The Distribution of Double Deficiencies in Pattern-Avoiding Permutations

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Abstract

We study the distribution of the number of double deficiencies (DD) in permutations of length n avoiding one or two patterns of length 3. Using structural decompositions of these avoidance classes—together with a lattice-path decomposition in the 321-avoiding case—we derive functional equations and convolution-type recurrences that efficiently compute the corresponding double-deficiency generating functions in all but one single-pattern case. In the 321-avoiding permutations, the resulting generating function is algebraic; we derive exact formulas for the mean and variance and prove that the distribution is close in total variation to $\mathrm{Bin}(n-2, 1/4)$, with an explicit convergence rate. We also identify a DD-preserving symmetry that yields DD-Wilf equivalences, reducing the number of two-pattern cases that need to be considered separately. For the resulting two-pattern classes, we obtain explicit recurrences, including C-finite relations.

1 Introduction

Pattern avoidance is a classical topic in enumerative combinatorics. Let

$$[n] = \{1, 2, \ldots, n\},$$

and let $\mathfrak{S}_n$ denote the set of all permutations of $[n]$. A permutation $\pi \in \mathfrak{S}_n$ is said to *contain* a pattern $\sigma \in \mathfrak{S}_k$ if some subsequence of π is order-isomorphic to σ; otherwise, π is said to *avoid* σ. For a set of patterns S, let $\mathrm{Av}_n(S)$ denote the set of permutations in $\mathfrak{S}_n$ avoiding every pattern in S. When $S = \{\sigma\}$, we write simply $\mathrm{Av}_n(\sigma)$.

For patterns of length 3, the ordinary enumeration of these avoidance classes is well understood. Simion and Schmidt systematically studied permutations avoiding subsets of $\mathfrak{S}_3$,

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determining the cardinalities of the resulting classes and giving several bijective results [17]. In particular, for every $\sigma \in \mathfrak{S}_3$, the single-pattern avoidance class $\mathrm{Av}_n(\sigma)$ is counted by the Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

The simultaneous avoidance of two length-3 patterns has also been studied extensively. For example, Vatter developed generating-tree methods for enumerating permutations avoiding two distinct patterns from $\mathfrak{S}_3$, possibly together with additional forbidden patterns [18].

A finer question is how permutation statistics are distributed within these classes. Ordinary enumeration alone does not capture such information: two avoidance classes may have the same cardinality but different distributions of a given statistic. Several refinements of length-3 pattern avoidance have therefore been studied. Robertson, Saracino, and Zeilberger refined the enumeration by the number of fixed points—indices k with $\pi(k) = k$—showing that the 132-, 213-, and 321-avoiding classes are equidistributed with respect to this statistic, as are the 231- and 312-avoiding ones [15]. Elizalde determined the joint distribution of fixed points and excedances (indices with $\pi(k) > k$) for permutations simultaneously avoiding subsets of $\mathfrak{S}_3$ [5], while Elizalde and Pak gave a bijection between 321- and 132-avoiding permutations that preserves both statistics [8]. Dokos, Dwyer, Johnson, Sagan, and Selsor studied this phenomenon more generally through *statistic-Wilf equivalence*, and in particular determined the equivalences arising from the inversion number and major index for sets of forbidden patterns contained in $\mathfrak{S}_3$ [4].

In this paper, the refining statistic is the number of *double deficiencies*. For $\pi \in \mathfrak{S}_n$, an index $k \in [n]$ is a double deficiency of π if

$$\pi(k) < k < \pi^{-1}(k),$$

and we denote the number of double deficiencies of π by $\mathrm{DD}(\pi)$. For example, consider the permutation $\pi = 4251736$. Since $\pi(6) = 3 < 6 < 7 = \pi^{-1}(6)$, the index $k = 6$ is a double deficiency. In fact, this is the only double deficiency of π, so $\mathrm{DD}(\pi) = 1$.

Definition 1. For a set of patterns S, define the double-deficiency generating function by

$$f_S(n, x) = \sum_{\pi \in \mathrm{Av}_n(S)} x^{\mathrm{DD}(\pi)}.$$

For each fixed $n \geq 2$, this is a polynomial in x of degree at most $n-2$, since neither $k = 1$ nor $k = n$ can be a double deficiency, as $\pi(1) < 1$ and $\pi^{-1}(n) > n$ are impossible, so at most $n-2$ of the n indices qualify.

For unrestricted permutations (i.e., when $S = \varnothing$), inversion transforms double deficiencies into double excedances. Under a version of Foata's fundamental transformation, double excedances correspond to occurrences of the consecutive pattern 123, which, by reversal or complement symmetry, are equidistributed with occurrences of the consecutive pattern 321, or equivalently, with proper double descents [6]. The study of double descents and related local permutation statistics goes back to Carlitz and Scoville [3]. Elizalde and Noy obtained the full generating function for the distribution of the number of occurrences of the

consecutive pattern 123, and hence, by symmetry, 321 [7, Theorem 4.1]. Consequently, their result also gives the unrestricted double-deficiency distribution:

$$\sum_{n\geq 0} f_{\varnothing}(n,x)\frac{z^n}{n!} = \frac{2r\exp\left(\frac{1-x+r}{2}z\right)}{1+x+r-(1+x-r)\exp(rz)}, \qquad r=\sqrt{(x-1)(x+3)}.$$

Fu later gave a context-free-grammar treatment and further refinements involving proper double descents [12].

For pattern-avoiding permutations, however, the distribution of double deficiencies has received much less attention than other classical permutation statistics such as fixed points, excedances, inversions, and major index. The main previous work on double deficiencies in pattern-avoiding permutations concerns 321-avoiding permutations. Motivated by a conjecture of Marczinzik from the representation theory of Nakayama algebras, Rubey and Stump studied double deficiencies using the Billey–Jockusch–Stanley bijection between Dyck paths and 321-avoiding permutations [1, 16]. In particular, they showed that 321-avoiding permutations with no double deficiencies are counted by the Motzkin numbers. More recently, Krityakierne, Thanatipanonda, and Zeilberger gave a shorter direct proof of the same enumeration [14].

Despite the existing results for unrestricted permutations and for the zero-double-deficiency case in Av(321), to the best of our knowledge there has been no systematic study of the full double-deficiency distribution for avoidance classes defined by one or two patterns of length 3. In Section 2, we focus on single-pattern avoidance. Using structural decompositions of these avoidance classes, together with a lattice-path decomposition in the 321-avoiding case, we derive functional equations and convolution-type recurrences for efficiently computing the associated generating functions in all but one single-pattern case. We also identify, in Section 3, a symmetry that preserves the number of double deficiencies. A composition of the usual reverse, complement, and inverse operations interchanges the patterns 132 and 213 while fixing the other four patterns of length 3. In the framework of statistic-Wilf equivalence, this yields a family of *DD-Wilf equivalences* and reduces the number of two-pattern avoidance classes that need to be investigated separately in Section 4. All Maple programs accompanying this work are available at `https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/DD3av.html`.

2 Single-Pattern Avoidance

We now study the double-deficiency generating functions for single-pattern avoidance classes of length 3. For convenience, write

$$f_{\sigma}(n,x) = f_{\{\sigma\}}(n,x), \qquad \sigma \in S_3.$$

2.1 The class Av(321)

The class Av(321) does not admit the block decomposition used later for Av(132), Av(213), and Av(231). We therefore use a lattice-path bijection under which double deficiencies

correspond to a single step type. Previous work on double deficiencies in 321-avoiding permutations uses the Billey–Jockusch–Stanley bijection with Dyck paths [1, 16]. For the full distribution, we use Elizalde's cycle-diagram formulation [6] of the Foata–Zeilberger correspondence [11], which gives a bijection with bicolored Motzkin paths.

Recall that a Motzkin path of length n is a lattice path from $(0,0)$ to $(n,0)$, with up-steps $U=(1,1)$, level steps $L=(1,0)$, and down-steps $D=(1,-1)$, that never passes below the x-axis. The number of such paths is the nth Motzkin number M_n.

In Elizalde's correspondence [6, Sections 3.1 and 4.2], a 321-avoiding permutation π is mapped to a bicolored Motzkin path in which level steps above the x-axis have two possible colors, while level steps on the x-axis have only one. The level steps on the x-axis correspond to fixed points of π, while the two types of level step above the x-axis correspond, respectively, to double excedances and double deficiencies. We adopt the convention of leaving fixed-point and double-excedance level steps uncolored and coloring double-deficiency level steps red. This gives a bijection between $\mathrm{Av}_n(321)$ and the set of all such paths of length n, under which the number of red level steps equals $\mathrm{DD}(\pi)$.

