Double Deficiency on Permutations with/without Pattern Avoidance

August 6, 2026

1 Introduction

Let S_n denote the set of all permutations of length n. A *double deficiency* (DD) of a permutation $\pi \in S_n$ is an index k satisfying

$$\pi(k) < k < \pi^{-1}(k).$$

Example. Consider the permutation

$$\pi = 4251736.$$

Since

$$\pi(6) = 3 < 6 < 7 = \pi^{-1}(6),$$

the only double deficiency is $k = 6$.

For $n \geq 0$, define

$$f(n, x) = \sum_{\pi \in S_n} x^{\mathrm{DD}(\pi)} = a_0(n) + a_1(n)x + \cdots + a_n(n)x^n,$$

where $a_k(n)$ denotes the number of permutations of length n having exactly k double deficiencies.

Example. For $n = 4$,

$$f(4, x) = 17 + 6x + x^2.$$

Thus, among the 24 permutations of length 4, seventeen have no double deficiencies, six have exactly one double deficiency, and one has exactly two.

While double deficiencies have been studied in several specific settings, much less is known about their distribution within permutation classes.

In this paper, we study the distribution of double deficiencies through their generating polynomials. We consider both unrestricted permutations and the six pattern-avoiding classes of length three. Although the six pattern-avoiding classes have the same cardinality, their distributions of double deficiencies differ substantially. In the following subsections, we determine the generating polynomials for five of the six patterns in S_3 and discuss the remaining case Av(123).

2 Generating Functions for Unrestricted Permutations

We begin with unrestricted permutations by representing them as canonical cycle words. This leads to a simple recurrence for the generating polynomials, which can be computed efficiently. Pattern-avoiding classes are considered in the following section.

Canonical cycle words

We identify permutations in canonical cycle notation with words via the following bijection. Given a permutation in cycle notation, write each cycle beginning with its largest element, order the cycles according to their largest elements in increasing order, and then erase the parentheses. We denote the resulting word by

$$\Phi(\pi) = a_1 a_2 \cdots a_n.$$

For example,

$$\pi = (2)(41)(7635) \qquad \Longrightarrow \qquad \Phi(\pi) = 2417635.$$

The inverse map is determined by the left-to-right maxima of the word. Namely, one inserts a left parenthesis immediately before each left-to-right maximum and closes each cycle immediately before the next left-to-right maximum.

For this permutation,

$$\pi = (2)(41)(7635)$$

has ordinary one-line notation

$$\pi = 4251736,$$

whose unique double deficiency is the index $k = 6$. In the canonical cycle representation $\Phi(\pi) = 241\underline{763}5$, this is precisely the middle entry of the consecutive decreasing triple 763.

The following proposition transforms the global condition $\pi(k) < k < \pi^{-1}(k)$ into a local condition in the canonical cycle representation.

Proposition 1. *Let*

$$\Phi(\pi) = a_1 a_2 \cdots a_n.$$

For $2 \le i \le n-1$, the value a_i is a double deficiency of π if and only if

$$a_{i-1} > a_i > a_{i+1}.$$

Equivalently, the double deficiencies of π are precisely the middle entries of three consecutive decreasing letters in $\Phi(\pi)$.

The preceding proposition reduces the double-deficiency condition to a local condition on the canonical cycle word. We now derive a recurrence for the generating polynomials $f(n,x)$ by distinguishing whether the first comparison in the canonical cycle word is an ascent or a descent.

For $1 \le l \le n$, let $U(l,n,x)$ and $D(l,n,x)$ denote the generating polynomials for canonical cycle words of length n whose first letter is l and whose first comparison is an ascent and a descent, respectively.

Suppose a word counted by $U(l,n,x)$ begins with

$$lj, \qquad j > l.$$

Delete the first letter and standardize the remaining word by subtracting 1 from every letter greater than l. The resulting word has length $n-1$, its first letter ranges from l to $n-1$, and no new double deficiency is created. The standardized word may begin with either an ascent or a descent. Hence

$$U(l,n,x) = \sum_{i=l}^{n-1} \big(U(i,n-1,x) + D(i,n-1,x) \big). \tag{1}$$

Now suppose a word counted by $D(l,n,x)$ begins with

$$lj, \qquad j < l.$$

After deleting the first letter and standardizing, the first letter ranges from 1 to $l-1$. If the standardized word begins with an ascent, no new double deficiency is created; if it begins with a descent, the first three letters of the original word form a decreasing triple, creating one new double deficiency. Therefore,

$$D(l,n,x) = \sum_{i=1}^{l-1} \big(U(i,n-1,x) + xD(i,n-1,x) \big). \tag{2}$$

The recurrences are completed by the boundary conditions

$$U(l,n,x) = D(l,n,x) = 0, \qquad l > n,$$

together with the initial condition $U(1,1,x)=1$.

