CHOMP, RECURRENCES, and CHAOS(?)

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Dedicated to Saber Elaydi on his 60th Birthday

Abstract. In this article, dedicated with admiration and friendship to Chaos and Difference (and hence Recurrence) Equations Guru Saber Elaydi, I give a new approach and a new algorithm for Chomp, David Gale's celebrated combinatorial game. This work is inspired by Xinyu Sun's "ultimate-periodicity" conjecture and by its brilliant proof by high-school-student Steven Byrnes. The algorithm is implemented in a Maple package `BYRNES` accompanying this article. By looking at the output, and inspired by previous work of Andries Brouwer, I speculate that Chomp is Chaotic, in a yet-to-be-made-precise sense, because the losing positions are given by "weird" recurrences.

Saber Elaydi

When Gerry Ladas asked me to write an article dedicated to Saber Elaydi, I hesitated, since while we both work on *difference equations*, our research interests (but not our mathematical philosophies!) are almost diametrically opposite. In particular, the subject of my current research, *Combinatorial Games*, seemed superficially to be far removed from Saber's research interests. On further thought, however, I realized that this current research of mine is not as far from Saber, after all, since *recurrences* feature in it prominently. But *recurrence equations* are almost synonymous with *difference equations* (see the next section), and Saber wrote *the* **Book**[E1] on this subject, a modern classic that I, and my students, thoroughly enjoyed. As I dwelt even deeper into my research, I also realized that **chaos**, or something like it, also arises naturally, and Saber is also a Chaos guru (cf. [E2]). So even though the *primary* "subject classification" of this article is "Combinatorial Games", the *secondary* ones are "Recurrence equations" and "Chaos", so I hope that Saber will find this interesting. Happy 60th Birthday, Saber!

Difference vs. Recurrence Equations

In most contexts, the notions of *difference equation* and of *recurrence equation* are identical, and the choice of which of them to use is merely cultural, the former preferred by numerical analysts and the latter by combinatorialists and number theorists.

Indeed a generic form for an r^{th}-order difference equation is

$$P(n, f(n), \Delta f(n), \Delta^2 f(n), \ldots, \Delta^r f(n)) \equiv 0 \quad ,$$

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while a generic form for an r^{th}-order recurrence equation is

$$Q(n, f(n), Ef(n), E^2 f(n), E^r f(n)) \equiv 0 \quad ,$$

where, as usual, $\Delta f(n) := f(n+1) - f(n)$ and $Ef(n) := f(n+1)$. Since $\Delta = E - 1$ and $E = \Delta + 1$, we can write, using the binomial theorem , $\Delta^k f(n) = (E-1)^k f(n)$ as a linear combination of $E^i f(n)$'s and $E^k f(n) = (\Delta + 1)^k f(n)$ as a linear combination of $\Delta^i f(n)$'s, thereby going from one form to the other.

This equivalence is still true if the *order* of the difference (alias recurrence) equation is infinite (i.e. to compute $f(n)$ we need *all* the previous values: $f(n-1), f(n-2), \ldots, f(1), f(0)$, but only if the function P is "algebraic" or "analytic" in some sense. In this article we will encounter *weird recurrences* that in addition to using *all* the previous values, also feature *mex* that inputs sets of integers and outputs non-negative integers.

Definition of mex: Given a set of non-negative integers S, $mex(S)$ is the *smallest* non-negative integer that does not belong to S.

For example, $mex(\{0, 1, 2, 5, 7\}) = 3$, and $mex(\{2, 3, 5, 7\}) = 0$.

Beware of Sequences Defined by mex

Exercise: Consider the sequence defined by the recurrence

$$a_i := mex(\{0, 1\} \cup \{ j a_r \; ; \quad j \geq 1 \, , \; 1 \leq r < i\}) \quad , \quad i \geq 1 \quad .$$

Prove some obvious properties of this sequence, for example that there are infinitely many i with $a_{i+1} - a_i = 2$.

How To Play CHOMP

David Gale's famous game of Chomp ([G]) starts out with an M by N chocolate bar, in which the leftmost-topmost square is poisonous. Players take turns picking squares. In his or her (or its) turn, a player must pick one of the remaining squares, and eat it along with all the squares that are "to its right and below it". Using matrix-notation with the poisonous square being entry $(1, 1)$, and the initial position consisting of the whole bar $\{(i, j) \; ; \; 1 \leq i \leq M, 1 \leq j \leq N\}$, then picking the square (i_0, j_0) means that one has to eat all the remaining squares (i, j) for which *both* $i \geq i_0$ and $j \geq j_0$ hold. The player that eats the poisonous (leftmost-topmost) square loses. Of course picking $(1, 1)$ kills you, so a non-suicidal player will not play that move unless it is forced to.

