# Commutation relations and Vandermonde determinants

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Abstract: We consider a certain decomposition of the matrix algebra $M_n(F)$, where $F$ is a field. The commutation relations of that decomposition yield an $n^2 \times n^2$ matrix $M^{M_n(F)}$, which determines the multilinear polynomial identities of $M_n(F)$. Thus if $char(F) = 0$, the matrix $M^{M_n(F)}$ determines the polynomial identities of $M_n(F)$. We show that $M^{M_n(F)}$ is very close to the tensor product of two $n \times n$ Vandermonde matrices. In particular this allows us to evaluate the determinant of $M^{M_n(F)}$.

# 1 Regular algebras

Let $A$ be an algebra over the field $F$.

**Definition 1.1.** [1, Definition 2.3] The decompositions

$$A = A_1 \oplus \cdots \oplus A_r \tag{1}$$

of the algebra $A$ as a direct sum of subspaces is called *regular* if it satisfies the following two conditions.

1) Given $n$ indices $1 \le i_1, \ldots, i_n \le r$, there exist elements $x_{i_j} \in A_{i_j}$, $1 \le j \le n$ such that $x_{i_1} \cdots x_{i_n} \ne 0$.

2) Given $1 \le i, j \le r$, there exist $0 \ne \theta_{i,j}^{(A)} \in F \setminus \{0\}$ such that for any $x_i \in A_i$ and $y_j \in A_j$

$$x_i y_j = \theta_{i,j}^{(A)} y_j x_i. \tag{2}$$

Then

$$M^A := (\theta_{i,j}^{(A)})_{1 \le i,j \le r} \tag{3}$$

is the matrix of the commutation relations of the regular decomposition (1).

Recall that algebras satisfying polynomial identities are called P.I. algebras. Two P.I. algebras are P.I. equivalent if they satisfy the same polynomial identities. For example, it can be shown that in characteristic zero, the quaternions algebra and the algebra of $2 \times 2$ are P.I. equivalent. The importance of regular decompositions stems from the following theorem.

**Theorem 1.2.** *[1, Theorem 3.1]*
*Let $A, B$ be algebras with regular decompositions $A = \oplus_{i=1}^r A_i$ and $B = \oplus_{i=1}^r B_i$ and with corresponding $r \times r$ matrices $M^A$ and $M^B$ of the commutation relations. Assume $M^B = P^{-1} M^A P$ where $P$ is an $r \times r$ permutation matrix. Then $A$ and $B$ satisfy the same multilinear identities. Hence $A$ and $B$ are P.I. equivalent when $char(F) = 0$.*

The main object studied in this paper is given in the following example.

**Example 1.3.** Let $A = M_n(F)$, the $n \times n$ matrices over $F$. Note first that with respect to the decomposition $M_n(F) = \oplus_{i,j=1}^n span_F\{e_{i,j}\}$, $M_n(F)$ is *not* regular. Here $e_{i,j}$ are the matrix units. However, $M_n(F)$ *is* regular – with respect to the following decomposition. Let $\xi$ be a primitive $n$-th root of 1 and assume that $\xi \in F$. Let $X_a = diag(\xi^{n-1}, \xi^{n-2}, \ldots, \xi, 1)$ and $X_b = e_{n,1} + \sum_{i=1}^{n-1} e_{i,i+1}$, then we obtain the following decomposition:

$$M_n(F) = \bigoplus_{i,j=0}^{n-1} A_{i,j} \quad \text{where} \quad A_{i,j} = span_F\{X_a^i X_b^j\}. \tag{4}$$

Indeed, since $1, \xi, \xi^2, \ldots, \xi^{n-1}$ are distinct, by a standard Vandermonde argument, the matrices $\{X_a^i X_b^j \mid 0 \leq i \leq n-1\}$ are linearly independent for any fixed $j$. It follows that $\{X_a^i X_b^j \mid 0 \leq i, j \leq n-1\}$ are linearly independent, which implies the decomposition (4). We verify that $M_n(F)$ is regular with respect to that decomposition: The relation $X_a X_b = \xi X_b X_a$ implies that

$$(X_a^i X_b^j)(X_a^k X_b^\ell) = \xi^{jk} X_a^{i+k} X_b^{j+\ell}, \tag{5}$$

which then implies that

$$(X_a^i X_b^j)(X_a^k X_b^\ell) = \xi^{jk-i\ell}(X_a^k X_b^\ell)(X_a^i X_b^j). \tag{6}$$