For $n \geq 2$, every canonical cycle word begins with either an ascent or a descent. Hence

$$f(n,x) = \sum_{l=1}^{n} \big(U(l,n,x) + D(l,n,x)\big) = U(1,n+1,x),$$

where the last equality follows from (1) with $l=1$.

An accelerated recurrence

Although the recurrences (1) and (2) are already efficient, each evaluation involves summing several previously computed values. These repeated summations can be eliminated by introducing cumulative generating polynomials.

Define

$$T(l,n,x) = \sum_{i=l}^{n} U(i,n,x), \qquad S(l,n,x) = \sum_{i=l}^{n} D(i,n,x).$$

Since

$$U(l,n,x) = T(l,n,x) - T(l+1,n,x),$$

and

$$D(l,n,x) = S(l,n,x) - S(l+1,n,x),$$

the recurrences (1) and (2) become

$$\begin{aligned} T(l,n,x) &= T(l+1,n,x) + T(l,n-1,x) + S(l,n-1,x), && (3)\\ S(l,n,x) &= S(l+1,n,x) + \big(T(1,n-1,x) - T(l,n-1,x)\big) \\ &\quad + x\big(S(1,n-1,x) - S(l,n-1,x)\big). && (4) \end{aligned}$$

The corresponding boundary conditions are obtained by taking cumulative sums of those for U and D. Finally,

$$f(n,x) = T(1,n,x) + S(1,n,x).$$

The cumulative formulation eliminates the repeated summations in (1) and (2), reducing each update to constant time once the previous values have been computed.

Using this accelerated recurrence, we computed the generating polynomial up to $n = 140$. Figure 1 shows that the distribution of the number of double deficiencies is remarkably close to the binomial distribution $\mathrm{Bi}(n-2, \frac{1}{6})$. This is consistent with the fact that each interior position in the canonical cycle word has probability $1/6$ of being the middle entry of three consecutive decreasing letters.

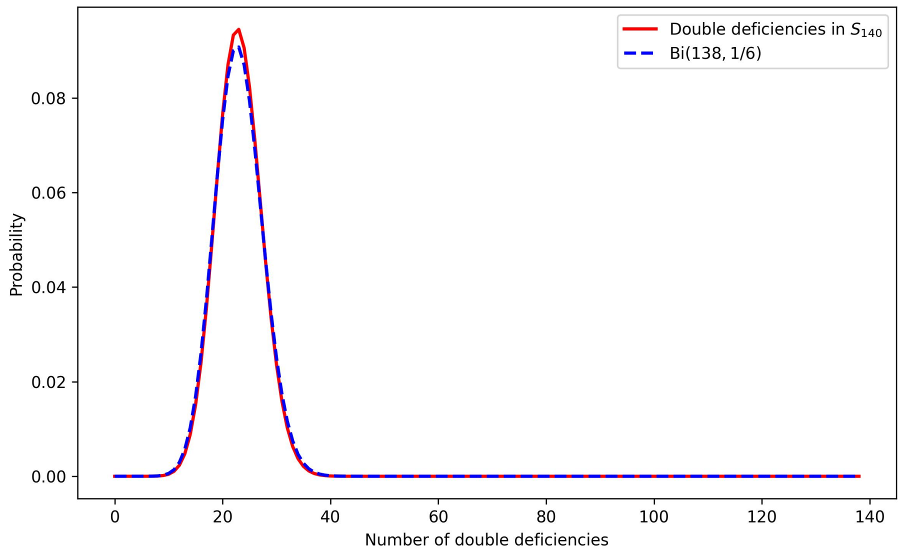


Figure 1: Probability distribution of the number of double deficiencies for unrestricted permutations of length 140, together with the binomial distribution $\mathrm{Bi}(138, \frac{1}{6})$.

Let I_i be the indicator variable that the ith letter of the canonical cycle word is the middle entry of three consecutive decreasing letters. Although $\Pr(I_i = 1) = 1/6$, the indicators are not independent. Indeed, $I_i = I_{i+1} = 1$ if and only if four consecutive letters are strictly decreasing, so

$$\Pr(I_i = 1, I_{i+1} = 1) = \frac{1}{24} > \frac{1}{36} = \Pr(I_i = 1) \Pr(I_{i+1} = 1).$$

Thus adjacent positions are positively correlated, and consequently the distribution is not exactly binomial.