For example, if $M = 4$ and $N = 3$, then the initial game-position is

$$\begin{matrix} X & X & X \\ X & X & X \\ X & X & X \\ X & X & X \end{matrix} \quad .$$

The first player may choose to play $(4, 3)$, in which case the game-position becomes

$$\begin{matrix} X & X & X \\ X & X & X \\ X & X & X \\ X & X & \end{matrix} ,$$

or he may choose to play $(2, 2)$, which shrinks the chocolate bar to

$$\begin{matrix} X & X & X \\ X & & \\ X & & \\ X & & \end{matrix} ,$$

and so on.

What is Chomp Good For?

Chomp is a *typical* example of a *combinatorial game*. The mathematical theory of combinatorial games is one of the most *gorgeous* subjects in the whole of mathematics, see [BCG] and [F] for nice expositions. Why study Combinatorial Games? If beauty is not a good enough reason for you, then there are numerous other reasons, since according to Steven Byrnes [By], "Combinatorial game theory has applications in fields such as complexity theory, artificial intelligence, error-correcting codes, algorithms, and surreal analysis, and of course, human recreation (chess, for example)." All true, of course, but Byrnes forgot to mention an even more important application, at least for the Byrnes family. It can pay your Harvard tuition! ([By] won *first prize* at the 2002 Siemens Westinghouse Competition, worth a $\$10^5$ scholarship).

Chomp Positions

Any Chomp position in a game that starts out with an $M \times N$ chocolate bar, can be described by an array of integers $(a_1, a_2, \ldots, a_M)$, with $N \geq a_1 \geq a_2 \geq \ldots \geq a_M \geq 0$. We will find it more convenient to set $b_i = a_i - a_{i+1}$ for $1 \leq i < M$ and $b_M = a_M$, and denote the same position with square brackets, and read it from bottom to top $[b_M, b_{M-1}, \ldots, b_1]$. Note that this means that there are b_i columns with exactly i squares. The advantage of this notation is that the b_i are independent non-negative integers.

3-Rowed Chomp

This was studied in my previous article [Z], whose "meta-motivation" was to give an example of "computer-generated research", where the computer first conjectures a pattern, and then proves it rigorously, *all by itself*. This work was extended, using both human and computer ingenuity by my student Xinyu Sun[S], first for computing the so-called *Sprague-Grundy function* (a.k.a. *nim-values*), and then for general Chomp, in which all but the two top-rows are fixed. Xinyu Sun[S] conjectured that the set of losing positions (and more generally the set of positions with any given nim-value) is either a finite set, or displays "ultimately-periodic" behavior, in a sense described below. This was further confirmed empirically, for the case of 3-rowed Chomp, by Andries

Brouwer[Br] for the bottom row as large as 90000. Brouwer also refuted the implicit conjecture in [Z] that for 3-rowed Chomp the period is always one (i.e. that the sequence is ultimately-constant). Brouwer found that when the bottom row has exactly 120 squares, then the period is 2, and found many other examples of larger periods later on.

Enter Steven Byrnes

But neither I, nor Xinyu Sun, nor Andries Brouwer, were able to *prove* Sun's ultimate-periodicity conjecture. Steve Byrnes's prize-winning article [By] is truly impressive!

In spite of Byrnes's breakthrough, the following is still open.

BIG PROBLEM: Give A **fast** (i.e. poly-log in $c + a + b$), characterization of the set of **losing** Chomp positions $[c, a, b]$.

Here I only consider the following more modest problem, already studied in [Z].

LITTLE PROBLEM: For *any* inputted integer C give a **fast** (poly-log in $a+b$) characterization of the set of *losing* Chomp positions $[c, a, b]$, with $c \leq C$.

In [Z], the *little problem* was solved for $C = 115$, and the algorithm could be extended, in principle, for any C, as long as the ultimate-periodicity phenomenon holds. Before Byrnes, we did not know for sure that it would always work, but now we do!

So why bother with yet another paper and yet another Maple package? First, our new Maple package `BYRNES` accompanying this paper, is much more efficient than the Maple package `Chomp3Rows` that accompanied [Z]. Second, `BYRNES` also computes Grundy function tables automatically. Third, this paper gives an easy *algorithmic* proof of Byrnes's theorem, much easier to follow than Byrnes's original proof. Finally, and most importantly, it is yet another example of 'taming the infinite' and using computers to discover and *prove* general theorems (i.e. infinitely many facts) by finitary means.

