The Composition Enumeration Reciprocity Theorem:
The Number of Compositions of n Into Parts $k\,(mod\ a)$ ($k<a$) Equals
The Number of Compositions of $n+a-k$ Into Parts $a\,(mod\ k)$ that are $\geq a$

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Theorem: See title .

Proof: Given a composition $(k+an_1, k+an_2, \dots, k+an_r)$ of n, construct the word, in the *alphabet* $\{k,a\}$, $w := ka^{n_1}ka^{n_2}\dots ka^{n_r}$ (whose sum is n). Now replace the first letter (k) of w by a, getting a word $w' := aa^{n_1}ka^{n_2}\dots ka^{n_r}$, still in the alphabet $\{k,a\}$ but whose sum is $n+a-k$, and now parse it as $w' = ak^{m_1}ak^{m_2}\dots ak^{m_s}$ that gives you the following composition of $n+a-k$ into parts that are $a(mod\ k)$ and $\geq a$:

$$(a+m_1k, \dots, a+m_sk) \quad . \quad \square$$

Remark: The case $k=1, a=2$ (with essentially the same bijective proof) was done in Drew Sills' nice article *Compositions, Parititions, and Fibonacci Numbers*, that appeared in the *Fibonacci Quarterly* **40** (2011), 348-354, available on-line at
`http://math.georgiasouthern.edu/~asills/comps/CompositionsFQRev.pdf` .

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