3 Pattern-Avoiding Permutations of Length Three

For a pattern $\mathcal{P}$, let

$$\mathrm{Av}_n(\mathcal{P}) = \{\pi \in S_n : \pi \text{ avoids } \mathcal{P}\}.$$

For $\mathcal{P} \in S_3$, define

$$f_{\mathcal{P}}(n, x) = \sum_{\pi \in \mathrm{Av}_n(\mathcal{P})} x^{\mathrm{DD}(\pi)}.$$

Thus, for example, $f_{123}(n, x)$ and $f_{321}(n, x)$ denote the generating polynomials for the classes $\mathrm{Av}_n(123)$ and $\mathrm{Av}_n(321)$, respectively.

It is well known that, for every $\mathcal{P} \in S_3$, the class $\mathrm{Av}_n(\mathcal{P})$ is enumerated by the

Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Although the six pattern-avoiding classes have the same cardinality, their distributions of double deficiencies differ substantially. In the following subsections, we determine the generating polynomials $f_{\mathcal{P}}(n,x)$ for each of the six patterns in S_3.

3.1 The class Av(321)

The connection between double deficiencies and 321-avoiding permutations was established by Rubey and Stump [3] via the Billey–Jockusch–Stanley bijection [1] between 321-avoiding permutations and Dyck paths. More recently, Krityakierne, Thanatipanonda, and Zeilberger [2] gave a short direct proof, using the same underlying bijection, that the 321-avoiding permutations with no double deficiencies are enumerated by the Motzkin numbers.

Recall that a Motzkin path of length n is a lattice path from $(0,0)$ to $(n,0)$, consisting of up-steps $U = (1,1)$, level steps $L = (1,0)$, and down-steps $D = (1,-1)$, which never passes below the x-axis. The number of such paths is the nth Motzkin number M_n, whose first few values are

$$1,\ 1,\ 2,\ 4,\ 9,\ 21,\ 51,\ 127, \ldots.$$

To keep track of double deficiencies, we introduce two bivariate generating functions. Let $F_1(z,x)$ denote the generating function for the bicolored Motzkin paths arising from 321-avoiding permutations, in which only level steps above the x-axis have two possible colors. Let $F_0(z,x)$ denote the generating function for bicolored Motzkin paths in which every level step has two possible colors. In both cases, z marks the path length and x marks one of the two colors of a level step.

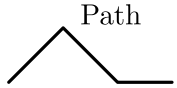
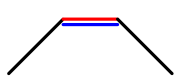
Path	Contribution to F_1	Contribution to F_0
	1	2
	1	2
	2	2
	1	$2^3 = 8$

Figure 2: The four Motzkin path shapes of length 3. In F_1, only level steps above the x-axis are colored, whereas in F_0 every level step has two possible colors.

Figure 2 illustrates the distinction between the two families of Motzkin paths. For paths of length 3, there are four underlying Motzkin path shapes. Since level steps on the x-axis are uncolored in F_1, whereas every level step is colored in F_0,

$$[z^3]F_1(z,1) = 5 = C_3, \qquad [z^3]F_0(z,1) = 14 = C_4.$$

More generally,

$$[z^n]F_1(z,1) = C_n, \qquad [z^n]F_0(z,1) = C_{n+1}.$$

Under the Foata–Zeilberger correspondence, every 321-avoiding permutation of length n is in bijection with a Motzkin path of length n counted by $F_1(z,x)$. Moreover, each double deficiency corresponds to one of the distinguished colors of a level step above the x-axis. Assigning weight z to each step and weight x to each distinguished level step, the path corresponding to π receives weight

$$z^n x^{\mathrm{DD}(\pi)}.$$

Summing these weights over all Motzkin paths, and hence over all 321-avoiding permutations, gives

$$F_1(z,x) = \sum_{n\geq 0} \sum_{\pi\in \mathrm{Av}_n(321)} x^{\mathrm{DD}(\pi)} z^n = \sum_{n\geq 0} f_{321}(n,x) z^n.$$

We first derive a functional equation for $F_0(z,x)$. Since every level step may be colored independently, F_0 admits a straightforward first-return decomposition. The relation between F_0 and F_1 will then allow us to determine F_1.

A first-return decomposition of a nonempty path counted by $F_0(z,x)$ yields

$$F_0(z,x) = 1 + (1+x)zF_0(z,x) + z^2F_0(z,x)^2,$$

since the path either begins with a level step, which may be of either color, contributing the factor $(1+x)z$, where the factor x records the distinguished color, followed by an arbitrary F_0-path, or begins with an up-step. In the latter case, the matching down-step uniquely determines the first return to the x-axis, decomposing the path into an arbitrary F_0-path enclosed between the up- and down-steps, followed by a second arbitrary F_0-path.

