The Absent-Minded Passengers Problem via Computer Algebra

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Abstract. In this *case study*, we illustrate the power of *experimental mathematics* and *symbolic computation*, by **discovering** interesting new facts about the so-called Absent-Minded Passengers Problem, extending recent work of Norbert Henze and Günter Last. Since we are absolutely certain that these new facts are indeed true, and proving is not nearly as much fun as discovering, we leave the proofs to the obtuse readers.

The Maple package. This article is accompanied by the Maple package `AMP.txt` that can be obtained, along with numerous input and output files, from the front of this article

`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/amp.html` .

The initial puzzle

The beautiful article [HL] is inspired by the following puzzle that appeared in the two delightful collections [B] and [W].

An airplane with $n \geq 2$ passengers is fully booked. Passengers are boarding in chronological order, according to the numbers on their boarding passes. The first passenger loses his boarding pass and picks one of the seats (uniformly) at random. Each subsequent passenger takes his or her seat if available, otherwise takes one of the remaining seats (uniformly) at random. What is the probability that the last passenger (i.e. passenger n) will sit in the correct seat?"

It is not too hard to see ([B][M][HL]) that the answer is $\frac{1}{2}$. It is proved in [HL] that, more generally, the probability that passenger i ($i \geq 2$) will sit in the correct seat is $\frac{n-i+1}{n-i+2}$. Even more generally, they proved that when the first k passengers are absent-minded, and $i > k$, that probability equals $\frac{n-i+1}{n-i+k+1}$.

A Generatingfunctionlogy Approach to the $k = 1$ case

A quicker way to handle the original case with only one absent-minded passenger is via *generating functions* (alias *weight-enumerators*).

Let the *weight* of a sitting arrangement (a certain permutation of length n) resulting from this process be the product of w_i over all passengers i sitting in the **wrong** seat. The initial *state* is when all the seats are empty. If, by pure luck, passenger 1 landed in seat 1, then the game is over, and the weight of that scenario is 1 since everyone landed in the right seat. Also the probability of that happening is $\frac{1}{n}$. Otherwise, passenger 1 will take seat i, with probability $\frac{1}{n}$, for *some* i between 2 and n. All the passengers, 2 through $i-1$ will each take their rightful seat, and we now have a situation where i has to pick one of the $n-i+1$ seats in the set $\{1, i+1, \ldots, n\}$. Let's call the initial state S_0 and the subsequent states S_i ($2 \leq i \leq n$)). Let $F_n(w_1, \ldots, w_n)$ be the

weight-enumerator of the set of all final sitting configurations that start at the initial state (our object of desire) and let A_i be the weight-enumerator of those that come from state S_i.

We have

$$F_n(w_1,\dots,w_n) = \frac{1}{n} + \frac{w_1}{n}\sum_{i=2}^{n} A_i \quad , \quad \textit{and}$$

$$A_i = \frac{w_i}{n-i+1}\left(1 + \sum_{j=i+1}^{n} A_j\right) \quad , \quad 2 \le i \le n \quad .$$

This equation follows from the fact that passenger i has $n-i+1$ equally likely choices , each of them resulting with him sitting in the wrong seat (hence the factor $\frac{w_i}{n-i+1}$ in the front). If he chose seat 1 then the game is over, since all the remaining passengers seat where they are supposed to. Otherwise he sits in seat j ($i < j \le n$), and we are in state S_j.

Hence, for $2 \le i \le n$,

$$\frac{n-i+1}{w_i}A_i \;=\; 1 \;+\; \sum_{j=i+1}^{n} A_j \,.$$

Replacing i by $i-1$, we have

$$\frac{n-i+2}{w_{i-1}}A_{i-1} \;=\; 1 \;+\; \sum_{j=i}^{n} A_j \,.$$

Subtracting, we get

$$\frac{n-i+2}{w_{i-1}}A_{i-1} - \frac{n-i+1}{w_i}A_i = A_i \quad ,$$

implying that

$$A_{i-1} = \frac{w_{i-1}}{w_i} \cdot \frac{w_i + n - i + 1}{n-i+2} \cdot A_i \quad .$$

