An elegant Multi-Integral that implies an even more elegant determinant identity of Dougherty and McCammond

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The first purpose of this note is to prove the following elegant identity.

Theorem A: Let $z_1, \ldots z_n$ be commuting indeterminates, let n be a positive integer, and let $a_1, \ldots, a_n$, and b be non-negative integers. Let $\bar{a} := \sum_{i=1}^{\lceil n/2 \rceil} a_{2i-1}$. Then

$$\int_0^{z_1} \cdots \int_0^{z_n} \prod_{i=1}^{n} x_i^b \prod_{1 \le j,k \le n} (x_j - z_k)^{a_k} \prod_{1 \le i < j \le n} (x_j - x_i)\, dx_n \ldots dx_1$$

$$= (-1)^{\bar{a}} \prod_{1 \le i < j \le n} (z_j - z_i)^{a_i + a_j + 1} \cdot \prod_{i=1}^{n} z_i^{a_i + b + 1} \cdot \frac{b! \prod_{i=1}^{n} a_i!}{(n + b + \sum_{i=1}^{n} a_i)!}. \qquad (1)$$

The second purpose is to deduce from it (and thereby give a shorter proof) of the following even more elegant identity, discovered, and first proved in [1].

Theorem B (Dougherty and McCammond): Let

$$p(Z) := \int_0^{Z} \prod_{i=1}^{n} (w - z_i)^{a_i}\, dw,$$

and let $\mathbf{J}(z_1, \ldots, z_n)$ be the $n \times n$ matrix whose (i,j) entry is $\mathbf{J}(z_1, \ldots, z_n)_{i,j} := \frac{\partial}{\partial z_i} p(z_j)$, then

$$\det \mathbf{J}(z_1, \ldots, z_n) = \frac{\prod_{i=1}^{n} a_i!}{(\sum_{i=1}^{n} a_i)!} \cdot \prod_{i=1}^{n} (-z_i)^{a_i} \cdot \prod_{\substack{1 \le i,j \le n \\ i \ne j}} (z_i - z_j)^{a_j}. \qquad (2)$$

Proof that A $\Rightarrow$ B: Let's rewrite the determinant and apply Cauchy's alternant formula so that

$$\det(\mathbf{J}) = \det \left(-a_i \int_0^{z_j} \prod_{k=1}^{n} (w - z_k)^{a_k} \frac{dw}{w - z_i} \right) = \det \left(-a_i \int_0^{z_j} \prod_{k=1}^{n} (x_j - z_k)^{a_k} \frac{dx_j}{x_j - z_i} \right)$$

$$= \prod_{i=1}^{n} (-a_i) \int_0^{z_1} \cdots \int_0^{z_n} \prod_{1 \le j,k \le n} (x_j - z_k)^{a_k} \cdot \det \left(\frac{1}{x_j - z_i} \right)_{i,j}^{1,n} dx_n \cdots dx_1$$

$$= \prod_{i=1}^{n} (-a_i) \int_0^{z_1} \cdots \int_0^{z_n} \prod_{1 \le j,k \le n} (x_j - z_k)^{a_k} \cdot \left[\frac{\prod_{1 \le i < j \le n} (z_i - z_j)(x_j - x_i)}{\prod_{1 \le i,j \le n} (x_j - z_i)} \right] dx_n \cdots dx_1$$

$$= \prod_{i=1}^{n} (-a_i) \cdot \prod_{1 \le i < j \le n} (z_i - z_j) \cdot \int_0^{z_1} \cdots \int_0^{z_n} \prod_{1 \le j,k \le n} (x_j - z_k)^{a_k - 1} \cdot \prod_{1 \le i < j \le n} (x_j - x_i)\, dx_n \cdots dx_1.$$

But by Theorem **A**, with $b=0$ and $(a_1,\ldots,a_n)$ replaced by $(a_1-1,\ldots a_n-1)$, this equals

$$= (-1)^{\bar{a}-\lceil n/2\rceil}\prod_{i=1}^{n}(-a_i)\prod_{1\le i<j\le n}(z_i-z_j)\prod_{1\le i<j\le n}(z_j-z_i)^{a_i+a_j-1}\prod_{i=1}^{n} z_i^{a_i}\frac{\prod_{i=1}^{n}(a_i-1)!}{(\sum_{i=1}^{n}a_i)!}$$

$$= \frac{\prod_{i=1}^{n}a_i!}{(\sum_{i=1}^{n}a_i)!}\prod_{i=1}^{n}(-z_i)^{a_i}\prod_{\substack{1\le i,j\le n\\ i\ne j}}(z_i-z_j)^{a_j}. \quad \square$$

Proof of Theorem A: The proof is by induction on n and b. When $n=1$ and $b=0$ this is saying that $\int_0^{z_1}(x_1-z_1)^{a_1}\,dx_1=(-1)^{a_1}\frac{(z_1)^{a_1+1}}{a_1+1}$, whose proof is left to the reader's five-year-old.