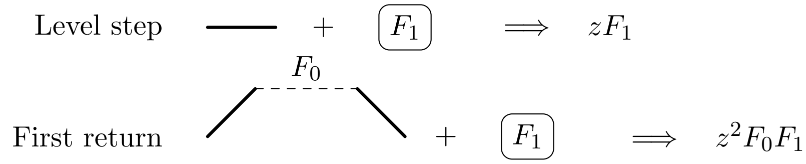


Figure 3: First-return decomposition for the generating function $F_1(z,x)$.

Figure 3 illustrates the first-return decomposition of a nonempty path counted by $F_1(z,x)$. Such a path either begins with an uncolored level step on the x-axis, followed by another F_1-path, or with an up-step followed by an excursion above the x-axis, a down-step, and another F_1-path. Since the excursion never touches the x-axis except at its endpoints, it is counted by $F_0(z,x)$. Therefore,

$$F_1(z,x) = 1 + zF_1(z,x) + z^2F_0(z,x)F_1(z,x).$$

Solving the quadratic equation for $F_0(z,x)$ gives

$$F_0(z,x) = \frac{1-(1+x)z-\sqrt{1-2z-2xz-3z^2+2xz^2+x^2z^2}}{2z^2}.$$

Substituting into the equation for $F_1(z,x)$ and rearranging yields

$$F_1(z,x) = \frac{2}{1-z+xz+\sqrt{1-2z-2xz-3z^2+2xz^2+x^2z^2}}.$$

This provides an explicit expression for the bivariate generating function of the distribution of double deficiencies over Av(321).

Coefficient recurrences

Although the closed-form expressions for $F_0(z,x)$ and $F_1(z,x)$ determine the generating polynomials, extracting coefficients directly from them becomes increasingly expensive as n grows. A more efficient approach is to equate coefficients in the functional equations.

Write

$$F_0(z,x) = \sum_{n\geq 0} g(n,x)z^n, \qquad F_1(z,x) = \sum_{n\geq 0} f_{321}(n,x)z^n,$$

where $g(n,x)$ and $f_{321}(n,x)$ denote the coefficient polynomials of F_0 and F_1, respectively.

Comparing coefficients in

$$F_0 = 1 + (1+x)zF_0 + z^2F_0^2$$

gives

$$g(0,x) = 1,$$

and, for $n \geq 1$,

$$g(n,x) = (1+x)g(n-1,x) + \sum_{i=2}^{n} g(i-2,x)\,g(n-i,x).$$

Similarly, comparing coefficients in

$$F_1 = 1 + zF_1 + z^2 F_0 F_1$$

gives

$$f_{321}(0, x) = 1,$$

and, for $n \geq 1$,

$$f_{321}(n, x) = f_{321}(n-1, x) + \sum_{i=2}^{n} g(i-2, x)\, f_{321}(n-i, x).$$

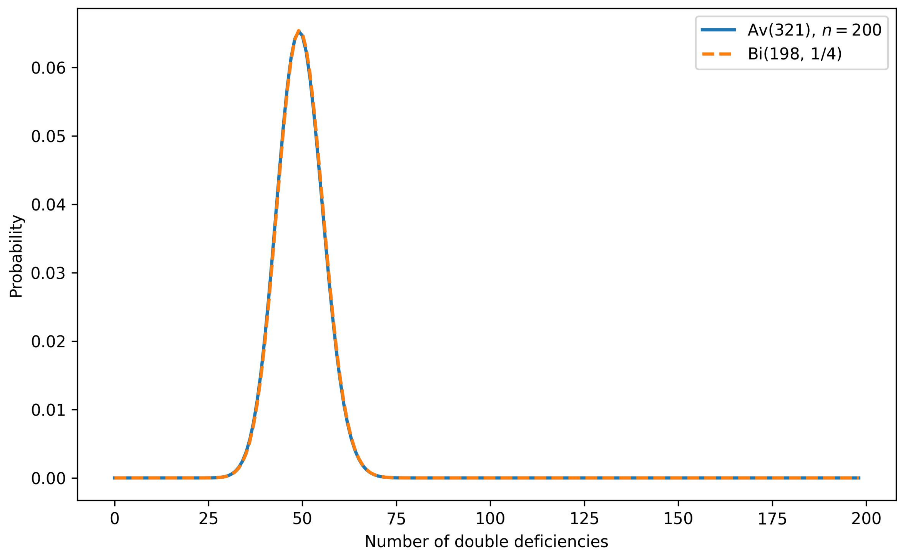


Figure 4: Probability distribution of the number of double deficiencies in $\mathrm{Av}_{200}(321)$, together with the binomial distribution $\mathrm{Bi}(198, \frac{1}{4})$.

