots\int_0^{z_1}\prod_{1\le j,k\le n}(x_j-z_k)^{a_k-1}\cdot V_n(\boldsymbol{x})\,\boldsymbol{dx}.
\end{aligned}$$

But by Theorem 2.1, with $b=0$ and $(a_1,\ldots,a_n)$ replaced by $(a_1-1,\ldots a_n-1)$, this equals

$$\begin{aligned}
\det(\mathbf{J}) &= (-1)^{\bar{\mathbf{a}}-\lceil n/2\rceil}\prod_{i=1}^n(-a_i)\prod_{1\le i<j\le n}(z_i-z_j)\prod_{1\le i<j\le n}(z_j-z_i)^{a_i+a_j-1}\prod_{i=1}^n z_i^{a_i}\frac{\prod_{i=1}^n(a_i-1)!}{(\sum_{i=1}^n a_i)!}\\
&= \frac{\prod_{i=1}^n a_i!}{(\sum_{i=1}^n a_i)!}\prod_{i=1}^n(-z_j)^{a_j}\prod_{\substack{1\le i,j\le n\\ i\neq j}}(z_i-z_j)^{a_j}. \quad \square
\end{aligned}$$

5 Proof of Theorem 2.1:

Proceed by induction on n and b. When $n=1$ and $b=0$, whose proof is left to the reader's five-year-old, the claim is saying $\int_0^{z_1}(x_1-z_1)^{a_1}\,dx_1 = (-1)^{a_1}\frac{z_1^{a_1+1}}{a_1+1}$.

Let's denote the statement of Theorem 2.1 by $\mathbf{A}(n,b)$.

Proof that $\mathbf{A}(n,b) \implies \mathbf{A}(n,b+1)$:

Let $\boldsymbol{a} = (a_1,\dots,a_n)$. We claim that *both* sides of Eq. (1), let's call them $L(\boldsymbol{a};b)$ and $R(\boldsymbol{a};b)$ respectively, satisfy the recurrence

$$X(\boldsymbol{a};b+1) = \sum_{i=1}^{n}\left(\prod_{\substack{j=1\\ j\neq i}}^{n}\frac{z_j}{z_j-z_i}\right)\cdot X(a_1,\dots,1+a_i,\dots,a_n;b) + \left(\prod_{j=1}^{n} z_j\right)\cdot X(\boldsymbol{a};b). \tag{3}$$

In other words, if you replace X by either L or R you get a true statement. Regarding the left-hand side of (1), in fact, this identity is already true if you replace X by the **integrand** of the left side of (1), since there are no x_i's in sight, it is still true when you integrate with respect to $x_1,\dots,x_n$. We leave both checks as pleasant exercises for the curious reader. □

Proof that $\mathbf{A}(n-1,b)$ for all $b \implies \mathbf{A}(n,0)$:

Fix $a_1,\dots,a_n$. Notations: $\boldsymbol{a} = (a_1,\dots,a_n), \boldsymbol{z} = (z_1,\dots,z_n), V_n(\boldsymbol{x}) = \prod_{1\le i<j\le n}(x_j - x_i)$ and $\hat{\boldsymbol{x}}^i = (x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$. Define the multi-variable polynomial

$$F(\boldsymbol{x},\boldsymbol{z}) := \prod_{1\le j,k\le n}(x_j - z_k)^{a_k}.$$

We claim (check! this reduces to the Laplace expansion $V_n(\boldsymbol{x}) = \sum_{i=1}^{n}(-1)^i V_{n-1}(\hat{\boldsymbol{x}}^i)x_i^{n-1}$) that

$$\left(n+\sum_{i=1}^{n} a_i\right) F(\boldsymbol{x},\boldsymbol{z})V_n(\boldsymbol{x}) = \sum_{i=1}^{n}(-1)^i V_{n-1}(\hat{\boldsymbol{x}}^i)\cdot\frac{\partial}{\partial x_i}\left[\prod_{j=1}^{n}(x_i - z_j)\cdot F(\boldsymbol{x},\boldsymbol{z})\right]. \tag{4}$$