Let's denote the statement of theorem **A** by $\mathbf{A}(n,b)$.

Proof that $\mathbf{A}(n,b)\Rightarrow\mathbf{A}(n,b+1)$

We claim that *both* sides of Eq. (1), let's call them $L(a_1,\ldots,a_n;b)$ and $R(a_1,\ldots,a_n;b)$ respectively, satisfy the recurrence

$$X(a_1,\ldots,a_n;b{+}1) = \sum_{i=1}^{n}\left(\prod_{\substack{j=1\\ j\ne i}}^{n}\frac{z_j}{z_j-z_i}\right)\cdot X(a_1,\ldots .a_i{+}1,\ldots,a_n;b)+\left(\prod_{j=1}^{n}z_j\right)\cdot X(a_1,\ldots,a_n;b). \tag{3}$$

In other words, if you replace X by either L or R you get a true statement. Regarding the left side of (1), in fact, this identity is already true if you replace X by the **integrand** of the left side of (1), since there are no x_i's in sight, it is still true when you integrate with respect to $x_1,\ldots,x_n$. We leave both checks as pleasant exercises for the reader. $\square$

Proof that $\mathbf{A}(n-1,b)$ for all b implies $\mathbf{A}(n,0)$

Fix $a_1,\ldots,a_n$. Let $V(x_1,\ldots,x_n):=\prod_{1\le i<j\le n}(x_j-x_i)$ and

$$F(x_1,\ldots x_n;z_1,\ldots,z_n):=\prod_{1\le j,k\le n}(x_j-z_k)^{a_k}.$$

We claim (check!) that

$$\left(n+\sum_{i=1}^{n}a_i\right)F(x_1,\ldots x_n;z_1,\ldots,z_n)V(x_1,\ldots,x_n)$$

$$=\sum_{i=1}^{n}(-1)^iV(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)\cdot\frac{\partial}{\partial x_i}\left[\prod_{j=1}^{n}(x_i-z_j)\cdot F(x_1,\ldots,x_n;z_1,\ldots,z_n)\right]. \tag{4}$$

Applying $\int_0^{z_1} \dots \int_0^{z_n} dx_n \cdots dx_1$, we get

$$\left(n + \sum_{i=1}^n a_i\right) \int_0^{z_1} \dots \int_0^{z_n} F(x_1, \dots x_n; z_1, \dots, z_n) V(x_1, \dots, x_n) dx_n \cdots dx_1$$

$$= \sum_{i=1}^n (-1)^i \int_0^{z_1} \dots \int_0^{z_n} V(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \cdot \frac{\partial}{\partial x_i} \left[\prod_{j=1}^n (x_i - z_j) \cdot F(x_1, \dots, x_n; z_1, \dots, z_n) \right] dx_n \cdots dx_1$$

$$= \sum_{i=1}^n (-1)^i \int_0^{z_1} \dots \int_0^{z_{i-1}} \int_0^{z_{i+1}} \dots \int_0^{z_n} V(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) dx_n \cdots dx_{i+1} dx_{i-1} \cdots dx_1$$

$$\times \int_0^{z_i} \frac{\partial}{\partial x_i} \left[\prod_{j=1}^n (x_i - z_j) \cdot F(x_1, \dots, x_n; z_1, \dots, z_n) \right] dx_i.$$

By the *Fundamental Theorem of Calculus*, we have

$$\int_0^{z_i} \frac{\partial}{\partial x_i} \left[\prod_{j=1}^n (x_i - z_j) \cdot F(x_1, \dots, x_n; z_1, \dots, z_n) \right] dx_i = \prod_{j=1}^n (x_i - z_j) \cdot F(x_1, \dots, x_n; z_1, \dots, z_n) \Big|_{x_i=0}^{x_i=z_i}$$

$$= \prod_{j=1}^n (x_i - z_j)^{a_j+1} \prod_{j=1}^n \prod_{\substack{1 \le i' \le n \\ i' \ne i}} (x'_i - z_j)^{a_j} \Big|_{x_i=0}^{x_i=z_i} = -\prod_{j=1}^n (-z_j)^{a_j+1} \prod_{j=1}^n \prod_{\substack{1 \le i' \le n \\ i' \ne i}} (x'_i - z_j)^{a_j}.$$