Set of Positions and Legal Moves

A combinatorial game is completely determined by its *set of legal positions*, S, and by a set-valued function $f : S \to 2^S$, assigning to any member of S a (possibly empty) subset of S where $f(s)$ is the set of positions reachable from s in *one* move.

For 3-rowed Chomp we have

$$S := \{[c, a, b] \ ; \ a, b, c \geq 0\} \backslash \{[0, 0, 0]\} \quad ,$$

and

$$f([c, a, b]) := \{[c, a, b - x] \ ; \ 0 < x \leq b\}$$
$$\cup \ \{[c, a - x, 0] \ ; \ 0 < x \leq a\}$$

$$\cup \{[c-x,0,0] \ ; \ 0 < x < c\}$$

$$\cup \{[c,a-x,b+x] \ ; \ 0 < x \leq a\}$$

$$\cup \{[c-x,0,a+b+x] \ ; \ 0 < x \leq c\}$$

$$\cup \{[c-x,a+x,b] \ ; \ 0 < x \leq c\} \quad . \qquad (LegalMoves)$$

The six sets on the right hand side of $(LegalMoves)$ correspond to chomping, respectively, at the
top row, "columns of 1's section",
top row, "columns of 2's section",
top row, "columns of 3's section".
middle row, "columns of 2's section",
middle row, "columns of 3's section", and
bottom row, "columns of 3's section" .

This description is equivalent to the usual description in terms of a *directed graph*. The set of vertices is S and $f(s) = \{t|s \to t\}$.

Yet another description of a game, more useful for our purposes, is to state the sets $g(t) := \{s|s \to t\}$, in other words for each position t, indicate those positions from which t can be reached in one (legal) move.

In the case of 3-rowed Chomp we have

if $a, b > 0$ then

$$g([c,a,b]) = \{[c,a,b{+}x] \ ; \ x > 0\} \bigcup \{[c,a{+}x,b{-}x] \ ; \ 0 < x \leq b\} \bigcup \{[c{+}x,a{-}x,b] \ ; \ 0 < x \leq a\} \quad , \qquad (Rule1)$$

while if $a > 0$ and $b = 0$ then

$$g([c,a,0]) = \{[c,a+x,y] \ ; \ x \geq 0, y \geq 0, x+y > 0\} \bigcup \{[c+x,a-x,0] \ ; \ 0 < x \leq a\} \quad , \qquad (Rule2)$$

while if $b > 0$ and $a = 0$ then

$$g([c,0,b]) = \{[c+x,y,b-x-y] \ ; \ x \geq 0, y \geq 0, 0 < x+y \leq b\} \quad , \qquad (Rule3)$$

and, finally when $a = 0$ and $b = 0$ then

$$g([c,0,0]) = \{[c+x,y,z] \ ; \ x \geq 0, y \geq 0, z \geq 0, 0 < x+y+z\} \quad . \qquad (Rule4)$$

We have the following consequences.

Lemma 1:

if $a, b > 0$ and $[c, a, b]$ is a loser then the following are *guaranteed* winners

$$\{[c, a, b + x] \ ; \ x > 0\} \tag{1.1}$$

$$\{[c, a + x, b - x] \ ; \ 0 < x \le b\} \tag{1.2}$$

$$\{[c + x, a - x, b] \ ; \ 0 < x \le a\} \quad . \tag{1.3}$$

Lemma 2: If $a > 0$ and $[c, a, 0]$ is a loser then the following are *guaranteed* winners

$$\{[c, a + x, y] \ ; \ x \ge 0 \ , \ y \ge 0 \ , \ x + y > 0\} \quad . \tag{2.1}$$

$$\{[c + x, a - x, 0] \ ; \ 0 < x \le a\} \quad . \tag{2.2}$$

Lemma 3: If $b > 0$ and $[c, 0, b]$ is a loser then the following are *guaranteed* winners

$$\{[c + x, y, b - x - y] \ ; \ x > 0 \ , \ y > 0 \ , \ 0 < x + y \le b\} \quad .$$

The 'Trivial' Case of 2-Rowed Chomp

2-rowed Chomp is really a piece of cake. Instead of using a real chocolate bar, we can play this game on the lattice N^2 with $[0, 0]$ removed. The Chomp position $[a, b]$ can be thought of the location of a counter. There are three kinds of legal moves:

$$[a, b] \to [a, b - x] \quad , \quad 1 \le x \le b \tag{$LegalMove1$}$$

$$[a, b] \to [a - x, b + x] \quad , \quad 1 \le x \le a \tag{$LegalMove2$}$$

$$[a, b] \to [a - x, 0] \quad , \quad 1 \le x < a \quad . \tag{$LegalMove3$}$$

If a player can't move (i.e. it is at position $[0, 1]$), then it lost. Henceforth we will call positions "points", and refer to "losing positions" and "winning positions" as *losers* and *winners* respectively.