Let $F_1(z,x)$ count these paths, with z marking length and x marking red level steps. Let $F_0(z,x)$ be the auxiliary series in which level steps on the x-axis may also be red. Thus

$$F_1(z,x)=\sum_{n\geq 0} f_{321}(n,x)z^n.$$

Path (drawn for F_1)	Contribution to F_1 at $x=1$	Contribution to F_0 at $x=1$
	1	2
	1	2
	2	2
	1	$2^3=8$

Figure 1: The four Motzkin path shapes of length 3, drawn for F_1: level steps on the x-axis must be uncolored, while those above it may be either uncolored or red, as the parallel segments indicate. In F_0, every level step may be either.

At $x=1$, Figure 1 gives

$$[z^3]F_1(z,1)=5=C_3, \qquad [z^3]F_0(z,1)=14=C_4.$$

More generally,

$$[z^n]F_1(z,1)=C_n, \qquad [z^n]F_0(z,1)=C_{n+1}. \tag{1}$$

The first identity is the Catalan enumeration of $\mathrm{Av}_n(321)$; both identities are verified in Remark 1. A first-return decomposition of an F_0-path gives

$$F_0(z,x)=1+(1+x)zF_0(z,x)+z^2F_0(z,x)^2. \tag{2}$$

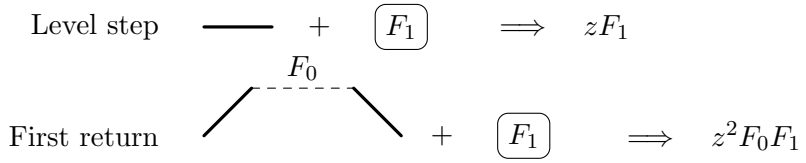

Figure 2: First-return decomposition for $F_1(z,x)$. The analogous decomposition for $F_0(z,x)$ is obtained by replacing the boxes labeled F_1 by F_0 and allowing the initial level step to be either uncolored or red. The two nonempty contributions then become $(1+x)zF_0$ and $z^2F_0^2$.

For F_1, the initial level step has weight z, while the portion enclosed by the first return is an F_0-path (Figure 2). Hence,

$$F_1(z,x) = 1 + zF_1(z,x) + z^2F_0(z,x)F_1(z,x). \tag{3}$$

Solving the quadratic equation for $F_0(z,x)$ gives

$$F_0(z,x) = \frac{1-(1+x)z-\sqrt{1-2z-2xz-3z^2+2xz^2+x^2z^2}}{2z^2}.$$

Substituting into the equation for $F_1(z,x)$ and rearranging yields

$$F_1(z,x) = \frac{2}{1-z+xz+\sqrt{1-2z-2xz-3z^2+2xz^2+x^2z^2}}.$$

This is the desired bivariate generating function over Av(321).

Remark 1. At $x=1$, the discriminant is $1-4z$, and

$$F_0(z,1) = \frac{1-2z-\sqrt{1-4z}}{2z^2}, \qquad F_1(z,1) = \frac{2}{1+\sqrt{1-4z}} = \frac{1-\sqrt{1-4z}}{2z} = C(z),$$

where $C(z) = \sum_{n\geq 0} C_n z^n$. Since $C = 1 + zC^2$,

$$F_0(z,1) = \frac{1-2z-\sqrt{1-4z}}{2z^2} = \frac{C(z)-1}{z} = C(z)^2.$$

This proves (1).

Coefficient recurrences. Equating coefficients in (2)–(3), write

$$F_0(z,x) = \sum_{n\geq 0} g(n,x)z^n, \qquad F_1(z,x) = \sum_{n\geq 0} f_{321}(n,x)z^n.$$

Then $g(0,x) = f_{321}(0,x) = 1$, and for $n \geq 1$,

$$g(n,x) = (1+x)g(n-1,x) + \sum_{i=2}^{n} g(i-2,x)\, g(n-i,x).$$

$$f_{321}(n,x) = f_{321}(n-1,x) + \sum_{i=2}^{n} g(i-2,x)\, f_{321}(n-i,x).$$

Figure 3, computed from this recurrence, suggests the binomial approximation proved next.

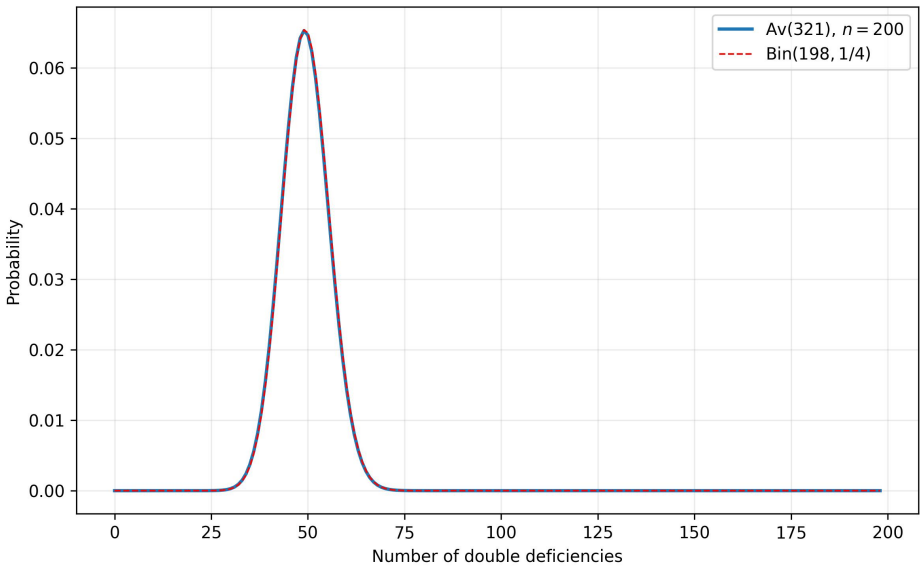

Figure 3: Probability distribution of the number of double deficiencies in $\mathrm{Av}_{200}(321)$, together with the binomial distribution $\mathrm{Bin}(198, \frac{1}{4})$.

A binomial approximation. For probability distributions μ and ν on $\mathbb{Z}$, write

$$d_{\mathrm{TV}}(\mu, \nu) = \frac{1}{2} \sum_{j \in \mathbb{Z}} |\mu\{j\} - \nu\{j\}| \,.$$

Proposition 1. *Let X_n denote the number of double deficiencies in a uniformly random permutation in $\mathrm{Av}_n(321)$. For $n \geq 2$, the support of X_n is $\{0, 1, \ldots, n-2\}$, and*

$$\mathbb{E}[X_n] = \frac{(n-1)(n-2)}{2(2n-1)}$$

and

$$\mathrm{Var}(X_n) = \frac{(n+1)(n-2)(3n^2-8n+3)}{2(2n-1)^2(2n-3)}.$$

Moreover,

$$d_{\mathrm{TV}}\left(\mathcal{L}(X_n), \mathrm{Bin}\left(n-2, \frac{1}{4}\right)\right) = O\left(n^{-1/2}(\log n)^{1/4}\right).$$

In particular, X_n is asymptotically normal.

The moment formulas follow by differentiating the functional equations and extracting coefficients, while the binomial approximation follows from singularity analysis applied to $F_1(z, x)$; both are proved in Appendix A.

A holonomic recurrence. Eliminating $g(n,x)$ yields the following third-order recurrence, valid for $n \geq 0$ as an identity in $\mathbb{Z}[x]$:

$$\begin{aligned}(n+4)x\, f_{321}(n+3,x) = &\Big((5x^2+9x-4)+(2x^2+3x-1)n\Big) f_{321}(n+2,x)\\ &-(x-1)\Big((x^2+8x+5)+(x^2+5x+2)n\Big) f_{321}(n+1,x)\\ &+(x-1)^2(x+3)(n+1) f_{321}(n,x),\end{aligned}$$

with initial conditions

$$f_{321}(0,x) = f_{321}(1,x) = 1, \qquad f_{321}(2,x) = 2.$$

At $x = 0$ the leading coefficient vanishes, and the identity reduces to

$$(n+4)M_{n+2} = (2n+5)M_{n+1} + 3(n+1)M_n,$$

the classical recurrence for the Motzkin numbers, consistent with $f_{321}(n,0) = M_n$.

2.2 The class Av(312)

Proposition 2. *If k is a double deficiency of π, then π contains an occurrence of the pattern 312.*

Proof. Suppose k is a double deficiency of π. Then $\pi(k) < k < \pi^{-1}(k)$. Among the first $k-1$ positions, neither the value $\pi(k)$ nor the value k can occur. Since there are $k-1$ positions but only $k-2$ remaining values from $\{1,\dots,k\}$, the pigeonhole principle implies that some position $i < k$ satisfies $\pi(i) > k$. Hence, $\pi(i) > k > \pi(k)$, so the entries $\pi(i)$, $\pi(k)$, k at positions i, k and $\pi^{-1}(k)$, respectively, form an occurrence of the pattern 312. □

Corollary 3. *For every n, every permutation in $\mathrm{Av}_n(312)$ has no double deficiencies. Consequently,*

$$f_{312}(n,x) = C_n,$$

where C_n denotes the nth Catalan number.