Since $A_n = w_n$, we have, for all $0 \le i \le n-1$,

$$A_{n-i} = \frac{w_{n-i}}{(i+1)!}(w_{n-i+1}+i)(w_{n-i+2}+i-1)\cdots(w_n+1) \quad .$$

In particular (take $i = n-1$)

$$A_1 = \frac{w_1}{n!}\prod_{i=2}^{n}(w_i + n + 1 - i) \quad .$$

Since $F_n(w_1,\dots,w_n) = \frac{1}{n} - \frac{w_1}{n} + A_1$, we have:

Theorem 1: The weight-enumerator of all sitting arrangements with one absent-minded passenger is

$$F_n(w_1,\dots,w_n) \;=\; \frac{1-w_1}{n} \;+\; \frac{w_1}{n!}\prod_{i=2}^{n}(w_i + n + 1 - i) \quad .$$

It follows that for $2 \leq i \leq n$:

$$F_n(\,1^{i-1}\,,\,w_i\,,\,1^{n-i}\,) \;=\; \frac{w_i+n+1-i}{n+2-i} \quad ,$$

implying that the probability that passenger i will sit in the **right** place is $\frac{n+1-i}{n+2-i}$ (the coefficient of w_i^0), as proved, via a different method in [HL], Equation 1. More generally for any subset S of $\{2,\dots,n\}$, setting $w_i = 1$ if $i \neq S$ and leaving w_i alone when $i \in S$, we get that the *marginalized* generating function equals

$$\prod_{i \in S} \frac{w_i+n+1-i}{n+2-i} \quad ,$$

implying that the probability that *all* members of S will sit in the **wrong** place is

$$\prod_{i \in S} \frac{1}{n+2-i} \quad ,$$

while the probability that they *all* sit in the **right** place is

$$\prod_{i \in S} \frac{n+1-i}{n+2-i} \quad .$$

Specializing all the w_i to be w we have an alternative proof of the following theorem in [HL].

Theorem 2: The probability generating function, let's call it $f_n(w)$, (a polynomial of degree n in w), whose coefficient of w^l is the probability that exactly l passengers sit in the wrong seat is

$$f_n(w) \;=\; \frac{1-w}{n} + \frac{w}{n!}\prod_{i=1}^{n-1}(w+i) \quad .$$

The question of a closed-form expression for the analogous probability generating function, let's call it $f_n^{(k)}(w)$, for the case where the first k passengers are absent-minded was left open in [HL]. The next theorem fills this gap. (Note that $f_n^{(1)}(w) = f_n(w)$).

Theorem 3: The probability generating function $f_n^{(k)}(w)$ (a polynomial of degree n in w), whose coefficient of w^l is the probability that exactly l passengers sit in the wrong seat when the first k passengers are absent-minded, is given by

$$f_n^{(k)}(w) \;=\; \frac{1}{n!}\sum_{r=0}^{k} r!\binom{k}{r} w^r\,(1-w)^{k-r}\prod_{i=1}^{n-k}(rw\,+\,i\,) \quad .$$

We don't believe that it is possible to conjecture this theorem by merely cranking out sufficiently many special cases and guessing a *pattern*. What we did was try and conjecture a generalization of Theorem 1, where one keeps track of the actual passengers that are sitting in the wrong seat. Let $F_n^{(k)}(w_1,\dots,w_n)$ be the multi-linear polynomial in $(w_1,\dots,w_n)$ whose coefficient of $w_{i_1}\dots w_{i_l}$

is the probability that the passengers in the set $\{i_1, \ldots, i_l\}$ **definitely** are *wrongly*-seated, and the complement is **definitely** seated in the *right* seats. (Note that $F_n^{(1)}(w_1, \ldots, w_n) = F_n(w_1, \ldots, w_n)$).

Using **dynamical programming** (see the source code for procedure `AnwkG(n,w,k)` in the Maple package `AMP.txt`), we generated lots of specific examples, that enabled us to discover the following generalization of Theorem 1.