Since any product of the matrices $X_a^i$ and $X_b^j$ is non zero, it follows that $M_n(F)$ is regular – with respect to the decomposition (4). The commutation relations given by (6) yield the $n^2 \times n^2$ matrix

$$M^{M_n(F)} = (\xi^{jk-i\ell})_{0\leq i,j,k,\ell\leq n-1}. \tag{7}$$

This matrix has row indices $(i, j)$ and column indices $(k, \ell)$, with $0 \leq i, j, k, \ell \leq n-1$. Replacing $\xi$ by the variable $x$ we obtain the $n^2 \times n^2$ matrix

$$B(x) = (x^{jk-i\ell})_{0\leq i,j,k,\ell\leq n-1}. \tag{8}$$

## 2 Tensor factorization of $M^{M_n(F)}$

We show that $B(x)$ of (8) is very close to the tensor product of two generic $n\times n$ Vandermonde matrices $V(x)$ and $V(y)$, and is obtained from such a tensor product by a natural row permutation and by a natural substitution. Since $M^{M_n(F)} = B(\xi)$, it follows that $M^{M_n(F)}$ admits a similar tensor decomposition. In fact, by (9) below, $B(x) = D(x, x^{-1})$, and $D(x, y)$ is obtained from the tensor product $V(x)\otimes V(y)$ by a natural row permutation. It follows that $M^{M_n(F)}$ is obtained by the same row permutation from $V(\xi) \otimes V(\xi^{-1})$. This is Theorem 2.2 below.

Consider the tensor product $A \otimes B$ of two $n \times n$ matrices $A$ and $B$. By definition $A \otimes B = (a_{j_1,j_2} \cdot b_{i_1,i_2})$. This is an $n^2 \times n^2$ matrix. Let its row indices be $(i_1, j_1)$ and column indices $(i_2, j_2)$, both ordered lexicographically, where $1 \leq, i_1, j_1, i_2, j_2 \leq n$. Then $A \otimes B$ is an array of $n \times n$ blocks, each block being an $n \times n$ matrix. Each such a block is determined by fixing some $i_1$ and $i_2$ and letting $1 \leq j_1, j_2 \leq n$, which is then the $(i_1, i_2)$ block $A \cdot b_{i_1,i_2}$ in $A \otimes B$. Thus, if $A = (a_{i_1,j_1})_{1\leq,i_1,j_1\leq n}$ and $B = (b_{i_2,j_2})_{1\leq i_2,j_2\leq n}$, then

$$A \otimes B = (a_{j_1,j_2} \cdot b_{i_1,i_2})_{1\leq,i_1,j_1,i_2,j_2\leq n}.$$

It is well known that when both $A$ and $B$ are $n \times n$ matrices,

$$\det(A \otimes B) = (\det(A) \cdot \det(B))^n$$

The key observation here is the following lemma.

**Lemma 2.1.** *Let* $V(x) = (x_i^{j-1})_{1\le i,j\le n}$ *denote a generic* $n\times n$ *Vandermonde matrix. Thus*

$$V(x)\otimes V(y) = (x_{j_1}^{j_2-1}\cdot y_{i_1}^{i_2-1})_{1\le i_1,j_1,i_2,j_2\le n}.$$

*Let*

$$D(x,y) = (x_{i_1}^{j_2-1}\cdot y_{j_1}^{i_2-1})_{1\le i_1,j_1,i_2,j_2\le n}. \tag{9}$$

*Then the matrix* $D(x,y)$ *is obtained from* $V(x)\otimes V(y)$ *by a row permutation. That row permutation is a product of* $\binom{n}{2}$ *disjoint transpositions (hence is an involution).*