Going back we have that the left side of Eq. (1), when $b = 0$, is

$$\frac{-1}{n + \sum_{i=1}^n a_i} \cdot \prod_{j=1}^n (-z_j)^{a_j+1} \cdot \sum_{i=1}^n (-1)^i \int_0^{z_1} \dots \int_0^{z_{i-1}} \int_0^{z_{i+1}} \dots \int_0^{z_n} V(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$$

$$\times \prod_{j=1}^n \prod_{\substack{1 \le i' \le n \\ i' \ne i}} (x'_i - z_j)^{a_j} dx_n \cdots dx_{i+1} dx_{i-1} \cdots dx_1$$

$$= \frac{-1}{n + \sum_{i=1}^n a_i} \cdot \prod_{j=1}^n (-z_j)^{a_j+1} \cdot \sum_{i=1}^n (-1)^i \int_0^{z_1} \dots \int_0^{z_{i-1}} \int_0^{z_{i+1}} \dots \int_0^{z_n} V(y_1, \dots, y_{n-1})$$

$$\times \prod_{j=1}^n \prod_{i=1}^{n-1} (y_i - z_j)^{a_j} dy_{n-1} \cdots dy_1.$$

We now claim that

$$\sum_{i=1}^{n}(-1)^{i-1}\int_0^{z_1}\dots\int_0^{z_{i-1}}\int_0^{z_{i+1}}\dots\int_0^{z_n} V(y_1,\dots,y_{n-1})\cdot\prod_{j=1}^{n}\prod_{i=1}^{n-1}(y_i-z_j)^{a_j}\,dy_{n-1}\cdots dy_1$$

$$=\int_{z_1}^{z_2}\dots\int_{z_1}^{z_n} V(y_1,\dots,y_{n-1})\cdot\prod_{j=1}^{n}\prod_{i=1}^{n-1}(y_i-z_j)^{a_j}\,dy_{n-1}\cdots dy_1. \qquad (5)$$

In order to prove this, notice that each of the integrands on the left, and the integrand on the right, are *anti-symmetric* in their arguments. Hence, for any given permutation of the integration variables, the effect is to multiple it by the sign of that permutation. Calling the common integrand $f(y_1,\dots,y_{n-1})$ and denoting $A_n(i)=Per(1,\dots,i-1,i+1,\dots n)$, we claim that

$$\sum_{i=1}^{n}(-1)^{i-1}\sum_{\pi\in A_n(i)} sgn(\pi)\int_0^{z_{\pi(1)}}\dots\int_0^{z_{\pi(i-1)}}\int_0^{z_{\pi(i+1)}}\dots\int_0^{z_{\pi(n)}} f(y_1,\dots,y_{n-1})dy_{n-1}\cdots dy_1$$

$$=\sum_{\pi\in A_n(1)} sgn(\pi)\int_{z_1}^{z_{\pi(2)}}\dots\int_{z_1}^{z_{\pi(n)}} f(y_1,\dots,y_{n-1})dy_{n-1}\cdots dy_1. \qquad (6)$$

Since both sides of Eq. (6) are $(n-1)!$ times the respective sides of Eq. (5), if we can prove (6), then (5) would follow.

But **surprise!**, Eq. (6) is valid for *any* integrand! It is just a relation between *regions* in R^{n-1} that is equivalent to an easy symmetric function identity, that we also leave as a pleasant exercise to the reader. Now make the change of variables $(y_1,\dots,y_{n-1})\rightarrow(y_1-z_1,\dots,y_{n-1}-z_1)$, thereby making it a case of $\mathbf{A}(n-1,b)$ with $b=a_1$; and $a_1,\dots,a_{n-1}$ replaced by $a_2,\dots,a_n$, respectively; and $z_1,\dots,z_{n-1}$ replaced by $z_2-z_1,\dots,z_n-z_1$, respectively. Plugging it in and simplifying, completes the induction. □ □

Comment: Readers that prefer not do the 'exercises' can convince themselves of all the claims, empirically, by playing with the Maple package `CritVal.txt` available from the front of this paper

`https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/crit.html` .

Reference

[1] M. Dougherty, J. M. McCammond, *Critical Points, critical values, and a determinant identity for complex polynomials*, Proc. Amer. Math. Soc. **148** (2020), 5722-5289.
`https://arxiv.org/abs/1908.10477` .

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