Once we know that $[a, b]$ is a loser, we know right away many winners, namely all those points from which $[a, b]$ may be reached in one move, i.e. using any of $(LegalMove1 - 3)$. It follows that we have the

Implied Winners Lemma: If $[a, b]$ is a loser, and $b > 0$, then

$$[a, b + x] \quad , \quad 1 \le x < \infty \quad , \tag{$ImW1$}$$

$$[a + x, b - x] \quad , \quad 1 \le x \le b \quad , \tag{$ImW2$}$$

are all winners. If $[a, 0]$ is a loser then

$$[a + x, y] \quad , \quad 0 \le x, y < \infty \quad , \quad x + y > 0 \quad , \tag{$ImW3$}$$

are all winners.

Let's try to find the set of losers in 2-rowed Chomp. We already know that $[0,1]$ is a loser, hence by $(ImW1)$, $[0,x]$, $x>1$ are winners, and so is $[1,0]$, by $(ImW2)$. Crossing out $[0,1]$ and all its implied winners, the minimal point is $[1,1]$ that must be a loser since it only leads to previously established winners. Now $[1,1]$'s implied winners are $[1,x]$, $x>1$, and $[2,0]$, and hence $[2,1]$ is a loser. By induction, if $[a,1]$ is a loser, then $[a,x]$, $x>1$ and $[a+1,0]$ are winners, and hence the minimal uncovered point, $[a+1,1]$, is a loser, since it only leads to $[a+1,0]$ $[a+1-x,x+1]$, $(x \geq 1)$, and $[x,0]$ $(x \leq a)$, which are all previously established winners. It follows that we have the "theorem" that the set of losing positions in 2-rowed Chomp is $\{[a,1]\ ;\ a \geq 0\}$.

2-Rowed Chomp with Instant Winners

Now consider a slight generalization. The positions and legal moves are the same, **but**, *by fiat*, the members of a certain (finite or infinite) set of points are designated *instant winners*, and if it is your turn to move, and the counter is on that point, you are declared the winner.

Let's describe this set (of points $[a,b]$) as a sequence of sets (of integers), I_a $(0 \leq a < \infty)$, where I_a is the set of b such that $[a,b]$ is an instant winner. If the set of instant winners is finite then there would be an a_0 such that $I_a = \emptyset$, for $a \geq a_0$, in other words I_a would be eventually the empty set.

Let's try to figure out how to determine the set of losers, in this, more general, game. Because of $(ImW1)$ it follows that for any given a there is at most one b such that $[a,b]$ is a loser. Let's denote it by L_a, if it exists. Because of $(ImW3)$, it follows that if $[a_0,0]$ is a loser then there are no losers with $a > a_0$. In this case there are only finitely many losers.

The problem of finding the set of losers is equivalent to determining the sequence of integers $\{L_a\}_{a=0}^{\infty}$. Fix an integer a, and suppose that we already know L_i for $i<a$. This means that $[i,L_i]$ are losers for $0 \leq i < a$. Because of $(ImW2)$,

$$[a, L_i-(a-i)] \quad , \quad 0 \leq i < a$$

are implied winners. Combining these winners-by-merit with the lazy *instant winners*, it follows that L_a is the *smallest non-negative* integer that is *not* in the set

$$I_a \cup \{L_{a-1}-1, L_{a-2}-2, \ldots, L_0-a\} \quad .$$

In particular if L_i exists for $i<a$ and is never 0, then L_i exists. We have just established the

Fundamental Recurrence for Two-Rowed Chomp with Instant Winners

Let $\{I_a\}_{a=0}^{\infty}$ be a sequence of sets of non-negative integers. Consider 2-rowed Chomp with the set of Instant Winners

$$\bigcup_{a=0}^{\infty} \{[a,b]\ ;\ b \in I_a\} \quad .$$

Then the set of *losers* is the set $\{[a, L_a] \; ; \; a \geq 0\}$, where L_a is given by the Recurrence

$$L_a = mex(I_a \cup \{L_{a-1} - 1, L_{a-2} - 2, \ldots, L_0 - a\}) \quad , \qquad (FundamentalRecurrence)$$

valid as long as $L_a > 0$. If and when $L_{a_0} = 0$, then the sequence $\{L_a\}_{a=0}^{\infty}$ terminates at $a = a_0$.