Figure 4 suggests that the distribution of the number of double deficiencies approaches the binomial distribution $\mathrm{Bi}(n-2, \frac{1}{4})$ as n increases. The class Av(321) is distinguished by its correspondence with bicolored Motzkin paths, which yields an explicit generating function for this distribution. Whether this correspondence also explains the observed asymptotic binomial behavior remains an interesting open question.

Conjecture. Let X_n denote the number of double deficiencies in a uniformly random 321-avoiding permutation of length n. Then the distribution of X_n is asymptotically binomial with parameters $n-2$ and $\frac{1}{4}$.

Using the coefficient recurrences, the generating polynomials can be computed efficiently for moderate values of n.

A holonomic recurrence

Eliminating the auxiliary sequence $g(n,x)$ yields a faster recurrence involving only $f_{321}(n,x)$. Using standard holonomic techniques, one obtains the following third-order recurrence:

$$\begin{aligned}(n+4)x\, f_{321}(n+3,x) = {} & \Big((5x^2+9x-4)+(2x^2+3x-1)n\Big) f_{321}(n+2,x) \\ & - (x-1)\Big((x^2+8x+5)+(x^2+5x+2)n\Big) f_{321}(n+1,x) \\ & + (x-1)^2(x+3)(n+1) f_{321}(n,x),\end{aligned}$$

with initial conditions

$$f_{321}(0,x) = f_{321}(1,x) = 1, \qquad f_{321}(2,x) = 2.$$

3.2 The class $\mathrm{Av}(312)$

Proposition 2. *Every double deficiency determines an occurrence of the pattern* 312*. Consequently,*

$$\mathrm{Av}_n(312) = \{\pi \in S_n : \mathrm{DD}(\pi) = 0\}.$$

Proof. Suppose k is a double deficiency of π. Then $\pi(k) < k < \pi^{-1}(k)$. Among the first $k-1$ positions, neither the value $\pi(k)$ nor the value k can occur. Since there are $k-1$ positions but only $k-2$ remaining values from $\{1,\dots,k\}$, the pigeonhole principle implies that some position $i < k$ satisfies $\pi(i) > k$. Hence, $\pi(i) > k > \pi(k)$, so the entries $\pi(i),\ \pi(k),\ k$ form an occurrence of the pattern 312. □

Corollary 3. *For every n,*

$$f_{312}(n,x) = C_n,$$

where C_n denotes the nth Catalan number.

Remark. The converse of the lemma is false. For example, the permutation 3412 contains the pattern 312 but has no double deficiency.

3.3 The class $\mathrm{Av}(132)$

We now turn to the first nontrivial Catalan class. The standard recursive decomposition of 132-avoiding permutations is well known, but ordinary standardization does not preserve the double-deficiency statistic. To overcome this obstacle, we introduce a shifted double-deficiency statistic that is preserved under standardization and yields a closed recurrence.

Shifted double deficiencies. Let $\pi \in S_n$, and let $c \in \mathbb{Z}$. We regard the local positions $1, 2, \ldots, n$ as having shifted labels $c+1, c+2, \ldots, c+n$. Thus, local position j has shifted label $j + c$.

The ordinary double-deficiency condition is $\pi(k) < k < \pi^{-1}(k)$. Replacing the position label k by the shifted label $j + c$ yields

$$\pi(j) < j + c < \pi^{-1}(j + c) + c,$$

where $1 \le j + c \le n$, since $j + c$ must be one of the values of the permutation.

The preceding discussion motivates the following definition.

Definition. Let $\pi \in S_n$ and $c \in \mathbb{Z}$. The number of *shifted double deficiencies* of π with shift c is

$$\mathrm{DD}_c(\pi) = \# \left\{ j \in \{1, \ldots, n\} : 1 \le j + c \le n,\ \pi(j) < j + c < \pi^{-1}(j + c) + c \right\}.$$

When $c = 0$, this reduces to the ordinary double-deficiency statistic, namely

$$\mathrm{DD}_0(\pi) = \mathrm{DD}(\pi).$$

Generating function. For integers $n \ge 0$ and $c \in \mathbb{Z}$, define

$$f(n, c, x) = \sum_{\pi \in \mathrm{Av}_n(132)} x^{\mathrm{DD}_c(\pi)}.$$

Thus $f(n, c, x)$ is the generating polynomial for shifted double deficiencies over 132-avoiding permutations of length n with shift c. The original generating polynomial is recovered by taking $c = 0$:

$$f_{132}(n, x) = f(n, 0, x).$$

Although the original problem corresponds to the case $c = 0$, recursive standardization naturally produces subproblems with nonzero shifts. The parameter c therefore allows all recursive subproblems to be described by the same generating function.

We now derive a recurrence for $f(n, c, x)$.