Remark 2. The converse of the proposition is false. For example, the permutation 3412 contains the pattern 312 but has no double deficiency.

2.3 The class Av(132)

Let $\pi \in \mathrm{Av}_n(132)$, with its maximum in position i, and write

$$\pi = L\, n\, R.$$

Avoidance of 132 forces every entry of L to exceed every entry of R; otherwise a, n, b, with $a \in L$, $b \in R$, and $a < b$, is a 132-pattern. Thus, as sets of values,

$$L = \{n-i+1,\dots,n-1\}, \qquad R = \{1,\dots,n-i\}.$$

After standardization, both blocks remain 132-avoiding. The statistic is not preserved, however, because positions and values may be shifted by different amounts; the following parameter records their difference.

Shifted double deficiencies. Suppose a block of length m occupies positions $p+1, \ldots, p+m$, has value set $\{q+1, \ldots, q+m\}$, and standardizes to $\beta \in \mathfrak{S}_m$. Then

$$\pi(p+j) = q + \beta(j), \qquad 1 \le j \le m.$$

At local position j, the global position is $k = p + j$, while the global value k becomes

$$k - q = p + j - q = j + c, \qquad c := p - q.$$

When $1 \le j + c \le m$,

$$\pi(k) = q + \beta(j) \qquad \text{and} \qquad \pi^{-1}(k) = p + \beta^{-1}(j + c).$$

Consequently,

$$\pi(k) < k < \pi^{-1}(k) \quad \Longleftrightarrow \quad \beta(j) < j + c < \beta^{-1}(j + c) + c.$$

This motivates the following definition.

Definition 2. Let $\pi \in \mathfrak{S}_n$ and $c \in \mathbb{Z}$. The number of *shifted double deficiencies* of π with shift c is

$$\mathrm{DD}_c(\pi) = \#\left\{j \in \{1, \ldots, n\} : 1 \le j + c \le n,\ \pi(j) < j + c < \pi^{-1}(j + c) + c\right\}.$$

Thus $\mathrm{DD}_0(\pi) = \mathrm{DD}(\pi)$.

Example. Let $\pi = 31245 \in \mathrm{Av}_5(132)$ and $c = 1$. Since $\pi^{-1} = 23145$, the shifted condition holds at precisely $j = 3, 4$:

$$\pi(3) = 2 < 3 + 1 < \pi^{-1}(4) + 1 = 5,$$
$$\pi(4) = 4 < 4 + 1 < \pi^{-1}(5) + 1 = 6.$$

Thus $\mathrm{DD}_1(\pi) = 2$, whereas $\pi(2) = 1 < 2 < \pi^{-1}(2) = 3$. Thus $\mathrm{DD}(\pi) = 1$.

Generating function. For integers $n \ge 0$ and $c \in \mathbb{Z}$, define

$$f(n, c, x) = \sum_{\pi \in \mathrm{Av}_n(132)} x^{\mathrm{DD}_c(\pi)}.$$

The ordinary generating function is $f_{132}(n, x) = f(n, 0, x)$.

If the indices of π are counted by DD_c, then those of a standardized block with offsets p, q are counted by DD_{c+p-q}; that is, the block carries shift $c + p - q$. For $\pi = L\,n\,R$, the offset differences are $-(n - i)$ for L and i for R, so the left and right blocks contribute $f(i - 1, c - (n - i), x)$ and $f(n - i, c + i, x)$, respectively.

Lemma 4. *Let $\pi = L\,n\,R \in \mathrm{Av}_n(132)$ with $\pi(i) = n$, and let $c \in \mathbb{Z}$. The two recursive factors account for every shifted double deficiency of π, except possibly the position $j = n - c$, whose shifted value is $j + c = n$. This position is a shifted double deficiency if and only if*

$$n \in \{c + 1, \ldots, c + i - 1\}.$$

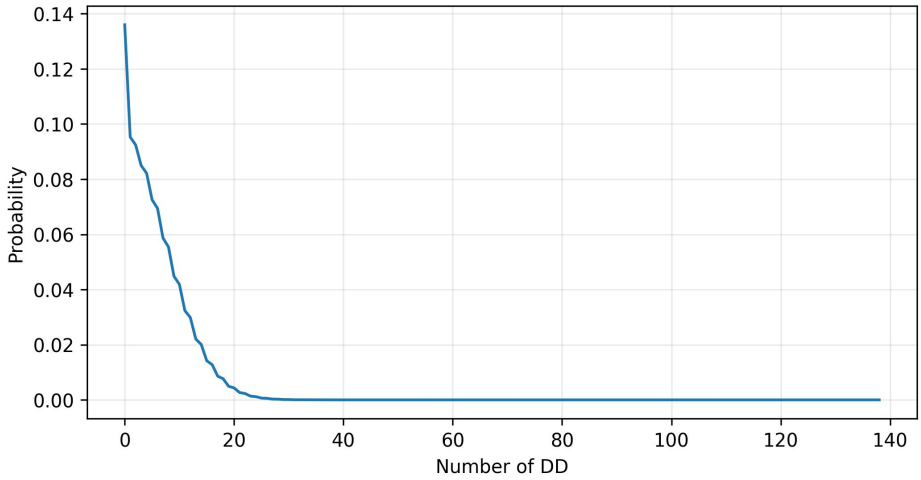

Figure 4: Probability distribution of the number of double deficiencies in $\mathrm{Av}_{140}(132)$.

Proof. The position i cannot contribute because $\pi(i) = n$. If a position lies in L while its shifted value lies in R, the first inequality fails; in the reverse situation, the shifted value occurs to the left and the second inequality fails. The only remaining shifted value is n, which forces $j = n - c$. Since $\pi^{-1}(n) = i$, its two inequalities are $\pi(n-c) < n < i + c$. They hold exactly when $n - c$ lies in L, namely when $1 \le n - c \le i - 1$: membership in L gives the first inequality, and $n - c < i$ gives the second. Equivalently, $n \in \{c + 1, \ldots, c + i - 1\}$. □

Proposition 5. *For every $n \ge 1$ and $c \in \mathbb{Z}$,*

$$f(n, c, x) = \sum_{i=1}^{n} x^{\Delta(n,c,i)} f\big(i - 1, c - (n - i), x\big) f(n - i, c + i, x),$$

where

$$\Delta(n, c, i) = \begin{cases} 1, & n \in \{c + 1, \ldots, c + i - 1\}, \\ 0, & \textit{otherwise}, \end{cases}$$

with initial condition

$$f(0, c, x) = 1.$$

In particular,

$$f_{132}(n, x) = f(n, 0, x).$$

Figure 4 shows the distribution obtained from the recurrence at $n = 140$.

The distribution is strongly asymmetric and concentrated near small values; Figure 5 shows the corresponding mean.

2.4 The class Av(213)

For this subsection, let

$$f(n, c, x) = \sum_{\pi \in \mathrm{Av}_n(213)} x^{\mathrm{DD}_c(\pi)}.$$

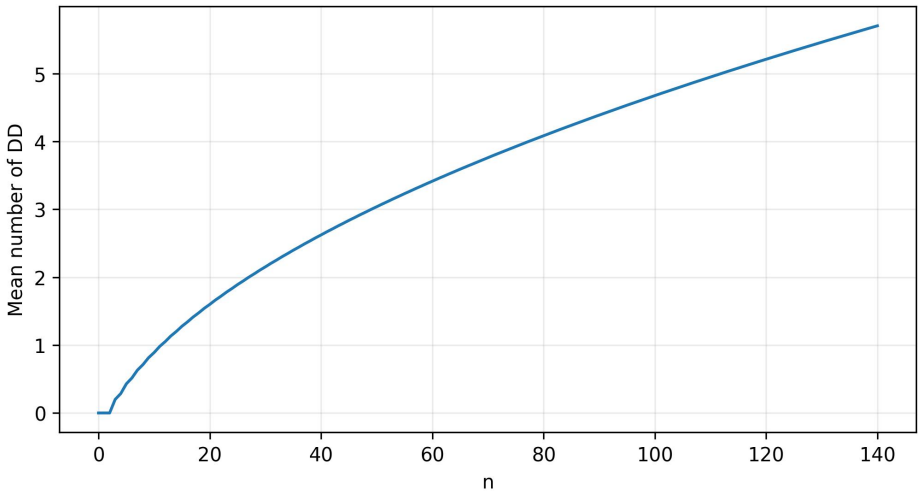

Figure 5: Mean number of double deficiencies in 132-avoiding permutations of length n, for $0 \leq n \leq 140$.