Example: Suppose that the set of instant winners consists of $[0, 0], [0, 1], [0, 2], [0, 3], [1, 0], [1, 1], [1, 5]$, and $[a, 0], [a, 1]$ $(a > 1)$. In other words $I_0 = \{0, 1, 2, 3\}, I_1 = \{0, 1, 5\}$, $I_a = \{0, 1\}(a \geq 2)$.

We have:

$$L_0 = mex(I_0) = mex(\{0, 1, 2, 3\}) = 4 \quad ,$$

$$L_1 = mex(I_1 \cup \{L_0 - 1\}) = mex(\{0, 1, 3, 5\}) = 2 \quad ,$$

$$L_2 = mex(I_2 \cup \{L_1 - 1, L_0 - 2\}) = mex(\{0, 1, 2\}) = 3 \quad ,$$

$$L_3 = mex(I_3 \cup \{L_2 - 1, L_1 - 2, L_0 - 3\}) = mex(\{0, 1, 2\}) = 3 \quad ,$$

$$L_4 = mex(I_4 \cup \{L_3 - 1, L_2 - 2, L_1 - 3, L_0 - 4\}) = mex(\{0, 1, 2\}) = 3 \quad .$$

Now we can already *guess* a pattern, $L_a = 3$ for $a \geq 2$. Let's try and prove it by induction using $(FundamentalRecurrence)$. We have just established it for $a = 2, 3, 4$. For $a \geq 5$ we have, by the inductive hypothesis, that the set of non-negative integers in

$$I_a \cup \{L_{a-1} - 1, L_{a-2} - 2, \ldots, L_0 - a\}$$

is really finite, namely

$$I_a \cup \{3 - 1, 3 - 2, 3 - 3\} = \{0, 1, 2\}$$

and its mex equals 3. □

The Ultimate-Periodicity Phenomenon

Now assume that the sequence of sets I_a describing the instant winners is not arbitrary, but is *ultimately-periodic*. In other words, starting at a certain place a_0, there is a (minimal) period p such that $I_a = I_{a+p}$ for $a > a_0$.

Now we are on more secure grounds. Ultimately-Periodic Sequences are *finite* objects, and hence meaningful. To describe such a sequence all we have to do is specify the non-periodic head $I_0, \ldots, I_{a_0}$ followed by the period $I_{a_0+1}, I_{a_0+2}, \ldots, I_{a_0+p}$ that keeps repeating.

Two Immediate Consequences of $(FundamentalRecurrence)$

Lemma Bounded: If the sets I_a are (uniformly) bounded, and $M - 1$ is an upper bound, (i.e. $max(I_a) \leq M - 1$ for all $a > 0$) then $L_a \leq M$, for all $a > 0$.

Proof: This follows immediately from the fact that $mex(S) \leq max(S) + 1$, and induction on a.

Ultimate-Periodicity Theorem: If I_a is ultimately-periodic then the sequence L_a either terminates (with the last value being 0), or else is ultimately-periodic.

Proof: Since I_a is ultimately-periodic the set of finite sets $\{I_a \ ; \ a > 0\}$ is finite, and hence bounded. Let $M-1$ be the (least) upper bound. By Lemma **Bounded**, $L_a \leq M$. Since negative integers in a set do not affect its *mex* (recall that $mex(S)$ is the smallest *non-negative* integer in the complement of S), the hitherto "infinite memory" recurrence ($FundamentalRecurrence$), where to know the value of L_a you have to remember *all* your past, now becomes a "finite memory" recurrence, where you only have to remember what happened in the last M days.

$$L_a = mex(I_a \cup \{L_{a-1} - 1, L_{a-2} - 2, \ldots, L_{a-M} - M\}) \quad . \quad (BoundedFundamentalRecurrence)$$

Introducing the 'states'

$$S_a := (I_a \ ; \ L_{a-1}, L_{a-2}, \ldots, L_{a-M})$$

($BoundedFundamentalRecurrence$) induces a well-defined function $F := S_a \to S_{a+1}$. Since L_a is bounded, and I_a is ultimately-periodic, it follows that there are only finitely many states. By the venerable **Pigeon-Hole** Principle, sooner or later we must visit a previously-visited 'state', i.e. there exists an a_0 and a q such that $S_{a_0} = S_{a_0+q}$. But once that happens, everything repeats itself with period q, and $S_{a_0} = S_{a_0+iq}$ for all $i > 0$. In particular, $L_{a+q} = L_a$ for all $a \geq a_0 + 1$. $\square$