Applying $\int_0^{z_n}\dots\int_0^{z_1}(\cdots)dx_1\cdots dx_n$, we get

$$\begin{aligned}
&\left(n+\sum_{i=1}^{n} a_i\right)\int_0^{z_n}\dots\int_0^{z_1} F(\boldsymbol{x},\boldsymbol{z})V_n(\boldsymbol{x})\,\boldsymbol{dx}\\
&=\sum_{i=1}^{n}(-1)^i\int_0^{z_n}\dots\int_0^{z_1} V_{n-1}(\hat{\boldsymbol{x}}^i)\cdot\frac{\partial}{\partial x_i}\left[\prod_{j=1}^{n}(x_i - z_j)\cdot F(\boldsymbol{x},\boldsymbol{z})\right]\boldsymbol{dx}\\
&=\sum_{i=1}^{n}(-1)^i\int_0^{z_n}\dots\int_0^{z_{i+1}}\int_0^{z_{i-1}}\dots\int_0^{z_1} V_{n-1}(\hat{\boldsymbol{x}}^i)\,\boldsymbol{d\hat{x}^i}\times\int_0^{z_i}\frac{\partial}{\partial x_i}\left[\prod_{j=1}^{n}(x_i - z_j)\cdot F(\boldsymbol{x},\boldsymbol{z})\right]dx_i.
\end{aligned}$$

By the *Fundamental Theorem of Calculus*, we have

$$\begin{aligned}
\int_0^{z_i}\frac{\partial}{\partial x_i}\left[\prod_{j=1}^{n}(x_i - z_j)\cdot F(\boldsymbol{x},\boldsymbol{z})\right]dx_i &= \prod_{j=1}^{n}(x_i - z_j)\cdot F(\boldsymbol{x},\boldsymbol{z})\Big|_{x_i=0}^{x_i=z_i}\\
&= \prod_{j=1}^{n}(x_i - z_j)^{a_j+1}\prod_{j=1}^{n}\prod_{\substack{1\le i'\le n\\ i'\neq i}}(x_{i'} - z_j)^{a_j}\Big|_{x_i=0}^{x_i=z_i}\\
&= -\prod_{j=1}^{n}(-z_j)^{a_j+1}\prod_{j=1}^{n}\prod_{\substack{1\le i'\le n\\ i'\neq i}}(x_{i'} - z_j)^{a_j}.
\end{aligned}$$

Going back, we have that the left side of Eq. (1), when $b = 0$, is

$$\frac{-1}{n+\sum_{i=1}^{n} a_i} \cdot \prod_{j=1}^{n} (-z_j)^{a_j+1} \cdot \sum_{i=1}^{n} (-1)^i \int_0^{z_n} \dots \int_0^{z_{i+1}} \int_0^{z_{i-1}} \dots \int_0^{z_1} V_{n-1}(\hat{\boldsymbol{x}}^i) \times \prod_{j=1}^{n} \prod_{\substack{1 \le i' \le n \\ i' \ne i}} (x_{i'} - z_j)^{a_j} \, \boldsymbol{d}\hat{\boldsymbol{x}}^i$$

$$= \frac{-1}{n+\sum_{i=1}^{n} a_i} \cdot \prod_{j=1}^{n} (-z_j)^{a_j+1} \cdot \sum_{i=1}^{n} (-1)^i \int_0^{z_n} \dots \int_0^{z_{i+1}} \int_0^{z_{i-1}} \dots \int_0^{z_1} V_{n-1}(y_1, \dots, y_{n-1}) \times \prod_{j=1}^{n} \prod_{i=1}^{n-1} (y_i - z_j)^{a_j} \, dy_1 \cdots dy_{n-1}.$$

We now claim that

$$\sum_{i=1}^{n} (-1)^{i-1} \int_0^{z_n} \dots \int_0^{z_{i+1}} \int_0^{z_{i-1}} \dots \int_0^{z_1} V_{n-1}(y_1, \dots, y_{n-1}) \cdot \prod_{j=1}^{n} \prod_{i=1}^{n-1} (y_i - z_j)^{a_j} \, dy_1 \cdots dy_{n-1}$$
$$= \int_{z_1}^{z_n} \dots \int_{z_1}^{z_2} V_{n-1}(y_1, \dots, y_{n-1}) \cdot \prod_{j=1}^{n} \prod_{i=1}^{n-1} (y_i - z_j)^{a_j} \, dy_1 \cdots dy_{n-1}. \tag{5}$$