If $\pi \in \mathrm{Av}_n(213)$ has its minimum in position i, write

$$\pi = L\,1\,R.$$

Avoidance of 213 gives, as sets of values,

$$L = \{n-i+2, \ldots, n\}, \qquad R = \{2, \ldots, n-i+1\}.$$

The standardized blocks remain 213-avoiding and have shifts $c-(n-i+1)$ and $c+i-1$. Cross-block contributions fail as in the Av(132) case. The separating position i contributes exactly when its shifted value $i+c$ lies in R, namely when $i+c \in \{2, \ldots, n-i+1\}$.

Proposition 6. *For every $n \geq 1$ and $c \in \mathbb{Z}$,*

$$f(n,c,x) = \sum_{i=1}^{n} x^{\Delta(n,c,i)} f\big(i-1, c-(n-i+1), x\big) f\big(n-i, c+i-1, x\big),$$

where

$$\Delta(n,c,i) = \begin{cases} 1, & i+c \in \{2, \ldots, n-i+1\}, \\ 0, & \textit{otherwise}, \end{cases}$$

with initial condition

$$f(0,c,x) = 1.$$

In particular,

$$f_{213}(n,x) = f(n,0,x).$$

Proof. The block shifts count all contributions internal to L and R, and the cross-block argument from the Av(132) case rules out the other crossings. At the separating position, occupied by 1, the condition is

$$1 < i+c < \pi^{-1}(i+c) + c.$$

It holds exactly when $i + c \in R = \{2, \ldots, n - i + 1\}$: then $1 < i + c$ and $\pi^{-1}(i + c) > i$, giving the two strict inequalities. Thus the factor $x^{\Delta(n,c,i)}$ records the only possible extra contribution, and summing over i gives the formula. □

The DD-preserving bijection described in Section 3 also shows that

$$f_{213}(n, x) = f_{132}(n, x).$$

2.5 The class Av(231)

For $n \geq 0$ and $c \geq 0$, set

$$f(n, c, x) = \sum_{\pi \in \mathrm{Av}_n(231)} x^{\mathrm{DD}_c(\pi)}.$$

If the maximum of $\pi \in \mathrm{Av}_n(231)$ is in position i, write

$$\pi = L\, n\, R.$$

Avoidance of 231 forces every entry of L below every entry of R, and hence, as sets of values,

$$L = \{1, \ldots, i - 1\}, \qquad R = \{i, \ldots, n - 1\}.$$

The two standardized blocks contribute $f(i-1, c, x)$ and $f(n-i, c+1, x)$. Their interaction is described next.

Lemma 7. *Let $\pi = L\, n\, R \in \mathrm{Av}_n(231)$, where $\pi(i) = n$, and suppose that $0 \leq c < n$. The shifted double deficiencies not counted internally by the two recursive factors are precisely the positions j satisfying*

$$1 \leq j \leq i - 1 \qquad \textit{and} \qquad i \leq j + c \leq n.$$

Their number is

$$\Delta(n, c, i) = \min\{c,\ i - 1,\ n - c,\ n - i + 1\}.$$

Proof. No new contribution can have its position in R: if $j + c < n$ it is internal to R, while if $j + c = n$ the value occurs at position $i < j$. The position i also fails because $\pi(i) = n$. Thus only a position in L whose shifted value lies in $R \cup \{n\}$ can give a new contribution, which is equivalent to

$$1 \leq j \leq i - 1 \qquad \text{and} \qquad i \leq j + c \leq n.$$

For such j, $\pi(j) \leq i - 1 < j + c$, and the value $j + c$ occurs to the right of j; hence both inequalities hold. Therefore,

$$\Delta(n, c, i) = \#\big([1, i - 1] \cap [i - c, n - c]\big).$$

The intersection has lower endpoint $\max(1, i - c)$ and upper endpoint $\min(i - 1, n - c)$. Comparing the two possible upper endpoints with the two possible lower endpoints gives

$$\Delta(n, c, i) = \min\{c, i - 1, n - c, n - i + 1\}.$$

□

Proposition 8. *For $n \geq 1$ and $0 \leq c < n$,*

$$f(n,c,x) = \sum_{i=1}^{n} x^{\Delta(n,c,i)} f(i-1,c,x) f(n-i,c+1,x),$$

where

$$\Delta(n,c,i) = \min\{c,\ i-1,\ n-c,\ n-i+1\}.$$

The initial and boundary conditions are

$$f(0,c,x) = 1$$

and

$$f(n,c,x) = C_n, \qquad c \geq n,$$

where C_n is the nth Catalan number. In particular,

$$f_{231}(n,x) = f(n,0,x).$$

Proof. Sum the decomposition in the lemma over i. The boundary condition follows because $c \geq n$ leaves no admissible shifted value, so every one of the $|\operatorname{Av}_n(231)| = C_n$ permutations has weight 1. □

Figure 6 shows the distribution computed at $n = 120$.

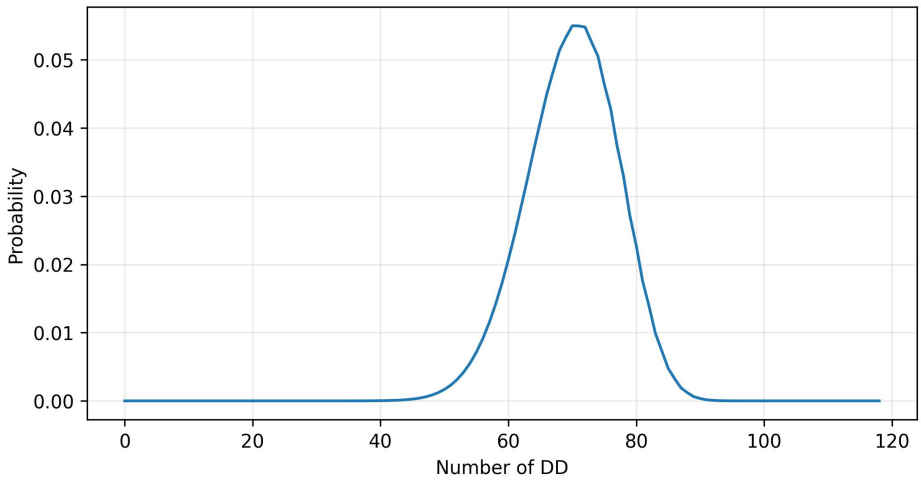

Figure 6: Probability distribution of the number of double deficiencies in $\operatorname{Av}_{120}(231)$.

Empirically, Av(231) has the largest mean among the six single-pattern classes; unlike the other recurrences, its interaction term may add several shifted deficiencies at once. Figure 7 shows this mean as a function of n.

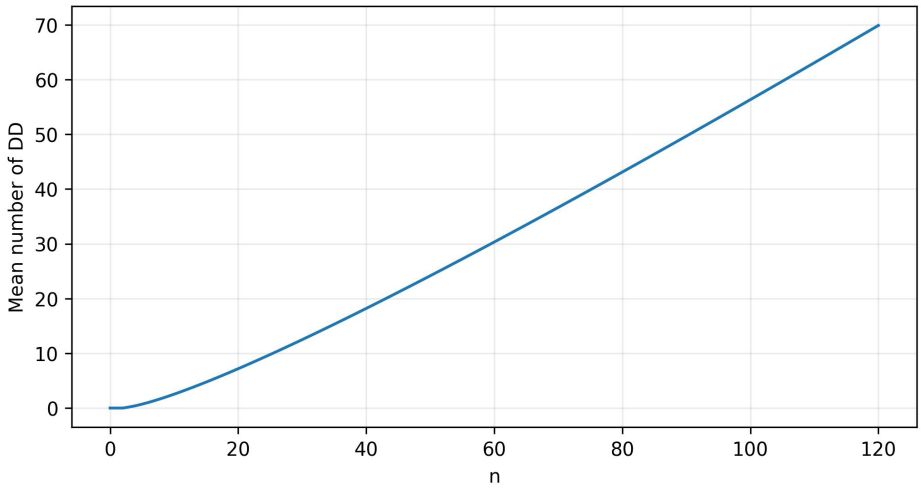

Figure 7: Mean number of double deficiencies in 231-avoiding permutations of length n, for $0 \leq n \leq 120$.

2.6 The class Av(123)

The class Av(123) does not appear to admit a decomposition analogous to those used for Av(132), Av(213), or Av(231). A path-based approach analogous in spirit to that used for Av(321) may be possible. However, the relevant indices do not align with the double-deficiency condition, and we have not obtained a closed recurrence for $f_{123}(n, x)$.