*Proof.* Recall that $V(x)\otimes V(y)$ has row indices $(i_1,j_1)$. Apply on $V(x)\otimes V(y)$ the row permutation $\sigma\in S_{n^2}$ given by $(i_1,j_1)\longrightarrow(j_1,i_1)$. Thus $x_{j_1}^{j_2-1}\cdot y_{i_1}^{i_2-1}\longrightarrow x_{i_1}^{j_2-1}\cdot y_{j_1}^{i_2-1}$ by $\sigma$, so $V(x)\otimes V(y)$ is changed by $\sigma$ into $D(x,y)$. Since $\sigma$ has $n$ fixed points $(r,r)$, it is the product of $(n^2-n)/2$ disjoint transpositions. $\square$

The above implies

**Theorem 2.2.** *Let* $\sigma\in S_{n^2}$ *denote the permutation given by* $\sigma((i,j))=(j,i)$. *It induces the corresponding row permutation on* $n^2\times n^2$ *matrices, which we also denote by* $\sigma$. *Let* $B(x)$ *and* $D(x,y)$ *be given by* (8) *and* (9). *Then*

$$D(x,y)=\sigma(V(x)\otimes V(y)),$$

*hence*

$$B(x)=D(x,x^{-1})=\sigma(V(x)\otimes V(x^{-1})),$$

*and therefore*

$$B(\xi)=\sigma(V(\xi)\otimes V(\xi^{-1})).$$

## 2.1 The determinants

We prove here

**Proposition 2.3.** *The matrix* $B(\xi)=M^{M_n(F)}$ *of* (7) *has the following determinant:*

$$det(M^{M_n(F)}) = det((\xi^{jk-i\ell})_{0\le i,j,k,\ell\le n-1}) = \pm n^{(n^2)}.$$

The proof is given below. Note first that by Theorem 2.2 and Lemma 2.1

$$\det(D(x,y)) = (-1)^{\binom{n}{2}}\cdot\det(V(x)\otimes V(y)) = (-1)^{\binom{n}{2}}(\det(V(x))\cdot\det(V(y)))^n. \tag{10}$$

**Corollary 2.4.** *Since* $V(x),V(y)$ *are Vandermonde matrices, it follows from* (10) *that*

$$\det(D(x,y)) = (-1)^{\binom{n}{2}}\left(\prod_{1\le i<j\le n}(x_i-x_j)\cdot\prod_{1\le i<j\le n}(y_i-y_j)\right)^n. \tag{11}$$

**Theorem 2.5.** *The matrix $B(x)$ of* (8) *has the following determinant:*

$$\det(B(x)) = (-1)^{\binom{n}{2}(n+1)} \cdot \left( \prod_{0\le i<j\le n-1} (x^i - x^j) \right)^{2n} \cdot \left( \frac{1}{\prod_{0\le i<j\le n-1} x^{i+j}} \right)^{n}. \tag{12}$$

*Proof.* Substitute $x_i \longrightarrow x^{i-1}$ and $y_i \longrightarrow 1/x^{i-1}$, then

$$x_{i_1}^{j_2-1} \cdot y_{j_1}^{i_2-1} \longrightarrow x^{(i_1-1)(j_2-1)-(j_1-1)(i_2-1)}.$$

Denote $i_1 - 1 = j,\ j_1 - 1 = i,\ i_2 - 1 = k$ and $j_2 - 1 = \ell$, then

$$x^{(i_1-1)(j_2-1)-(j_1-1)(i_2-1)} = x^{jk-i\ell},$$

$0 \le i, j, k, \ell$, hence $D(x, y) \longrightarrow B(x)$, see (8), and $\det(D(x, y)) \longrightarrow \det(B(x))$. By these same substitutions,

$$\prod_{1\le i<j\le n} (x_i - x_j) \longrightarrow \prod_{0\le i<j\le n-1} (x^i - x^j).$$

Similarly,

$$\prod_{1\le i<j\le n} (y_i - y_j) \longrightarrow \prod_{0\le i<j\le n-1} \frac{x^j - x^i}{x^{i+j}} = (-1)^{\binom{n}{2}} \cdot \prod_{0\le i<j\le n-1} \frac{x^i - x^j}{x^{i+j}}$$