In order to prove this, notice that each of the integrands on the left-hand side, and the integrand on the right-hand side, are *anti-symmetric* in their arguments. Hence, for any given permutation of the integration variables, the effect is to multiple it by the sign of that permutation. Calling the common integrand $f(y_1, \dots, y_{n-1})$ and denoting $A_n(i) = Per(1, \dots, i-1, i+1, \dots n)$, we claim that

$$\sum_{i=1}^{n} (-1)^{i-1} \sum_{\pi \in A_n(i)} sgn(\pi) \int_0^{z_{\pi(n)}} \dots \int_0^{z_{\pi(i+1)}} \int_0^{z_{\pi(i-1)}} \dots \int_0^{z_{\pi(1)}} f(y_1, \dots, y_{n-1}) dy_1 \cdots dy_{n-1}$$
$$= \sum_{\pi \in A_n(1)} sgn(\pi) \int_{z_1}^{z_{\pi(n)}} \dots \int_{z_1}^{z_{\pi(2)}} f(y_1, \dots, y_{n-1}) dy_1 \dots dy_{n-1}. \tag{6}$$

Since both sides of Eq. (6) are $(n-1)!$ times the respective sides of Eq. (5), if we can prove (6), then (5) would follow.

But **surprise!**, Eq. (6) is valid for *any* integrand! It is just a relation between *regions* in $\mathbb{R}^{n-1}$ that is equivalent to an easy symmetric function identity, that we also leave as a pleasant exercise to the reader. Now make the change of variables $(y_1, \dots, y_{n-1}) \to (y_1 - z_1, \dots, y_{n-1} - z_1)$, thereby making it a case of $\mathbf{A}(n-1, b)$ with $b = a_1$; and $a_1, \dots, a_{n-1}$ replaced by $a_2, \dots, a_n$, respectively; and $z_1, \dots, z_{n-1}$ replaced by $z_2 - z_1, \dots, z_n - z_1$, respectively. Plugging it in and simplifying, completes the induction. □ □

Remark: Readers that prefer not do the 'exercises' can convince themselves of all the claims, empirically, by playing with the Maple package `CritVal.txt` available from the front of this paper

`https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/crit.html`

6 Appendix: proof of the "exercises"

Proof of Equation (3):

Let $\boldsymbol{e}_i = (0,\dots,0,1,0,\dots,0)$ be the i-th unit vector. So, the recurrence (3) reads as

$$X(\boldsymbol{a};b+1) = \sum_{i=1}^{n} X(\boldsymbol{a}+\boldsymbol{e}_i;b)\cdot \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{z_j}{z_j - z_i} + X(\boldsymbol{a};b)\cdot\prod_{i=1}^{n} z_i.$$

The integrand on the left-hand side of (1) satisfies this recurrence because it reduces to (check!)

$$\prod_{i=1}^{n}\frac{x_i}{z_i} = 1 + \sum_{i=1}^{n}\left(\frac{x_i - z_i}{z_i}\right)\cdot\prod_{\substack{j=1\\ j\neq i}}^{n}\frac{x_j - z_i}{z_j - z_i}. \tag{7}$$

Induct on n. The case $n=1$ is obvious. Let $w = x_1$ and treat (7) as a *linear* equation in w:

$$\frac{w}{z_1}\cdot\prod_{i=2}^{n}\frac{x_i}{z_i} = 1 + \frac{w-z_1}{z_1}\prod_{j=2}^{n}\frac{x_j - z_1}{z_j - z_1} + \sum_{i=2}^{n}\frac{x_i - z_i}{z_i}\cdot\frac{w - z_i}{z_1 - z_i}\cdot\prod_{\substack{j=2\\ j\neq i}}^{n}\frac{x_j - z_i}{z_j - z_i}.$$