Open Problem 1. *Find an efficient method for computing the generating functions*

$$f_{123}(n, x) = \sum_{\pi \in \mathrm{Av}_n(123)} x^{\mathrm{DD}(\pi)}.$$

3 DD-Wilf equivalences

For a set of patterns S, recall that

$$f_S(n, x) = \sum_{\pi \in \mathrm{Av}_n(S)} x^{\mathrm{DD}(\pi)}$$

is the generating function for the distribution of double deficiencies over $\mathrm{Av}_n(S)$. Following the standard terminology of statistic-Wilf equivalence [4], we say that two sets of patterns S and T are *DD-Wilf equivalent* if

$$f_S(n, x) = f_T(n, x)$$

for every $n \geq 0$.

To establish several DD-Wilf equivalences, we use the following standard transformations of permutations [17]. For $\pi \in \mathfrak{S}_n$, let $R(\pi)$, $C(\pi)$, and $I(\pi)$ denote its reverse, complement, and inverse, respectively, defined by

$$R(\pi)(i) = \pi(n + 1 - i), \qquad C(\pi)(i) = n + 1 - \pi(i), \qquad I(\pi) = \pi^{-1}.$$

We consider the composition

$$\Phi = C \circ R \circ I.$$

Explicitly,

$$\Phi(\pi)(i) = n + 1 - \pi^{-1}(n + 1 - i).$$

For the proof below, let

$$\mathcal{D}(\pi) = \left\{ k \in [n] : \pi(k) < k < \pi^{-1}(k) \right\}$$

denote the set of double-deficiency indices of π. Thus, $\mathrm{DD}(\pi) = |\mathcal{D}(\pi)|$.

Proposition 9. *For every $\pi \in \mathfrak{S}_n$,*

$$\mathrm{DD}(\Phi(\pi)) = \mathrm{DD}(\pi).$$

More precisely,

$$\mathcal{D}(\Phi(\pi)) = \{\, n + 1 - k : k \in \mathcal{D}(\pi) \,\}.$$

Proof. Let $\rho = \Phi(\pi) = C(R(I(\pi)))$. Then $\rho(i) = n + 1 - \pi^{-1}(n + 1 - i)$. Since $\rho^{-1}(i) = n + 1 - \pi(n + 1 - i)$, the index i is a double deficiency of ρ precisely when

$$n + 1 - \pi^{-1}(n + 1 - i) < i < n + 1 - \pi(n + 1 - i).$$

Set $k = n + 1 - i$. The preceding inequalities are equivalent to $\pi(k) < k < \pi^{-1}(k)$. Thus,

$$i \in \mathcal{D}(\rho) \quad \Longleftrightarrow \quad n + 1 - i \in \mathcal{D}(\pi).$$

Hence, $\mathcal{D}(\Phi(\pi)) = \{\, n + 1 - k : k \in \mathcal{D}(\pi) \,\}$, and therefore $\mathrm{DD}(\Phi(\pi)) = \mathrm{DD}(\pi)$. □

We now determine how Φ acts on patterns of length 3. Let $\sigma \in \mathfrak{S}_3$ and $\pi \in \mathfrak{S}_n$. Since the reverse, complement, and inverse operations are invertible and carry pattern occurrences to pattern occurrences,

$$\pi \text{ contains } \sigma \quad \Longleftrightarrow \quad \Phi(\pi) \text{ contains } \Phi(\sigma).$$

For $S \subseteq \mathfrak{S}_3$, define

$$\Phi(S) = \{\Phi(\sigma) : \sigma \in S\}.$$

It follows that Φ restricts to a bijection

$$\Phi : \mathrm{Av}_n(S) \longrightarrow \mathrm{Av}_n(\Phi(S)).$$

On the six patterns of length 3, Φ acts as follows:

σ	123	321	132	213	231	312
$\Phi(\sigma)$	123	321	213	132	231	312

Thus Φ fixes 123, 321, 231, and 312, and interchanges 132 and 213.

Corollary 10. *For every*

$$S \subseteq \{123, 321, 231, 312\}$$

and every $n \geq 0$*,*

$$f_{S\cup\{132\}}(n, x) = f_{S\cup\{213\}}(n, x).$$

Hence, $S \cup \{132\}$ *and* $S \cup \{213\}$ *are DD-Wilf equivalent.*

Proof. Since Φ fixes every pattern in S and interchanges 132 and 213, $\Phi\big(S \cup \{132\}\big) = S \cup \{213\}$. It therefore gives a bijection $\Phi : \mathrm{Av}_n(S \cup \{132\}) \longrightarrow \mathrm{Av}_n(S \cup \{213\})$. Moreover, the preceding proposition shows that $\mathrm{DD}(\Phi(\pi)) = \mathrm{DD}(\pi)$. Thus, the bijection preserves the exponent in the DD generating function, proving $f_{S\cup\{132\}}(n, x) = f_{S\cup\{213\}}(n, x)$. □

In particular, taking $S = \varnothing$ yields $f_{132}(n, x) = f_{213}(n, x)$, so the patterns 132 and 213, whose avoidance classes were studied in Sections 2.3 and 2.4, are DD-Wilf equivalent. This grouping differs from the fixed-point classification recalled in Section 1.

In the next section, we apply the corollary to avoidance classes defined by two forbidden patterns of length 3.

4 Two-pattern avoidance

Since

$$\mathrm{Av}_n(\sigma, \tau) = \mathrm{Av}_n(\sigma) \cap \mathrm{Av}_n(\tau),$$

we refine the preceding one-pattern decompositions by imposing the second avoidance condition on their blocks.

The Erdős–Szekeres theorem [9] implies that every permutation of length at least 5 contains either an increasing or a decreasing subsequence of length 3. Consequently, $\mathrm{Av}_n(123, 321) = \varnothing, n \geq 5$. Hence,

$$f_{\{123,321\}}(n, x) = 0, \qquad n \geq 5,$$

so this class is finite.

By Proposition 2, every double deficiency implies an occurrence of the pattern 312. Hence, if $312 \in S$, then every permutation in $\mathrm{Av}_n(S)$ has no double deficiencies, and therefore

$$f_S(n, x) = |\,\mathrm{Av}_n(S)|.$$

By Section 3, replacing 132 by 213 (or vice versa) does not change the double-deficiency generating function when the other forbidden pattern is one of $123, 321, 231$, or 312. Hence, after excluding the trivial class $\mathrm{Av}(123, 321)$ and the five classes containing 312, it suffices to consider the six classes

$$\{321, 231\}, \quad \{321, 132\}, \quad \{132, 231\}, \quad \{132, 123\}, \quad \{132, 213\}, \quad \{123, 231\}.$$

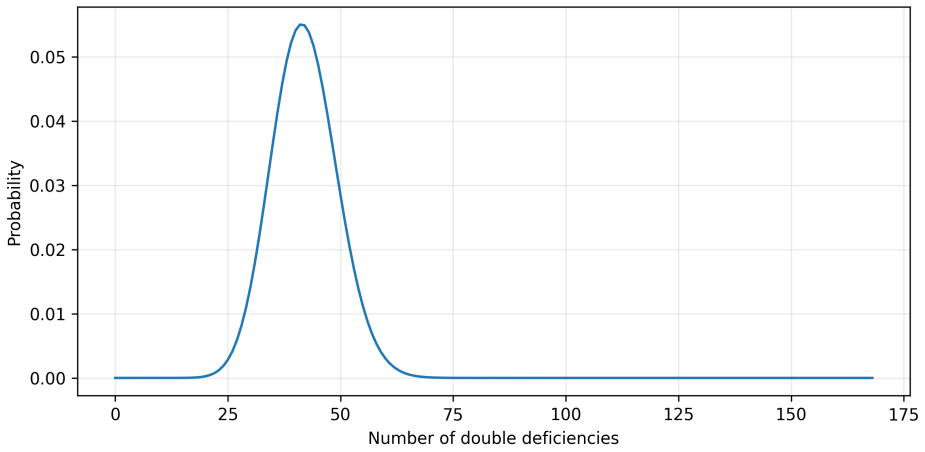

Figure 8: Probability distribution of the number of double deficiencies in $\mathrm{Av}_{170}(321, 231)$.

4.1 The class $\mathrm{Av}(321, 231)$

Let $S = \{321, 231\}$. In the 231-decomposition $\pi = L\,n\,R$, with $\pi(i) = n$, the additional avoidance of 321 forces $R = [i, i+1, \ldots, n-1]$, while $L \in \mathrm{Av}_{i-1}(S)$. Hence,

$$\pi = [L, n, i, i+1, \ldots, n-1].$$

For $i < n$, precisely $j = i+1, \ldots, n-1$ are new double deficiencies, since $\pi(j) = j-1$ and $\pi^{-1}(j) = j+1$; thus the contribution is $x^{n-i-1} f_S(i-1, x)$. For $i = n$, it is $f_S(n-1, x)$.