A Posteriori Justification

How to turn this into an algorithm? The 'theoretical' upper bound for the period is enormous, but is hardly (and perhaps never) achieved. Once an ultimately-periodic sequence of sets I_a is given in the form

$$[I_0, I_1, \ldots, I_{b_0}][I_{b_0+1}, I_{b_0+2}, \ldots, I_{b_0+p}]^\infty \quad ,$$

just keep computing L_a using ($BoundedFundamentalRecurrence$). Suppose that your computer detects that, after a certain place a_0, the same segment $L_{a_0+1}, L_{a_0+2}, \ldots, L_{a_0+q}$ keeps repeating (say 10 times). Then the computer is justified in *guessing* that the $\{L_a\}_{a=0}^\infty$ equals.

$$[L_0, L_1, \ldots, L_{a_0}][L_{a_0+1}, L_{a_0+2}, \ldots, L_{a_0+q}]^\infty \quad .$$

In order to *prove* this conjecture you only have to check it for the **finite** number of cases $0 \leq a \leq max(a_0, b_0) + M + lcm(p, q)$, since later on things start to 'repeat themselves'.

Analogy Example 1: Guess the decimal representation of 1/3, and then prove it rigorously.

Solution: Compute 1/3 to ten-decimal-digits accuracy, guess that $1/3 = 0.3333333333\ldots$, and then prove it rigorously by summing an infinite geometric series:

$$3\sum_{i=1}^{\infty}(1/10)^i = 3\frac{1}{10-1} = \frac{1}{3} \quad .\square$$

Analogy Example 2: Guess the continued-fraction representation of $\sqrt{7}$, and then prove it rigorously.

Solution: Using Maple, or, if you wish, pencil-and-paper, compute `convert(evalf(sqrt(7)),confrac);`, getting `[2,1,1,1,4,1,1,1,4,1,1,1,4,1,1,1,4]`. Now *conjecture* that $\sqrt{7} = [2,(1,1,1,4)^\infty]$. Let's call the right side x. Then $x = 2 + 1/y$, where $y = [(1,1,1,4)^\infty]$. This means $y = 1 + 1/(1 + 1/(1 + 1/(4 + 1/y)))$, from which you can get a quadratic equation satisfied by y, that implies the quadratic equation satisfied by x, that turns out to be $x^2 - 7 = 0$. $\square$

Note that the current algorithm for proving the conjectured ultimately-periodic sequence L_a from the input ultimately-periodic sequence of Instant Winners I_a is perfectly valid even if we did not have the *a priori* assurance that it is *always* guaranteed to work. The Ultimate-Periodicity Theorem *guarantees* that we are bound to succeed at the end, even though some of our initial guesses may prove to be wrong.

This is also true in the two elementary 'analogy examples' above. A well-known, elementary, and very easy theorem (that uses the pigeon-hole principle!) asserts that any rational number has either a terminating or an ultimately-periodic decimal expansion. This guarantees that the 'empirical algorithm' is going to work for any *rational number*.

Analogously for Example 2. A theorem of Lagrange states that any quadratic irrationality has an ultimately-periodic continued fraction, and the proof also uses recurrences and the pigeon-hole principle. Hence we have an *a priori* guarantee that this will work for the square-root of *any* (non-perfect-square) integer.

Back to Three-Rowed Chomp

Recall our **Little Problem** of determining **fast**, all the losers $[c,a,b]$ for $c \le C$, where C is a fixed, given integer.

Because of the first part of ($Rule1$), it follows that for any given c and a, there is *at most* one b such that $[c,a,b]$ is a losing position. Let's define $B_c(a)$ to be that b (if it exists, otherwise it is undefined).

It follows that knowledge of the set of losers in 3-rowed Chomp is the same as *knowing* the sequence of sequences B_c, and knowing it for $c \le C$ is the same as knowing its first $C+1$ terms.

We already know B_0! This is just the case of 2-rowed Chomp, and we found that $B_0 = 1^\infty$. Suppose that we already know B_c for $c < C$ and are interested in B_C. Now observe that B_C is just the loser-sequence for a 2-rowed Chomp (since the c is fixed at C) with lots of Instant Winners! These Instant Winners are those positions implied by losers from $c < C$ via (1.3), (2.2) and Lemma **3**.

