Appendix to Dr. Z.'s 140th Opinion:
Some Mathematical Footnotes to Zvi Artstein's Masterpiece "Mathematics and the Real World"

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Introduction: In `http://www.math.rutgers.edu/~zeilberg/Opinion140.html` , I expressed my great admiration of Zvika Artstein's masterpiece[A], and the lessons that can be learned from his penetrating and insightful book. The purpose of the present appendix is to list a few mathematical footnotes. I also include a (very short!) list of possible errors.

Personal Remark: Back in 1981, Zvika and I were colleagues at the Weizmann Institute of Science. Already then Zvika was interested in "meta-mathematics", i.e. in the big picture about the nature of mathematics, and *mathematical activity*, and I remember his great enthusiasm about the Davis-Hersh (now) classic, *The Mathematical Experience*[DH]. Artstein's new book may be considered an early-21st-century analog, and I am sure that it too is destined to become a classic.

• (pp. 34-36) About pattern-matching.

The subway sequence is sequence 54 in Neil Sloane's famous OEIS namely `https://oeis.org/A000054` . Zvika should have mentioned the OEIS, one of the greatest tools for "pattern-matching". I disagree with Wittgenstein (and many others) that logically a finite sequence can be continued in many ways. Using *Occam's razor*, and only using part of the known sequence, as "training data", so that the remaining known part can be checked, it is often clear-cut what is the most natural continuation. Of course, it is *ansatz-dependent*. See my article [Z1].

• (p. 53) Regarding the proof that $\sqrt{2}$ is irrational, see my note [Z3].

• (p. 188), Galileo was asked why gamblers put higher stakes on the sum of three (fair, 6-faced) dice adding up to 10 rather than 9, since the set of integer-partitions of 10 into 3 strictly positive parts,

$$\{6+3+1 \quad , \quad 6+2+2 \quad , \quad 5+4+1 \quad , \quad 5+3+2 \quad , \quad 4+4+2 \quad , \quad 4+3+3\} \quad ,$$

has the same number of elements as those adding-up to 9,

$$\{6+2+1 \quad , \quad 5+3+1 \quad , \quad 5+2+2 \quad , \quad 4+4+1 \quad , \quad 4+3+2 \quad , \quad 3+3+3\} \quad .$$

As told by Artstein, Galileo correctly replied that one has to talk about *compositions*. Indeed if you roll m (fair) k-faced dice, the probability generating function (in q) for compositions is

$$\frac{1}{k^m}\left(\sum_{i=1}^{k} q^i\right)^m \quad ,$$

that is (trivially!) **strictly** unimodal. The probability generating function for integer-partitions, on the other hand is given by the polynomial

$$\frac{1}{\binom{m+k-1}{m}}\, q^m \binom{m+k-1}{m}_q \quad ,$$

where $\binom{a}{b}_q := ((1-q^a)\cdots(1-q^{a-b+1}))/((1-q)\cdots(1-q^b))$.

For the Galileo case, the former generating polynomial is

$$\frac{1}{6^3}\,(\sum_{i=1}^{6} q^i)^3 =$$

$$\frac{1}{216}(q^{18}+3\,q^{17}+6\,q^{16}+10\,q^{15}+15\,q^{14}+21\,q^{13}+25\,q^{12}+27\,q^{11}+27\,q^{10}+25\,q^9+21\,q^8+15\,q^7+10\,q^6+6\,q^5+3\,q^4+q^3) \quad ,$$

while the latter is

$$\frac{1}{\binom{3+6-1}{3}}\,q^3\binom{3+6-1}{3}_q = \frac{1}{56}\,q^3\binom{8}{3}_q =$$

$$\frac{1}{56}(q^{18}+q^{17}+2\,q^{16}+3\,q^{15}+4\,q^{14}+5\,q^{13}+6\,q^{12}+6\,q^{11}+6\,q^{10}+6\,q^9+5\,q^8+4\,q^7+3\,q^6+2\,q^5+q^4+q^3) \quad .$$

Thanks to James Joseph Sylvester, the generating function for integer partitions of positive integers into m parts and largest part $\leq k$ is unimodal, and this fact was given a combinatorial proof by Kathy O'Hara([O]). In [Z2], I adapted it into a 'high-school algebra proof'. However these generating polynomials, the so-called *Gaussian polynomials*, are **not** always *strictly* unimodal, as the above Galileo 'paradox' exemplifies, since there is a 'plateau' at the middle (and at the two ends). Recently Igor Pak and Greta Panova [PP] proved (see [Za] for a nice elementary proof) that (with very few exceptions, that they list, including the above case considered by Galileo) it is *strictly* unimodal, so with larger-faced dice and/or more of them, this 'paradox' would not have arisen.

• (p. 216) Using Bayes' Rule.

For the 'person in the street' it is more instructive to use nicer numbers. Suppose that there are 100 people, out of whom it is known that exactly one person is sick and that the rest are healthy. It is also known that the out of the 99 healthy people, exactly one would be diagnosed sick, and the sick person would definitely be diagnosed sick.