If $w = z_1$, equality holds by the induction assumption. Putting $w=0$, turns the task into

$$\prod_{i=2}^{n}\frac{x_i - z_1}{z_i - z_1} = 1 + \sum_{i=2}^{n}\frac{x_i - z_i}{z_i - z_1}\cdot\prod_{\substack{j=2\\ j\neq i}}^{n}\frac{x_j - z_i}{z_j - z_i}$$

which is *again* the inductive step with $x_i \to x_i - z_1$ and $z_i \to z_i - z_1$. So, the first claim holds. Now, we focus on the right-hand side of (1) and show it, too, fulfils (3). This tantamount

$$\frac{1+b}{n+b+1+\sum_{i=1}^{n} a_i} = 1 + \sum_{i=1}^{n}(-1)^{\delta_i}\frac{1+a_i}{n+b+1+\sum_{i=1}^{n} a_i}\cdot(-1)^{i-1}$$

where $\delta_i = 1$ if i is odd; $\delta_i = 0$, otherwise. The justification is immediate. $\square$

Proof of Equation (4):

After computing the derivatives and cancelling out the term $F(\boldsymbol{x},\boldsymbol{z})$, the claim boils down to

$$\begin{aligned}\left(n + \sum_{i=1}^{n} a_i\right) V_n(\boldsymbol{x}) &= \sum_{i=1}^{n}(-1)^i\, V_{n-1}(\hat{\boldsymbol{x}}^i)\cdot\sum_{j=1}^{n}(1+a_j)\prod_{\substack{k=1\\ k\neq j}}^{n}(x_i - z_k)\\ &= \sum_{j=1}^{n}(1+a_j)\ \sum_{i=1}^{n}(-1)^i\, V_{n-1}(\hat{\boldsymbol{x}}^i)\cdot\prod_{\substack{k=1\\ k\neq j}}^{n}(x_i - z_k).\end{aligned} \tag{8}$$

If we consider the sum $\sum_{i=1}^{n}(-1)^i\, V_{n-1}(\hat{\boldsymbol{x}}^i)\cdot x_i^m$, it is recognized as the determinant of the Vandermonde matrix $\boldsymbol{M}$ where the last row is replaced by the vector $(x_1^m,\dots,x_n^m)$. This, however, is well-known to be $V_n(\boldsymbol{x})\cdot s_{m-n+1}$ where s_μ is the *Schur polynomial*. Notice that $\prod_{\substack{k=1\\ k\neq j}}^{n}(x_i - z_k)$ is a polynomial of degree $n-1$, in x_i. But, $s_{m-n+1} = 1$ if $m = n-1$; and $s_{m-n+1} = 0$ for $m < n-1$. Thus, equation (8) becomes trivial $(n + \sum_{i=1}^{n} a_i)\, V_n(\boldsymbol{x}) = \sum_{j=1}^{n}(1+a_j)\, V_n(\boldsymbol{x})$. $\square$

Proof of Equation (5):

Eq. (5) has been elucidated, neatly, using a symmetrizing process as depicted by Eq. (6). Here, we offer yet another verification. To this end, start with the right-hand side of Eq. (5) by replacing each integral with $\int_{z_1}^{z_i} = \int_0^{z_i} - \int_0^{z_1}$ and expand the product $(\int_0^{z_n} - \int_0^{z_1}) \cdots (\int_0^{z_2} - \int_0^{z_1})$. Based on the fact that the integrand is an anti-symmetric, any multi-integral involving repeated evaluation $\int_0^{z_1}\int_0^{z_1}$ vanishes while out-of-order pairs, such as $\int_0^{z_3}\int_0^{z_1}\int_0^{z_2}$, changes sign where *reordered*, that is, $-\int_0^{z_3}\int_0^{z_2}\int_0^{z_1}$. On account of this, we find the $(n-1)$-tuple multi-integrals

$$\begin{aligned}
\int_{z_1}^{z_n} \cdots \int_{z_1}^{z_2} &= \left(\int_0^{z_n} - \int_0^{z_1}\right) \cdots \left(\int_0^{z_2} - \int_0^{z_1}\right) \\
&= \int_0^{z_n} \cdots \int_0^{z_1} - \sum_{i=2}^{n} \int_0^{z_n} \cdots \int_0^{z_{i-1}} \int_0^{z_1} \int_0^{z_{i+1}} \cdots \int_0^{z_2} \\
&= \sum_{i=1}^{n} (-1)^{i-1} \int_0^{z_n} \cdots \int_0^{z_{i-1}} \int_0^{z_{i+1}} \cdots \int_0^{z_1}.
\end{aligned}$$

The proof follows. □

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