Recursive decomposition. Let $\pi \in \mathrm{Av}_n(132)$, and suppose that its maximum entry occurs in position i. Write

$$\pi = L\, n\, R,$$

where L and R denote the entries to the left and right of the maximum value.

Since π avoids 132, every entry of L is larger than every entry of R. Indeed, if there were $a \in L$ and $b \in R$ with $a < b$, then the entries a, n, b would form an occurrence of the pattern 132, a contradiction. Consequently,

$$L = \{n - i + 1, \ldots, n - 1\}, \qquad R = \{1, \ldots, n - i\},$$

as sets of values. After standardization, both blocks are again 132-avoiding permutations.

The left block has length $i - 1$. Standardization subtracts $n - i$ from every value. To preserve the shifted double-deficiency condition, the shift parameter must decrease by the same amount. Hence the left block contributes

$$f(i - 1, c - (n - i), x).$$

The right block has length $n - i$ and already uses the values

$$1, \ldots, n - i,$$

so no value standardization is required. Since it begins immediately after the first i positions, its shift becomes $c + i$. Hence the right block contributes

$$f(n - i, c + i, x).$$

It remains to determine how the shifted double deficiencies of π are distributed between the two recursive blocks and the maximum value. The following lemma provides the answer.

Lemma 4. *Every shifted double deficiency of $\pi = L\,n\,R$ is counted internally by one of the two recursive factors, except for one additional shifted double deficiency corresponding to the maximum value n. This additional shifted double deficiency occurs if and only if*

$$n \in \{c + 1, \ldots, c + i - 1\}.$$

Proof. Suppose first that the shifted label $j + c$ is a value of the right block. If j lies in the left block, then every entry of the left block is larger than every value of the right block, so $\pi(j) > j + c$. Hence the first inequality in

$$\pi(j) < j + c < \pi^{-1}(j + c) + c$$

fails. Therefore any shifted double deficiency whose shifted label is a right-block value is contained entirely in the right block.

Similarly, suppose that $j + c$ is a value of the left block. If j lies in the right block, then the value $j + c$ occurs in the left block, so $\pi^{-1}(j + c) < j$. Adding c gives

$\pi^{-1}(j+c)+c < j+c$, and the second inequality in the shifted double-deficiency condition fails. Therefore any shifted double deficiency whose shifted label is a left-block value is contained entirely in the left block.

The only value not belonging to either block is the maximum value n. The shifted label n occurs at the local position $j = n-c$. It is a shifted double deficiency precisely when $\pi(n-c) < n < \pi^{-1}(n)+c$. Since $\pi^{-1}(n) = i$, the shifted double-deficiency condition becomes $\pi(n-c) < n < i+c$. The second inequality is equivalent to $n-c \leq i-1$. Hence $n-c \neq i$. Since the value n occurs only at position i, it follows that $\pi(n-c) \neq n$. As n is the maximum value of the permutation, $\pi(n-c) < n$. Therefore n is a shifted double deficiency precisely when $1 \leq n-c \leq i-1$, that is, when the shifted label n lies in the left block. Equivalently,

$$n \in \{c+1, \dots, c+i-1\}.$$

□

The recurrence may now be stated as follows.

Proposition 5. *For every $n \geq 1$ and $c \in \mathbb{Z}$,*

$$f(n,c,x) = \sum_{i=1}^{n} x^{\Delta(n,c,i)} f\big(i-1, c-(n-i), x\big) f(n-i, c+i, x),$$

where

$$\Delta(n,c,i) = \begin{cases} 1, & n \in \{c+1, \dots, c+i-1\}. \\ 0, & \textit{otherwise}, \end{cases}$$

with initial condition

$$f(0,c,x) = 1.$$

In particular,

$$f_{132}(n,x) = f(n,0,x).$$

Using the above recurrence, we compute the probability generating function for $n = 140$. The resulting probability distribution is shown in Figure 5.

Unlike the distributions for the other pattern-avoiding classes, the distribution for Av(132) is noticeably asymmetric, with most of its mass concentrated near smaller numbers of double deficiencies. To better illustrate its behavior, Figure 6 plots the mean number of double deficiencies as a function of n.

3.4 The class Av(213)

We use the shifted double-deficiency statistic introduced for 132-avoidance.