So let's define the sequence of those Instant Winners $\mathcal{W}_C(a)$ as the set of b such that $[C,a,b]$ is a winner implied by some L_c with $c < C$. $\mathcal{W}_C(a)$ can be naturally partitioned as

$$\mathcal{W}_C(a) = \bigcup_{c=0}^{C-1} \mathcal{V}_{C,c}(a) \quad ,$$

where $\mathcal{V}_{C,c}(a)$ is the set of b such that $[C, a, b]$ is reachable from *some* loser $[c, a', b']$. $\mathcal{V}_{C,c}(a)$, in turn, can be written conveniently as

$$\mathcal{V}_{C,c}(a) \;=\; \mathcal{V}'_{C,c}(a) \cup \mathcal{V}''_{C,c}(a),$$

where $\mathcal{V}'_{C,c}(a)$ is the contribution from Lemma 3, and $\mathcal{V}''_{C,c}(a)$ is the contribution from (1.3) and (2.2).

From Lemma 3, (1.3), and (2.2), we have,

Crucial Facts

for $a \geq 0$:

$$\mathcal{V}'_{C,c}(a) = \{B_c(0) - (C-c) - a\} \cap \mathbf{Z}_{\geq 0} \quad ,$$

$$\mathcal{V}''_{C,c}(a) = \{B_c(a + C - c)\} \quad .$$

Note that each of these sets is either empty or a singleton.

We are now ready to give a new proof of

Byrnes's Theorem: For every C, B_C is ultimately-periodic.

Proof: By the Ultimate-Periodicity Theorem it suffices to show that for each non-negative integer C, the sequence of sets $\{\mathcal{W}_C(a)\}_{a=0}^{\infty}$ is ultimately-periodic.

Note that if $\{S(a)\}_{a=0}^{\infty}$ and $\{T(a)\}_{a=0}^{\infty}$ are ultimately-periodic sequences of sets so is

$$\{(S \cup T)(a)\}_{a=0}^{\infty} := \{S(a) \cup T(a)\}_{a=0}^{\infty} \quad .$$

We proceed by induction on C. We already know that it is true for $C = 0$. By induction, the sequence of sets $\{\mathcal{V}''_{C,c}(a)\}_{a=0}^{\infty}$ is ultimately-periodic, and $\{\mathcal{V}'_{C,c}(a)\}_{a=0}^{\infty}$ is even better, it is ultimately empty. Hence their union, $\{\mathcal{V}_{C,c}(a)\}_{a=0}^{\infty}$ is ultimately-periodic for all $c < C$, and hence so is their point-wise *union* over $c < C$, $\{\mathcal{W}_C(a)\}_{a=0}^{\infty}$ $\square$

Algorithm

Now that we know that B_c is ultimately-periodic, or finite, and hence is uniquely determined by finite data (the non-periodic beginning, and the periodic part), we can treat it as a *single* concrete *computational* object rather then as an *infinite sequence*.

It turns out that $\mathcal{W}_c$ is not that different than $\mathcal{W}_{c-1}$, hence it would be desirable to find a *recurrence* scheme that takes advantage of previously computed values.

Let's write

$$\mathcal{W}'_C(a) := \bigcup_{c=0}^{C-1} \mathcal{V}'_{C,c}(a) \quad ,$$

$$\mathcal{W}''_C(a) := \bigcup_{c=0}^{C-1} \mathcal{V}''_{C,c}(a) \quad .$$

Of course, $\mathcal{W}_C(a) = \mathcal{W}'_C(a) \cup \mathcal{W}''_C(a)$.

Now

$$\mathcal{W}''_C(a) := \bigcup_{c=0}^{C-1} \mathcal{V}''_{C,c}(a) = \left(\bigcup_{c=0}^{C-2} \mathcal{V}''_{C,c}(a) \right) \cup \ \mathcal{V}''_{C,C-1}(a)$$

$$= \left(\bigcup_{c=0}^{C-2} \{B_c(a+1+(C-1)-c)\} \right) \ \cup \ \{B_{C-1}(a+1)\}$$

$$= \mathcal{W}''_{C-1}(a+1) \cup \{B_{C-1}(a+1)\} \quad .$$

We also have

$$\mathcal{W}'_C(a) := \bigcup_{c=0}^{C-1} \mathcal{V}'_{C,c}(a) = \left(\bigcup_{c=0}^{C-2} \mathcal{V}'_{C,c}(a) \right) \cup \mathcal{V}'_{C,C-1}(a) =$$

$$\bigcup_{c=0}^{C-2} \{B_c(0) - ((C-1)-c) - a - 1\} \cup \{B_{C-1}(0) - 1 - a\} =$$

$$OneLess(\mathcal{W}'_{C-1}(a)) \cup \{B_{C-1}(0) - 1 - a\} \quad ,$$

where $OneLess(S) := \{s-1 \ ; \ s \in S\}$.

In the above we make the convention that negative numbers are discarded.