Summing over all possible positions of n, we obtain

$$f_S(n, x) = f_S(n-1, x) + \sum_{i=1}^{n-1} x^{n-i-1} f_S(i-1, x).$$

The corresponding C-finite recurrence is, for $n \geq 0$,

$$f_S(n+2, x) = (x+1) f_S(n+1, x) - (x-1) f_S(n, x),$$

where $f_S(0, x) = 1$, $f_S(1, x) = 1$, and $f_S(2, x) = 2$.

The mean is $(n-2)/4$ for $n \geq 2$, and Figure 8 shows the distribution at $n = 170$.

4.2 The class $\mathrm{Av}(321, 132)$

Let $S = \{321, 132\}$ and use the 132-decomposition $\pi = L\,n\,R$, where $\pi(i) = n$. If $i < n$, avoidance of 321 forces both blocks to be increasing: a descent in R, preceded by n, forms a 321-pattern, while a descent in L, followed by any entry of the nonempty block R, does the same. Thus,

$$\pi = [n-i+1, n-i+2, \ldots, n-1, n, 1, 2, \ldots, n-i].$$

For this permutation, positions before or at i fail the first inequality. In the right block, $\pi(j) = j - i$; moreover $\pi^{-1}(j) = i + j$ exactly when $j \leq n - i$. Hence, the double deficiencies

are $j = i+1, \ldots, n-i$, numbering $\max\{0, n-2i\}$. If $i = n$, deleting the final maximum leaves an arbitrary member of $\mathrm{Av}_{n-1}(S)$ and creates no new deficiency. Therefore,

$$f_S(n,x) = f_S(n-1,x) + \sum_{i=1}^{n-1} x^{\max\{0,n-2i\}},$$

with $f_S(0,x) = 1$. In particular,

$$f_S(n,1) = \binom{n}{2} + 1.$$

The corresponding C-finite recurrence is, for $n \geq 0$,

$$\begin{aligned} f_S(n+5,x) = (x+2)f_S(n+4,x) - 2xf_S(n+3,x) - 2f_S(n+2,x) \\ + (2x+1)f_S(n+1,x) - xf_S(n,x), \end{aligned}$$

where $f_S(0,x) = 1, f_S(1,x) = 1, f_S(2,x) = 2, f_S(3,x) = x+3, f_S(4,x) = x^2+x+5$ and $f_S(5,x) = x^3+x^2+2x+7$.

For the remaining four classes, we use the shift parameter $c \geq 0$ and write

$$f_S(n,c,x) = \sum_{\pi \in \mathrm{Av}_n(S)} x^{\mathrm{DD}_c(\pi)}.$$

Thus,

$$f_S(n,x) = f_S(n,0,x).$$

4.3 The class $\mathrm{Av}(132, 231)$

Let $S = \{132, 231\}$ and write $\pi = L\,n\,R$. The two avoidance conditions require every entry of L to be, respectively, larger and smaller than every entry of R. Hence, L and R cannot both be nonempty. Thus, for $n \geq 2$,

$$\pi = nR \qquad \text{or} \qquad \pi = L\,n,$$

with the nonempty block in $\mathrm{Av}_{n-1}(S)$. The first form contributes $f_S(n-1, c+1, x)$, with no extra contribution by Lemma 7 with $i = 1$. In the second, the same lemma with $i = n$ allows one additional shifted deficiency precisely when

$$1 \leq j \leq n-1 \qquad \text{and} \qquad j + c = n.$$

Thus this can occur only at $j = n - c$. Define

$$\triangle = \begin{cases} 1, & n \in \{c+1, c+2, \ldots, c+n-1\}, \\ 0, & \text{otherwise.} \end{cases}$$

When $\triangle = 1$, this candidate does satisfy both inequalities: since $j = n - c \in L$, $c \geq 1$, and $\pi^{-1}(n) = n$,

$$\pi(j) < j + c = n < \pi^{-1}(n) + c = n + c.$$

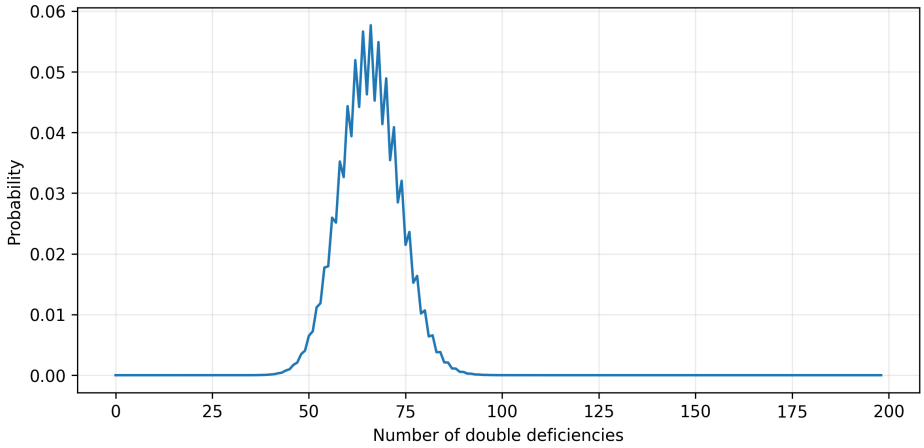

Figure 9: Probability distribution of the number of double deficiencies in $\mathrm{Av}_{200}(132, 231)$.

The two forms therefore give, for $n \geq 2$ and $n > c$,

$$f_S(n, c, x) = f_S(n-1, c+1, x) + x^{\triangle} f_S(n-1, c, x).$$

Together with $f_S(0, c, x) = f_S(1, c, x) = 1$, the boundary values are

$$f_S(n, c, x) = 2^{n-1}, \qquad 1 \leq n \leq c.$$

Here $|\mathrm{Av}_n(S)| = 2^{n-1}$, and no shifted value is admissible when $n \leq c$. In particular, $f_S(n, c, 1) = 2^{n-1}$ for $n \geq 1$.

At $c = 0$, the resulting C-finite recurrence is, for $n \geq 1$,

$$f_S(n+3, x) = (x+1) f_S(n+2, x) - (x-2) f_S(n+1, x) - 2 f_S(n, x),$$

where $f_S(0, x) = 1$, $f_S(1, x) = 1$, $f_S(2, x) = 2$ and $f_S(3, x) = x + 3$.

Figure 9 shows the distribution at $n = 200$.

4.4 The class $\mathrm{Av}(132, 123)$

Let $S = \{132, 123\}$. In the 132-decomposition $\pi = L\, n\, R$, with $\pi(i) = n$, avoidance of 123 forces

$$L = [n-1, n-2, \ldots, n-i+1].$$

Indeed, an increasing pair in L would complete a 123-pattern with n; meanwhile $R \in \mathrm{Av}_{n-i}(S)$. Thus

$$\pi = [n-1, n-2, \ldots, n-i+1, n, R].$$

The suffix contributes $f_S(n-i, c+i, x)$. The decreasing block has no internal contribution: if $j + c$ is also in L, then $\pi^{-1}(j+c) + c = \pi(j)$. As in the $\mathrm{Av}(132)$ case, the maximum adds one shifted deficiency exactly when $j = n - c$ lies in L, i.e., when $c + 1 \leq n \leq c + i - 1$. Define

$$\triangle_i = \begin{cases} 1, & n \in \{c+1, c+2, \ldots, c+i-1\}, \\ 0, & \text{otherwise.} \end{cases}$$

Then, for $n \geq 1$ and $0 \leq c < n$,

$$f_S(n, c, x) = \sum_{i=1}^{n} x^{\triangle_i} f_S(n - i, c + i, x).$$

Together with $f_S(0, c, x) = f_S(1, c, x) = 1$, the boundary values are

$$f_S(n, c, x) = |\operatorname{Av}_n(132, 123)| = 2^{n-1}, \qquad 1 \leq n \leq c.$$

Also $f_S(n, c, 1) = 2^{n-1}$ for $n \geq 1$. At $c = 0$, the corresponding C-finite recurrence is, for $n \geq 1$,

$$f_S(n + 3, x) = f_S(n + 2, x) + 4f_S(n + 1, x) - 4f_S(n, x),$$

where $f_S(0, x) = 1, f_S(1, x) = 1, f_S(2, x) = 2$ and $f_S(3, x) = x + 3$.