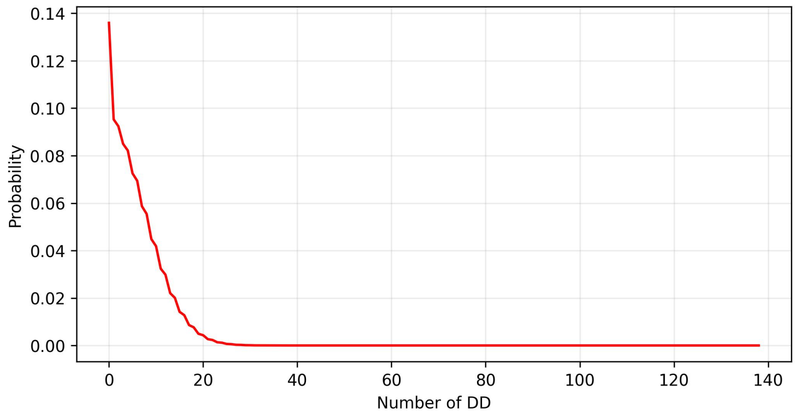


Figure 5: Probability distribution of $f(140, 0, x)$.

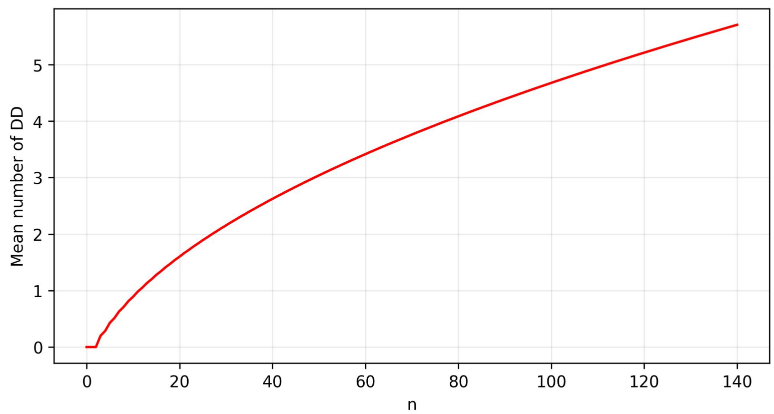


Figure 6: Mean number of double deficiencies in 132-avoiding permutations of length n, for $0 \leq n \leq 140$.

Recursive decomposition. Let $\pi \in \mathrm{Av}_n(213)$, and suppose that the minimum value 1 occurs in position i. Write

$$\pi = L\,1\,R.$$

Since π avoids 213, every entry of L is larger than every entry of R. Hence

$$L = \{n - i + 2, \dots, n\}, \qquad R = \{2, \dots, n - i + 1\}.$$

The left block has length $i - 1$. Standardizing it subtracts $n - i + 1$ from every entry, so its shift changes from c to $c - (n - i + 1)$. Thus the left recursive factor is

$$f(i - 1, c - (n - i + 1), x).$$

The right block has length $n - i$. Standardizing it subtracts 1 from every value, while its local positions begin after the first i positions. Hence its shift becomes $c + i - 1$, and the right recursive factor is

$$f(n-i, c+i-1, x).$$

As in the proof for 132-avoidance, every shifted double deficiency is counted by one of the two recursive factors, except possibly one involving the minimum value 1. The shifted label of 1 is $i+c$. Therefore an additional shifted double deficiency is created precisely when this shifted label belongs to the right block R.

Define

$$\Delta(n,c,i) = \begin{cases} 1, & i+c \in R, \\ 0, & \text{otherwise.} \end{cases}$$

This gives the following recurrence.

Proposition 6. *For every $n \geq 1$,*

$$f(n,c,x) = \sum_{i=1}^{n} x^{\Delta(n,c,i)} f\big(i-1, c-(n-i+1), x\big) f\big(n-i, c+i-1, x\big),$$

where

$$\Delta(n,c,i) = \begin{cases} 1, & i+c \in \{2, \ldots, n-i+1\}, \\ 0, & \textit{otherwise}, \end{cases}$$

with initial condition

$$f(0,c,x) = 1.$$

In particular,

$$f_{213}(n,x) = f(n,0,x).$$

Proof. The proof is identical to that of Proposition 2.2. The only difference is that the maximum value n is replaced by the minimum value 1, whose shifted label is $i+c$. The two recursive factors account for all shifted double deficiencies internal to the two blocks, while the factor $x^{\Delta(n,c,i)}$ records the possible additional shifted double deficiency created by the minimum value. □