Theorem 4: Let $e_r(w_1, \ldots, w_k)$ be the coefficient of X^r in $\prod_{j=1}^{k}((1-w_j)X + w_j)$ (these are variants of the elementary symmetric functions). Then, if $n \geq k$, we have

$$F_n^{(k)}(w_1, \ldots, w_n) \,=\, \frac{1}{n!} \sum_{r=0}^{k} r! e_{k-r}(w_1, \ldots, w_k) \cdot \prod_{j=k+1}^{n} (r w_j + n + 1 - j) \quad .$$

Theorem 3 follows from Theorem 4 by setting all the w_j's to be w. Note that if we plug-in all the w_j's, *except* w_i, to be 1, but leave w_i alone, we rederive the fact, proved in [HL] another way, that the probability of the event (if $i > k$) 'Passenger i sitting in the **right** seat' is $\frac{n-i+1}{n-i+k+1}$, Another consequence of our Theorem 4 is Theorem 3 in [HL] , that states that these events are independent.

By differentiating the expression for $f_n^{(k)}(w)$, given in Theorem 3, with respect to w, and plugging-in $w = 1$ we find (as [HL] already did) that the expectation is $k(1 + \sum_{i=k+1}^{n-1} \frac{1}{i})$. By differentiation twice, and doing some manipulatorics, one can get the expression for the variance established in [HL]. The advantage of our Theorem 3 is that we can keep going and derive explicit expressions for higher moments. Carsten Schneider's `Sigma` package [S1][S2] should be helpful here.

The First Eight Moments of the Random Variable 'Number of Passengers Sitting in the Wrong Seat' for the original case of One absent-minded passenger

We are too lazy to find higher moments for the general case of k absent-minded passengers, but we did it for the original case of $k = 1$.

Let X_n be that random variable. The expectation $E[X_n]$, that equals $f_n'(1)$ is easily seen (by *logarithmic differentiation*) to be $\sum_{i=1}^{n-1} \frac{1}{i}$, the *Harmonic number* H_{n-1}. This is already mentioned in [HL], where they also derived an explicit expression for the variance (for arbitrary k),

It is convenient to introduce the notation

$$Hn[r] := \sum_{i=1}^{n-1} \frac{1}{i^r} \quad .$$

Note that the upper limit is $n-1$ rather than the customary n. This way the formulas are much simpler.

Theorem 5: Let X_n be the random variable "number of passengers sitting in the wrong seat" where there is one absent-minded passenger, and n passengers altogether. Then, denoting by

$m_r(X_n)$ the r^{th} moment about the mean, we have (please pardon the *computereze*)

$$E[X_n] \,=\, \sum_{i=1}^{n-1} \frac{1}{i} \quad .$$

$$Var[X_n] \,=\, (n*Hn[1]-n*Hn[2]+2*Hn[1])/n \quad .$$

$$m_3(X_n) \,=\, (n*Hn[1]-3*n*Hn[2]+2*n*Hn[3]-3*Hn[1]^2+6*Hn[1]-3*Hn[2])/n \quad .$$

$$m_4(X_n) \,=\, (3*n*Hn[1]^2-6*n*Hn[1]*Hn[2]+3*n*Hn[2]^2+4*Hn[1]^3+n*Hn[1]-7*n*Hn[2]+12*n*Hn[3]$$

$$-6*n*Hn[4]-6*Hn[1]^2+14*Hn[1]-18*Hn[2]+8*Hn[3])/n \quad .$$

$$m_5(X_n) \,=\, (-5*Hn[1]^4+10*n*Hn[1]^2-40*n*Hn[1]*Hn[2]+20*n*Hn[1]*Hn[3]$$

$$+30*n*Hn[2]^2-20*n*Hn[2]*Hn[3]+10*Hn[1]^3+n*Hn[1]-15*n*Hn[2]+50*n*Hn[3]-60*n*Hn[4]$$

$$+24*n*Hn[5]+5*Hn[1]^2-30*Hn[1]*Hn[2]+15*Hn[2]^2+30*Hn[1]-75*Hn[2]+80*Hn[3]-30*Hn[4])/n \quad .$$