4.5 The class $\operatorname{Av}(132, 213)$

Let $S = \{132, 213\}$. In the 132-decomposition $\pi = L\,n\,R$, avoidance of 213 forces L to be increasing (a descent there completes a 213-pattern with n). Hence,

$$\pi = [n - i + 1, n - i + 2, \ldots, n - 1, n, R].$$

Here $R \in \operatorname{Av}_{n-i}(S)$, contributing $f_S(n - i, c + i, x)$. There are no cross-block contributions: a position in the prefix paired with a shifted value in R fails the first inequality, while a position in R paired with a shifted value in the prefix fails the second. Within the fixed prefix, $\pi(j) = n-i+j$, and for an admissible shifted value in that prefix, $\pi^{-1}(j+c) = j+c-(n-i)$. Both strict inequalities therefore hold exactly when $n - i < c$. The position i itself cannot contribute, so in this case the contributing positions are $j = 1, \ldots, \min\{i - 1, n - c\}$. Since $n - i < c$ is equivalent to $n - c \leq i - 1$, $\min\{i - 1, n - c\} = n - c$. Therefore,

$$\triangle_i = \begin{cases} n - c, & i > n - c, \\ 0, & i \leq n - c. \end{cases}$$

Thus, for $n \geq 1$ and $0 \leq c < n$,

$$f_S(n, c, x) = \sum_{i=1}^{n} x^{\triangle_i} f_S(n - i, c + i, x).$$

Together with $f_S(0, c, x) = f_S(1, c, x) = 1$, the boundary values are

$$f_S(n, c, x) = |\operatorname{Av}_n(132, 213)| = 2^{n-1}, \qquad 1 \leq n \leq c.$$

Also $f_S(n, c, 1) = 2^{n-1}$ for $n \geq 1$. At $c = 0$, the corresponding C-finite recurrence is, for $n \geq 1$,

$$f_S(n + 4, x) = (x + 1)f_S(n + 3, x) - (x - 4)f_S(n + 2, x) - 4(x + 1)f_S(n + 1, x) + 4x\, f_S(n, x),$$

where $f_S(0, x) = 1, f_S(1, x) = 1, f_S(2, x) = 2, f_S(3, x) = x + 3$ and $f_S(4, x) = x^2 + 7$.

4.6 The class $\mathrm{Av}(123, 231)$

Let $S = \{123, 231\}$ and use the 231-decomposition with maximum in position i. For $0 \leq c < n$, the case $i = 1$ contributes $f_S(n-1, c+1, x)$. If $i > 1$, avoidance of 123 forces both blocks to be decreasing: an ascent in L, together with n, gives a 123-pattern, while $1 \in L$, so an ascent in R, together with that 1, also gives a 123-pattern. Thus the permutation is

$$\pi = [i-1, i-2, \ldots, 1, n, n-1, n-2, \ldots, i].$$

The blocks have no internal shifted deficiencies: within either consecutive decreasing block, $\pi^{-1}(j+c) + c = \pi(j)$, so the two strict inequalities cannot both hold. By Lemma 7, the remaining positions form $[1, i-1] \cap [i-c, n-c]$, so the number of them is

$$\triangle_i = \min\{c, i-1, n-c, n-i+1\}.$$

Thus, for $n \geq 1$ and $0 \leq c < n$,

$$f_S(n, c, x) = f_S(n-1, c+1, x) + \sum_{i=2}^{n} x^{\min\{c, i-1, n-c, n-i+1\}}.$$

Together with $f_S(0, c, x) = f_S(1, c, x) = 1$, the boundary values are

$$f_S(n, c, x) = \binom{n}{2} + 1, \qquad n \leq c.$$

Here $|\mathrm{Av}_n(S)| = \binom{n}{2} + 1$. Also $f_S(n, c, 1) = \binom{n}{2} + 1$. At $c = 0$, the corresponding C-finite recurrence is, for $n \geq 0$,

$$\begin{aligned} f_S(n+8, x) = {} & f_S(n+7, x) + 2f_S(n+6, x) + (x-2)f_S(n+5, x) \\ & - (x+1)f_S(n+4, x) - (2x-1)f_S(n+3, x) \\ & + 2x\, f_S(n+2, x) + x\, f_S(n+1, x) - x\, f_S(n, x), \end{aligned}$$

where $f_S(0, x) = 1$, $f_S(1, x) = 1$, $f_S(2, x) = 2$, $f_S(3, x) = x + 3$, $f_S(4, x) = 2x + 5$, $f_S(5, x) = 5x + 6$, $f_S(6, x) = x^2 + 6x + 9$, $f_S(7, x) = 2x^2 + 10x + 10$ and $f_S(8, x) = 5x^2 + 10x + 14$.

5 Conclusion

The results show that the behavior of double deficiencies varies substantially across length-3 avoidance classes, even when the underlying classes have the same ordinary enumeration. For Av(321) the bivariate generating function is algebraic and the distribution admits an explicit binomial approximation; for Av(312) the statistic vanishes identically; and for Av(132), Av(213) and Av(231) the shifted double-deficiency statistic yields convolution-type recurrences that make the distributions computationally accessible. Together with the simplifications of Section 4, the DD-preserving symmetry of Section 3 reduces the two-pattern analysis to six cases, each of which satisfies a C-finite recurrence.

In the 321-avoiding case, Proposition 1 gives exact formulas for the mean and the variance and shows that $d_{\mathrm{TV}}(\mathcal{L}(X_n), \mathrm{Bin}(n-2, \frac{1}{4})) = O(n^{-1/2}(\log n)^{1/4})$. The logarithmic factor arises from the way the total-variation sum is split, and numerical evidence suggests that the true order is $n^{-1/2}$; determining the exact order, and the corresponding constant, would sharpen the result.

Several questions remain. A comparable recurrence for the double-deficiency generating functions of 123-avoiding permutations has not been obtained (Open Problem 1). The asymptotic behavior of the distributions for the remaining single-pattern classes is also open: the data suggest that the distribution for Av(231) is nearly symmetric, whereas the one for Av(132) is strongly skewed, with a mean that grows sublinearly in n rather than linearly as in Av(321) and Av(231). Finally, it would be natural to investigate double deficiencies in avoidance classes defined by patterns of length 4 and, more generally, longer patterns.

A Proof of Proposition 1

Support. Every red level step lies above the x-axis, so the path contains at least one up-step before it and at least one down-step after it; thus $X_n \le n-2$. Conversely, the path consisting of an up-step, $n-2$ level steps, and a down-step has all of its level steps above the x-axis, and coloring exactly j of them red realizes every $j \in \{0, 1, \dots, n-2\}$.

Moments. For $n = 2$, we have $X_2 = 0$, so the stated formulas hold immediately. We may therefore assume $n \ge 3$. Write $S = \sqrt{1-4z}$ and $C = C(z) = (1-S)/(2z)$, so that $C = 1 + zC^2$, $S = 1 - 2zC$, and, by Remark 1, $F_1(z,1) = C$ and $F_0(z,1) = C^2$. Taking the first and second derivatives of (2) and (3) with respect to x, evaluating at $x = 1$, and using $z^2C^2 = z(C-1)$, which gives the coefficients $1 - 2z - 2z^2C^2 = S$ and $1 - z - z^2C^2 = 1/C$, yields

$$\left.\frac{\partial F_0}{\partial x}\right|_{x=1} = \frac{zC^2}{S}, \qquad \left.\frac{\partial F_1}{\partial x}\right|_{x=1} = \frac{z^3C^4}{S}, \qquad \left.\frac{\partial^2 F_0}{\partial x^2}\right|_{x=1} = \frac{2z^2C^2}{S^2} + \frac{2z^4C^4}{S^3},$$
$$\left.\frac{\partial^2 F_1}{\partial x^2}\right|_{x=1} = \frac{2z^4C^4}{S^2} + \frac{2z^6C^6}{S^3} + \frac{2z^6C^7}{S^2}.$$

Since

$$[z^m]C^k(1-4z)^{-1/2} = \binom{2m+k}{m}$$

[13, eq. (5.72)], the formula for $\left.\partial F_1/\partial x\right|_{x=1}$ gives

$$\mathbb{E}[X_n] = \frac{1}{C_n}[z^n]\frac{z^3C^4}{S} = \frac{1}{C_n}\binom{2n-2}{n-3} = \frac{(n-1)(n-2)}{2(2n-1)}.$$

Substituting $C = (1-S)/(2z)$ in the formula for $\left.\partial^2 F_1/\partial x^2\right|_{x=1}$ and reducing every even power of S using $S^2 = 1-4z$ gives

$$\left.\frac{\partial^2 F_1}{\partial x^2}\right|_{x=1} = \frac{9 - 26z + 26z^2 - 6z^3 - z^{-1}}{(1-4z)^{3/2}} + \Big(\frac{1}{z} - 3 + 2z\Big).$$