Invoking *the principle of equal a priori probabilities*, let's make the *default assumption* that each of the 100 people is equally likely to be really sick (of course, if you neither smoke nor drink, and the other do, this assumption is false), and the equally unjustified assumption that each of the 99 healthy people are equally likely to be diagnosed sick.

Regardless of assumptions, there are always two people diagnosed sick. Under the above, default, assumptions, your *a priori* chance of being sick is $\frac{1}{100}$ and your chance of being healthy and being diagnosed sick is $\frac{99}{100}\cdot\frac{1}{99}=\frac{1}{100}$. So you are equally likely to be really sick and be healthy but pronounced sick. Of course with probability $\frac{98}{100}$ you are neither. But granted that you were unfortunate enough to be diagnosed sick, you still have a chance of $\frac{1}{2}$ of being healthy. Nevertheless, it is still depressing, since your chance of being healthy dropped from $\frac{99}{100}$ to $\frac{1}{2}$.

• (p. 221) I agree more with Gigerenzer than with Artstein about the proper way to train physicians and others not to abuse probability. Nowadays one can use *computer simulations* that are very convincing, and would not have to torture poor physicians with 'logical analysis' over their heads.

For example, for simulating one instance of the above scenario with $N-1$ healthy persons, and one sick one, the following Maple code

```
A:=proc(N) local rs,fs,ra,ra1,i,H: ra:=rand(1..N): ra1:=rand(1..N-1):  rs:=ra():
H:={ seq(i,i=1..N) } minus {rs}:  fs:=H[ra1()]:[rs,fs]:end:
```

inputs N and picks at random the *really* sick person, `rs` (that only G-d knows that he is sick), and the healthy person who was (wrongly) diagnosed sick. Using procedure `A(N)`, one can simulate it many (`K`) times, using the following procedure, `Khole(N,K)`:

```
Khole:=proc(N,K) local T,S,i,lu:  for i from 1 to N do T[i]:=0:  S[i]:=0:  od:
for i from 1 to K do lu:=A(N): T[lu[1]]:=T[lu[1]]+1:  S[lu[2]]:=S[lu[2]]+1:
od:  [seq([T[i],S[i]],i=1..N)]: end:
```

It outputs the list $[[a_i, b_i]]$, $1 \leq i \leq n$, where a_i is the number of times, out of the K trials, that person i was truly sick, and b_i is the number of times he was healthy but diagnosed sick. For example, by typing

```
Khole(10,10000);
```

one gets something, like (since we use a (pseudo)-random number generator, the actual numbers differ each time)

```
[[966, 1001], [996, 983], [998, 984], [994, 991], [1023, 956], [982, 1081], [964, 990],
[983, 989], [998, 1040], [1096, 985]] ,
```

indicating that each of the ten persons roughly gets a false positive and a true positive an equal number of times. Now this is more convincing, to most people, than 'logical analysis'.

• (p. 223), last paragraph. Indeed *over-fitting* is a dangerous disease in the human quest for 'patterns', and it reminds me of the Hebrew poet David Avidan's beautiful poem *yippui koach*, that says

"*... the knowledge that there is* **no** *justification [for our existence], and [nevertheless] searching for it all the time*"

• (p. 321): The calculus nightmare. The whole (so-called) 'rigorization' of the infinitesimal calculus can be done much more painlessly using discrete calculus, see [Z4].

• (p. 356): Regarding errors, my colleague, Abbas Bahri[B], has pointed out five serious gaps in deep (and famous) mathematical results that are universally considered as proved.

• (pp. 402-403): Artstein is a bit too hard, and overly pedantic, what Israelis call a *yeke*, in nit-picking on an article published in an Israeli high-school education journal. By 'reverse engineering' the (admittedly erroneous) solution of the original writer, it is clear that what the latter meant

was: 'Suppose that you found out that the family has at least two daughters'. Then the original solution is correct (corresponding to Artstein's second scenario). By 'slip of the pen', the criticized writer transforms it into:' you see two girls playing outside', a fact that implies that there are at least two girls in the family, but of course is not equivalent to it.

Whenever there is "missing information", there usually are natural 'default assumptions'. For example for the famous Monty Hall problem (p. 404), the most natural assumption is that Monty is *always* obligated to open a door with a goat.

A Very Short Errata

• p. 97, line 19 from top: 1,4,6,16 → 1,4,9, 16

• p. 126, line 3 from the bottom: $\frac{nct}{L} \to \frac{nc}{L}$

• p. 160, line 15 from top: Ernest Rutherford (1871-1927) → Ernest Rutherford (1871-1937)

• p. 186, line 13 from top: die → dice

• p, 227, subtitles, line 4: expected win → expected to win

• p. 298, line 4 from bottom:

exponential number of steps as a function of the number →

exponential number of steps as a function of the **logarithm** (or number of digits) of the number

• p. 299: KSA is more commonly known as AKS

• p. 300, line 8 from top: $k^{2m} \to k^m$

• p. 319, line 7 from bottom: was thwarted → has thwarted

• p. 338, line 15 from top: Gödel was, for many years only a "long-term visitor" at the Institute for Advanced Study. He became an official permanent member as late as 1953.

• p. 338, line 5 from bottom: always be theorems → always be statements

• p. 403, line 4 from top: $P(B|A) \to P(A|B)$

References

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