$$m_6(X_n) \,=\, (6*Hn[1]^5+15*n*Hn[1]^3-45*n*Hn[1]^2*Hn[2]+45*n*Hn[1]*Hn[2]^2-15*n*Hn[2]^3-15*Hn[1]^4$$

$$+25*n*Hn[1]^2-180*n*Hn[1]*Hn[2]+220*n*Hn[1]*Hn[3]-90*n*Hn[1]*Hn[4]+195*n*Hn[2]^2$$

$$-300*n*Hn[2]*Hn[3]+90*n*Hn[2]*Hn[4]+40*n*Hn[3]^2+20*Hn[1]^3+n*Hn[1]-31*n*Hn[2]$$

$$+180*n*Hn[3]-390*n*Hn[4]+360*n*Hn[5]-120*n*Hn[6]+90*Hn[1]^2-330*Hn[1]*Hn[2]$$

$$+120*Hn[1]*Hn[3]+225*Hn[2]^2$$

$$-120*Hn[2]*Hn[3]+62*Hn[1]-270*Hn[2]+520*Hn[3]-450*Hn[4]+144*Hn[5])/n \quad .$$

For $m_7(X_n)$ and $m_8(X_n)$ see the web-page

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oAMP2a.txt` .

Sketch of the Proof: The above theorem was discovered by **pure guessing**, using an *ansatz* with *undetermined coefficients* featuring the quantities $Hn[r]$, that are the partial sums of $\zeta(r)$ for $r \geq 2$. They are all solutions of complicated recurrences and hence can be rigorously proved using Carsten Schneider's amazing `Sigma` package [S1][S2]. Since we are sure that they are true, we did not bother to actually do it.

The web-page also has asymptotic expansions for these quantities, confirming, via elementary means, that X_n is **asymptotically normal**, up to the 8^{th} moment, and one can easily go far beyond. This fact was proved using 'advanced' probability in [HL].

Recurrences for $f_n^{(k)}(w)$ for $k = 1, 2, 3, 4$

It is useful to have recurrences for these quantities.

Theorem 6: The probability generating function for the random variable, 'the number of passengers sitting in the wrong seat where the first k passengers are absent-minded', $f_n^{(k)}(w)$ satisfy the following linear recurrences.

For $f_n^{(1)}(w)$ (alias $f_n(w)$) we have

$$\frac{n\,(n+w)\,f_n^{(1)}(w)}{(2+n)\,(1+n)} - \frac{(2\,n+w+1)\,f_{n+1}^{(1)}(w)}{2+n} + f_{n+2}^{(1)}(w) = 0$$

For $f_n^{(2)}(w)$ we have

$$-\frac{n\,(n+2\,w)\,(n+w)\,f_n^{(2)}(w)}{(n+4)\,(n+3)\,(2+n)} + \frac{\left(3\,n^2+6\,nw+2\,w^2+3\,n+3\,w+1\right)f_{n+1}^{(2)}(w)}{(n+4)\,(n+3)} \quad ,$$

$$-3\,\frac{(n+w+1)\,f_{n+2}^{(2)}(w)}{n+4} + f_{n+3}^{(2)}(w) = 0 \quad .$$

For $f_n^{(3)}(w)$ we have

$$\frac{n\,(n+2\,w)\,(n+3\,w)\,(n+w)\,f_n^{(3)}(w)}{(n+5)\,(n+4)\,(n+3)\,(n+6)} - \frac{(3\,w+1+2\,n)\left(2\,n^2+6\,nw+2\,w^2+2\,n+3\,w+1\right)f_{n+1}^{(3)}(w)}{(n+5)\,(n+4)\,(n+6)}$$

$$+\frac{\left(6\,n^2+18\,nw+11\,w^2+12\,n+18\,w+7\right)f_{n+2}^{(3)}(w)}{(n+6)\,(n+5)} - 2\,\frac{(2\,n+3\,w+3)\,f_{n+3}^{(3)}(w)}{n+6} + f_{n+4}^{(3)}(w) = 0 \quad .$$

For a recurrence for $f_n^{(4)}(w)$ see the web-page

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oAMP1.txt` .

One can easily go further. In general $f_n^{(k)}(w)$ satisfies a linear recurrence equation of order $k+1$.

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