The parenthesized part does not contribute to $[z^n]$ for $n \geq 2$. Since

$$[z^m](1-4z)^{-3/2} = (2m+1)\binom{2m}{m},$$

expressing the resulting central binomial coefficients as multiples of $\binom{2n}{n}$ yields

$$\mathbb{E}\big[(X_n)_2\big] = \frac{(n-2)(n-3)(n^2-3n+6)}{4(2n-1)(2n-3)}, \qquad (u)_2 = u(u-1).$$

The stated variance follows from

$$\mathrm{Var}(X_n) = \mathbb{E}[(X_n)_2] + \mathbb{E}[X_n] - \mathbb{E}[X_n]^2.$$

Singular expansion. Write $p_n(x) = [z^n]F_1(z,x)$, so that $p_n(x)/p_n(1)$ is the probability generating function of X_n and $p_n(1) = C_n$. The discriminant in the expression for F_1 factors as

$$1 - 2z - 2xz - 3z^2 + 2xz^2 + x^2z^2 = \big(1-(x+3)z\big)\big(1-(x-1)z\big) =: D(z,x),$$

and rationalizing gives

$$F_1(z,x) = \frac{\big(1+(x-1)z\big) - \sqrt{D(z,x)}}{2z\big(x-(x-1)z\big)}.$$

Hence, the only possible singularities of $z \mapsto F_1(z,x)$ are the branch points $(x+3)^{-1}$ and $(x-1)^{-1}$, together with the possible pole $x/(x-1)$; the apparent singularity at $z=0$ is removable. Fix $\varepsilon > 0$ sufficiently small and put

$$\mathcal{N} = \{x \in \mathbb{C} : |x-1| \leq \varepsilon\}.$$

For $x \in \mathcal{N}$, the last two possible singularities have modulus at least 1. Therefore,

$$\rho(x) = \frac{1}{x+3}$$

is the unique dominant singularity, and $F_1(\,\cdot\,,x)$ is Δ-analytic there, uniformly for $x \in \mathcal{N}$. Put $\eta = 1-(x+3)z$. The rationalized form gives

$$F_1(z,x) = \mathcal{R}(z,x) - \frac{\sqrt{1-(x-1)z}}{2z\big(x-(x-1)z\big)}\eta^{1/2},$$

where

$$\mathcal{R}(z,x) = \frac{1+(x-1)z}{2z\big(x-(x-1)z\big)}.$$

The factor multiplying $\eta^{1/2}$ is analytic at $z=\rho(x)$, uniformly for $x \in \mathcal{N}$, and its value there is

$$\frac{\sqrt{1-(x-1)\rho(x)}}{2\rho(x)\big(x-(x-1)\rho(x)\big)} = \frac{(x+3)^{3/2}}{(x+1)^2}.$$

Its Taylor expansion at $z = \rho(x)$ therefore gives

$$F_1(z,x) = \mathcal{R}(z,x) - \frac{(x+3)^{3/2}}{(x+1)^2}\eta^{1/2} + O\big(\eta^{3/2}\big),$$

uniformly for $x \in \mathcal{N}$.

Although $\mathcal{R}$ has a simple pole at $z = 0$, its principal part is $1/(2xz)$, which does not contribute to $[z^n]$ for $n \geq 0$. After subtracting this principal part, $\mathcal{R}$ is analytic in a disk whose radius is uniformly larger than $|\rho(x)|$. Hence, its coefficients of nonnegative powers are exponentially smaller than $(x+3)^n$. Applying uniform singularity analysis [10, Chapter VI and Section IX.7] to $F_1 - \mathcal{R}$ and adding back the contribution of $\mathcal{R}$ therefore gives

$$p_n(x) = \frac{(x+3)^{3/2}}{2\sqrt{\pi}\,(x+1)^2}(x+3)^n n^{-3/2}\big(1 + O(n^{-1})\big),$$

uniformly for $x \in \mathcal{N}$. At $x = 1$, this reduces to $C_n \sim \frac{4^n}{\sqrt{\pi}\,n^{3/2}}$. Dividing by $p_n(1)$, we obtain

$$\frac{p_n(x)}{p_n(1)} = A(x)B(x)^n\big(1 + O(n^{-1})\big),$$

where

$$A(x) = \frac{(x+3)^{3/2}}{2(x+1)^2}, \qquad B(x) = \frac{x+3}{4}.$$

Characteristic functions. Let

$$\phi_n(t) = \frac{p_n(e^{it})}{p_n(1)} \qquad \text{and} \qquad \psi_n(t) = B(e^{it})^{n-2}.$$

The latter is the characteristic function of $\mathrm{Bin}(n-2, \frac{1}{4})$, whose probability generating function is $B(x)^{n-2}$. Put $h = AB^2$. Then h is analytic near $x = 1$ and

$$h(1) = A(1)B(1)^2 = 1.$$

Therefore, $\phi_n(t) = \psi_n(t)\big(h(e^{it}) + O(n^{-1})\big)$ uniformly for $|t| \leq \delta$, where $\delta > 0$ is sufficiently small that $e^{it} \in \mathcal{N}$ whenever $|t| \leq \delta$. Since $|h(e^{it}) - 1| = O(|t|)$ and

$$\big|B(e^{it})\big|^2 = \frac{10 + 6\cos t}{16} = 1 - \frac{3}{4}\sin^2\frac{t}{2} \leq e^{-3t^2/(4\pi^2)},$$

we obtain

$$|\phi_n(t) - \psi_n(t)| \leq K\big(|t| + n^{-1}\big)e^{-cnt^2}, \qquad |t| \leq \delta,$$

where $K, c > 0$ are independent of n and t.

For $\delta \leq |t| \leq \pi$ and $x = e^{it}$, the three possible singularities listed above have moduli

$$|x+3|^{-1} \geq \frac{1}{4} + \gamma, \qquad |x-1|^{-1} \geq \frac{1}{2}, \qquad \left|\frac{x}{x-1}\right| \geq \frac{1}{2}$$

for some $\gamma = \gamma(\delta) > 0$. Hence, $F_1(z, e^{it})$ is analytic and uniformly bounded on $|z| \le R$ for some $R > \frac{1}{4}$. Cauchy's estimate, together with

$$p_n(1) = C_n \sim \frac{4^n}{\sqrt{\pi}\, n^{3/2}},$$

shows that $\phi_n(t)$ is uniformly exponentially small on this range. The same holds for $\psi_n(t)$, since $|B(e^{it})| < 1$ uniformly there. Parseval's identity applied to $\phi_n - \psi_n$ now gives

$$\sum_{j\in\mathbb{Z}} \big(\mathbb{P}(X_n = j) - \mathbb{P}\big(\mathrm{Bin}(n-2, \tfrac{1}{4}) = j\big)\big)^2 = O\big(n^{-3/2}\big).$$

Conclusion. For real s sufficiently close to 0, the same uniform estimate gives

$$\mathbb{E}[e^{sX_n}] \le K \exp\big(n\Lambda(s)\big), \qquad \Lambda(s) = \log B(e^s).$$

Moreover,

$$\Lambda'(0) = \frac{1}{4}, \qquad \Lambda''(0) = \frac{3}{16},$$

and hence

$$\Lambda(s) = \frac{s}{4} + \frac{3s^2}{32} + O(s^3).$$

The same bound holds for $\mathrm{Bin}(n-2, \frac{1}{4})$, whose moment generating function is exactly $e^{(n-2)\Lambda(s)}$. The Cramér–Chernoff method [2, Section 2.2], applied with $s = 2\sqrt{\frac{\log n}{n}}$ for the upper tail and its negative for the lower tail, therefore shows that both distributions put mass $O(n^{-1})$ outside

$$I_n = \left\{ j : \left| j - \frac{n}{4} \right| \le \sqrt{n \log n} \right\}.$$

Since $\#I_n = O(\sqrt{n \log n})$, splitting the total-variation sum at I_n and applying the Cauchy–Schwarz inequality inside I_n gives

$$\begin{aligned} 2\, d_{\mathrm{TV}}\Big(\mathcal{L}(X_n), \mathrm{Bin}\big(n-2, \tfrac{1}{4}\big)\Big) &\le (\#I_n)^{1/2} \left(\sum_{j\in\mathbb{Z}} \big(\mathbb{P}(X_n = j) - \mathbb{P}\big(\mathrm{Bin}(n-2, \tfrac{1}{4}) = j\big)\big)^2 \right)^{1/2} \\ &\quad + O(n^{-1}) \\ &= (\#I_n)^{1/2} O(n^{-3/4}) + O(n^{-1}) \\ &= O\big(n^{-1/2} (\log n)^{1/4}\big), \end{aligned}$$

as required. □

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