Together with (11) it implies that

$$\det(B(x)) = (-1)^{\binom{n}{2}} \left( \prod_{0\le i<j\le n-1} (x^i - x^j) \cdot (-1)^{\binom{n}{2}} \cdot \prod_{0\le i<j\le n-1} \frac{(x^i - x^j)}{x^{i+j}} \right)^{n} =$$

$$(-1)^{\binom{n}{2}(n+1)} \cdot \left( \prod_{0\le i<j\le n-1} (x^i - x^j) \right)^{2n} \cdot \left( \frac{1}{\prod_{0\le i<j\le n-1} x^{i+j}} \right)^{n}.$$

□

Note that

$$\sum_{0\le i<j\le n-1} i + j = \frac{(n-1)^2 n}{2},$$

hence

$$\left( \frac{1}{\prod_{0\le i<j\le n-1} x^{i+j}} \right)^{n} = \left( \frac{1}{x} \right)^{\frac{(n-1)^2 n^2}{2}}.$$

Thus (12) can be rewritten as

$$\det(B(x)) = (-1)^{\binom{n}{2}(n+1)} \cdot \left( \prod_{0 \le i < j \le n-1} (x^i - x^j) \right)^{2n} \cdot \left( \frac{1}{x} \right)^{\frac{(n-1)^2 n^2}{2}}. \tag{13}$$

**The proof of Proposition 2.3**

*Proof.* Let $\xi$ be a primitive $n$-th root of unit. Substituting $x \longrightarrow \xi$ in $B(x)$ we obtain the matrix $B(\xi)$ of (7). Since $\xi^n = 1$, the factor

$$\left( \frac{1}{\xi} \right)^{\frac{(n-1)^2 n^2}{2}}$$

equals 1 and can be discarded. Thus by (13) we need to show that

$$\left( \prod_{0 \le i < j \le n-1} (\xi^i - \xi^j) \right)^{2n} = \pm n^{(n^2)}. \tag{14}$$

Taking $n$-th root of both sides, we need to prove that for some integer $r$,

$$\left( \prod_{0 \le i < j \le n-1} (\xi^i - \xi^j) \right)^{2} = \prod_{0 \le i < j \le n-1} (\xi^i - \xi^j)^2 = \pm \xi^r n^n.$$

Rewrite the left side (up to a $\pm$ sign) as

$$\prod_{0 \le i \ne j \le n-1} (\xi^i - \xi^j). \tag{15}$$

Given $0 \le i \ne j \le n-1$, write the factor $\xi^i - \xi^j$ as $\xi^i - \xi^j = \xi^i(1 - \xi^{j-i})$ if $i < j$, and as $\xi^i - \xi^j = \xi^i(1 - \xi^{n-(i-j)})$ if $i > j$. Thus, for each $1 \le a \le n-1$ the factor $1 - \xi^a$ is obtained in (15) from $1 - \xi^a,\ \xi - \xi^{a+1},\ \ldots,\ \xi^{n-a-1} - \xi^{n-1}$ ($n - a$ factors) and from $\xi^{n-a} - 1,\ \xi^{n-a+1} - \xi,\ \ldots,\ \xi^{n-1} - \xi^{a-1}$ ($a$ factors). After pulling out the appropriate power of $\xi$, it follows that for each $1 \le a \le n-1$, the factor $1 - \xi^a$ appears in (15) exactly $n$ times. Also, for that $1 \le a \le n-1$ we pull out the factors $\xi^i,\ 0 \le i \le n-1$, namely we pull out

$$\xi^{0+1+2+\cdots+n-1} = \xi^{\frac{n(n-1)}{2}}.$$

Since there are $n-1$ $a$'s, in total we pull out

$$\xi^{\frac{n(n-1)^2}{2}}.$$

It follows that

$$\prod_{0 \le i < j \le n-1} (\xi^i - \xi^j)^2 = \pm \xi^{\frac{n(n-1)^2}{2}} \left( \prod_{i=1}^{n-1} (1 - \xi^i) \right)^n. \tag{16}$$