The Recurrence Scheme

Now let's define three **macros** on ultimately-periodic sequences of sets.

The first one is **Losers** that inputs an ultimately-periodic sequence of sets and outputs an ultimately-periodic, or finite, sequence of integers, by employing ($FundamentalRecurrence$).

The second one is **OneLess** defined by $OneLess(A)_i := OneLess(A_i)$. Of course it preserves the ultimate-periodicity property.

The third macro is **Chop** that is just a left-shift, discarding the first member. $Chop(A)_i = A_{i+1}$.

Using these three macros, we can rephrase the above scheme without any mention of the argument a, and just treat B_c, $\mathcal{W}'_c$, and $\mathcal{W}''_c$ as *objects* on their own right. The union of two sequences of sets is defined as the sequence of unions: $(S \cup T)_i := S_i \cup T_i$.

Initial Conditions: $W'_0 = \{0\}\emptyset^\infty$, $W''_0 = \emptyset^\infty$.

Recurrence: For $c \geq 0$

$$B_c = Losers(\mathcal{W}'_c \cup \mathcal{W}''_c) \quad ,$$

$$W'_{c+1} = OneLess(W'_c) \cup [\{B_c(0)-1\}, \{B_c(0)-2\}, \ldots, \{0\}][\emptyset]^\infty \quad ,$$

$$W''_{c+1} = Chop(W''_c) \cup Chop(B_c) \quad .$$

The Maple package BYRNES

Everything here is implemented in the Maple package `BYRNES` available from the author's website. It may be downloaded from

`http://www.math.rutgers.edu/~zeilberg/mamarim/ mamarimhtml/byrnes.html` ,

where you can also find a data file called `ChompData`, and detailed instructions.

The main procedures are `ChompLosers` and `ChompGrudny` .

The Sprague-Grundy Values

Losing positions are those for which the Sprague-Grundy function (nim-value) equals 0. The above can be repeated, with little modification, to compute the set of positions that have any given nim-value. The only difference is that for any given g, one has to add to the set of implied "Instant Winners", the positions that have been already shown to have smaller nim-values. It follows by induction on g that these too yield ultimately-periodic sequences. This is implemented in procedure `ChompGrundy` in `BYRNES`. Read the on-line help there.

Chaotic Behavior

As first noted by Brouwer[Br] for the losing positions, and Sun[S] for the values of the nim-values, larger periods then 1 do occur, and while at the beginning they are rather rare, eventually they take over, and the periods get rather large. Also the set of c for which B_c is a finite sequence (i.e. it ends with a 0) is rather mysterious. It seems that we have 'chaotic' behavior, but in a vague, yet-to-be-made-precise, sense.

Conclusion: Chomp is Locally Trivial But NOT Globally Trivial

What Xinyu Sun conjectured, Byrnes proved, and I reproved is that 3-Rowed Chomp is *locally* trivial, since for every fixed number of squares at the bottom row, c, the sequence coding the set of losing positions, B_c, is *ultimately-periodic* or finite.

But it seems that as a function of c, B_c is far from trivial. But then again, maybe it conforms to yet-another-ansatz, perhaps reminiscent of Wythoff's game (see [F]), but probably much more complicated.

REFERENCES

[BCG] E. R. Berlekamp, J. H. Conway, and R. K. Guy, *"Winning Ways for Your Mathematical Plays"*, Academic Press, London, 1982.

[Br] A. Brouwer, *Chomp*, on-line document, `http://www.win.tue.nl/~aeb/games/chomp.html` .

[By] S. Byrnes, *Poset Games Periodicity*, preprint.
Available from: `http://www.geocities.com/sbyrnes321/pgp.html` .

[E1] S. Elaydi, *"An Introduction to Difference Equations"*, 2nd edition, Springer Verlag, 1999.

[E2] S. Elaydi, *"Discrete Chaos"*, CRC Press, 1999.

[F] A. S. Fraenkel, *Scenic Trails Ascending from Sea-Level Nim to Alpine Chess*, in: "*Games of No Chance*", R. J. Nowakowski, ed., Cambridge University Press, 1996.

[G] D. Gale, *A Curious Nim-type game*, Amer. Math. Monthly **81** (1974), 876-879.

[S] X.Sun, *Improvements on Chomp*, Integers **2** (2002), G1, 8 pp. (electronic).

[Z] D. Zeilberger, *Three-rowed Chomp*, Adv. in Appl. Math.**26** (2001), 168–179. Available on-line from: `http://www.math.rutgers.edu/~zeilberg/mamarim/ mamarimhtml/chomp.html` .