How many coin tosses would you need until you get n Heads OR m Tails

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Abstract: We harness both human ingenuity and the power of symbolic computation to study the number of coin tosses until reaching n Heads OR m Tails. We also talk about the closely related problem of reaching n Heads AND m Tails.

Preface

If you toss a coin whose probability of Heads is p, until you reach n Heads, you should expect to make n/p coin tosses, and the variance and higher moments are easily derived from the explicit probability generating function, (as usual $q := 1 - p$)

$$\sum_{k=0}^{\infty} \binom{n+k-1}{n-1} (px)^n (qx)^k \, = \, \left(\frac{px}{1-qx} \right)^n \quad ,$$

which is essentially the *negative-binomial distribution* (note that usually one only counts the number of Tails until you reach n Heads, but we are interested in the total number of coin-tosses, so we add the n Heads. From this probability generating function we can extract not only the expectation, but by repeated differnting with respect to x, and plugging in $x = 1$ we can explicit expressions of as many as desired *factorial moments*, that in turn gives the *moments*, and from them the *central moments*. Then we can compute the *scaled central moments*, take the limit as $n \to \infty$ and prove that for a **fixed** p it tends to the good old Normal Distribution.

But what if you are not *Headist*? What if you like Tails just as much, and stop as soon as you get n Heads OR m Tails? Another interesting stopping rule is to make **both** Heads and Tails happy and keep tossing until you get n Heads AND m Tails. Now things are not as nice and simple. Nevertheless, using Wilf-Zeilberger algorithmic proof theory [PWZ], we can derive the next-best thing, linear recurrences that enable very fast comptation of these quantites. In the case of a fair coin and tossing until getting n Heads or n Tails, we get, explicit expressions not only for the expectation and variance, but for many moments, and prove that their centralized-scaled tend, rather than the standard normal distribution, $N(0, 1)$, to those of $-|N(0, 1)|$.

While we (or rather our computer) can go pretty far, we can not prove it for *all* moments. In the second part of this article we will prove it completely.

References

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