So it suffices to prove that

$$\prod_{i=1}^{n-1}(1-\xi^i)=n.$$

To prove this, note that, since $\xi$ is primitive, $\xi,\xi^2,\ldots,\xi^{n-1}$ are the roots of the monic polynomial $(x^n-1)/(x-1)$, hence

$$\prod_{i=1}^{n-1}(x-\xi^i)=(x^n-1)/(x-1)=\sum_{i=0}^{n-1}x^i.$$

Now substitute $x\longrightarrow 1$. $\square$

**Remark 2.6.** Let $\xi_1$ and $\xi_2$ be two primitive $n$-th roots of 1, then

$$\prod_{0\le i<j\le n-1}(\xi_1^i-\xi_1^j)=\pm\prod_{0\le i<j\le n-1}(\xi_2^i-\xi_2^j). \tag{17}$$

*Proof.* There is an integer $r>0$ such that $\xi_2=\xi_1^r$, and since both are primitive, $(r,n)=1$. Thus

$$\prod_{0\le i<j\le n-1}(\xi_2^i-\xi_2^j)=\prod_{0\le i<j\le n-1}(\xi_1^{ri}-\xi_1^{rj}).$$

Since $(r,n)=1$, the product on the right contains all the factors $\xi_1^k-\xi_1^\ell,\ 0\le k<\ell\le n-1$, each factor with a $\pm$ sign. $\square$

# 3 A generalization

More generally, given $0\le a,b\in\mathbb{Z}$, denote

$$D(x,a,b)=(x^{ajk+bi\ell})_{0\le i,j,k,\ell\le n-1}. \tag{18}$$

**Proposition 3.1.** *The above matrix $D(x,a,b)$ has the following determinant:*

$$det(D(x,a,b))=\left(\prod_{0\le i<j\le n-1}(x^{ai}-x^{aj})\right)^n\cdot\left(\prod_{0\le i<j\le n-1}(x^{bi}-x^{bj})\right)^n \tag{19}$$

*Proof.* Start with (11) and apply the substitution $x_i\longrightarrow x^{a(i-1)}$ and $y_i\longrightarrow x^{b(i-1)}$. $\square$

**Proposition 3.2.** *Let $\xi$ be a primitive $n$-th root of unity and let $a,b$ be integers. If either $(a,n)>1$ or $(b,n)>1$ then* $\det(D(\xi,a,b))=0$. *If both $(a,n)=(b,n)=1$ then*

$$\det(D(\xi,a,b))=\pm n^{(n^2)}$$

*Proof.* By (19)

$$det(D(\xi,a,b)) = \left(\prod_{0\le i<j\le n-1} (\xi^{ai}-\xi^{aj})\right)^n \cdot \left(\prod_{0\le i<j\le n-1} (\xi^{bi}-\xi^{bj})\right)^n \tag{20}$$

Assume $(a,n)=d>1$, then $(\xi^a)^{\frac{n}{d}}=1$, and since $0<\frac{n}{d}\le n-1$, the product

$$\prod_{0\le i<j\le n-1} (\xi^{ai}-\xi^{aj})$$

contains the factor $1-(\xi^a)^{\frac{n}{d}}=0$ hence is zero. Similarly if $(b,n)>1$.

If both $(a,n)=(b,n)=1$ then $\xi_1=\xi^a$ and $\xi_2=\xi^b$ are both primitive. By (17), the two factors of (20) differ by a $\pm$ sign, so by (20) and by (14)

$$det(D(\xi,a,b)) = \pm\left(\prod_{0\le i<j\le n-1} (\xi^{i}-\xi^{j})\right)^{2n} = \pm n^{(n^2)}.$$

$\square$

# 4 Minors

Since the minors of a generic Vandermonde matrix are Schur polynomials multiplied by the product of differences of the relevant variables (times some monomial), analogous statements can be proved for any minor of the considered $n^2\times n^2$ matrix, and one should get expressions featuring Schur polynomials evaluated at roots of unity.

# References


[1] Regev, A. and Seeman, T., $\mathbb{Z}_2$-graded tensor products of p.i. algebras, *J. Algebra* **291** (2005), 274–296.