3.5 The class Av(231)

We again use the shifted double-deficiency statistic. For $n \geq 0$ and $c \geq 0$, define

$$f(n,c,x) = \sum_{\pi \in \mathrm{Av}_n(231)} x^{\mathrm{DD}_c(\pi)}.$$

Thus

$$f_{231}(n,x) = f(n,0,x).$$

Recursive decomposition. Let $\pi \in \mathrm{Av}_n(231)$, and suppose that its maximum entry n occurs in position i. Write

$$\pi = L\,n\,R.$$

Since π avoids 231, every entry of L is smaller than every entry of R. Indeed, if $a \in L$ and $b \in R$ satisfied $a > b$, then a, n, b would form an occurrence of 231. Hence, as sets of values,

$$L = \{1, \ldots, i-1\}, \qquad R = \{i, \ldots, n-1\}.$$

The left block already uses the values $\{1, \ldots, i-1\}$, so no standardization of its values is required. Its shift remains c, and its contribution is

$$f(i-1, c, x).$$

The right block has length $n-i$ and uses the values $i, \ldots, n-1$. Standardizing this block subtracts $i-1$ from every value. A local position r in the standardized right block corresponds to position $i+r$ in π, whose shifted label is $i+r+c$. After subtracting $i-1$, this becomes $i+r+c-(i-1) = r+c+1$. Thus the standardized right block has shift $c+1$, and its contribution is

$$f(n-i, c+1, x).$$

It remains to count the shifted double deficiencies arising from the interaction between the two blocks.

Lemma 7. *Let*

$$\pi = L\,n\,R \in \mathrm{Av}_n(231), \qquad \pi(i) = n,$$

and suppose that $0 \le c < n$. *The shifted double deficiencies not counted internally by the two recursive factors are precisely the local positions* j *satisfying*

$$1 \le j \le i-1 \qquad \textit{and} \qquad i \le j+c \le n.$$

Their number is

$$\Delta(n, c, i) = \min\{c,\ i-1,\ n-c,\ n-i+1\}.$$

Proof. The shifted double-deficiency condition at local position j is $\pi(j) < j+c < \pi^{-1}(j+c)+c$. By construction of the shifted statistic, standardization preserves every shifted double deficiency whose position j and tested value $j+c$ belong to the same block. These are therefore counted by the corresponding recursive factor. It remains only to determine the additional shifted double deficiencies created by interactions between the two blocks.

Since $c \geq 0$, a position in the right block cannot test a value in the left block. Indeed, if $j > i$, then $j+c \geq j > i$, whereas every value of L is at most $i-1$. Hence $j+c \notin L$.

Now suppose $j = i$, so that j is the position of the maximum value n. If $i+c > n$, then the tested value $i+c$ is not a value of the permutation. Otherwise, $\pi(i) = n \geq i+c$, so the inequality $\pi(i) < i+c$ fails. Hence $j = i$ cannot be a shifted double deficiency.

Since a position in the right block cannot test a value in the left block, and the position $j = i$ cannot be a shifted double deficiency, the only remaining possibility is that the position j lies in the left block while the tested value $j + c$ lies in $R \cup \{n\}$. That is,

$$1 \leq j \leq i-1 \qquad \text{and} \qquad i \leq j+c \leq n.$$

Whenever these inequalities hold, $\pi(j) \leq i - 1 < j + c$. Moreover, the value $j + c$ occurs in $R \cup \{n\}$, hence strictly to the right of j. Thus $j < \pi^{-1}(j+c)$, and therefore

$$j + c < \pi^{-1}(j + c) + c.$$

Consequently, $\Delta(n, c, i) = \#\big([1, i-1] \cap [i-c, n-c]\big)$. Let $a = i-1$, and $b = n-i+1$, so that $a+b = n$. Then, $\Delta(n, c, i) = \#\big([1, a] \cap [a+1-c, n-c]\big)$, the overlap is itself an interval, $\big[\max(1, a+1-c),\ \min(a, n-c)\big]$. Hence,

$$\Delta(n, c, i) = \min(a, n-c) - \max(1, a+1-c) + 1.$$

To simplify this expression, observe that there are only two possibilities for each endpoint.

For the left endpoint,

$$\max(1, a+1-c) = \begin{cases} a+1-c, & c \leq a, \\ 1, & c > a. \end{cases}$$

For the right endpoint,

$$\min(a, n-c) = \begin{cases} a, & c \leq b, \\ n-c, & c > b, \end{cases}$$

where

$$b = n - a = n - i + 1.$$

Combining these two possibilities yields four cases:

$$\Delta(n, c, i) = \begin{cases} a - (a+1-c) + 1 = c, & c \leq a,\ c \leq b, \\ (n-c) - (a+1-c) + 1 = b, & c \leq a,\ c > b, \\ a - 1 + 1 = a, & c > a,\ c \leq b, \\ (n-c) - 1 + 1 = n - c, & c > a,\ c > b. \end{cases}$$

Therefore, $\Delta(n,c,i) = \min\{c, a, b, n-c\}$. Substituting $a = i-1$, and $b = n-i+1$ gives $\Delta(n,c,i) = \min\{c,\ i-1,\ n-c,\ n-i+1\}$.

□

The recurrence may now be stated as follows.

Proposition 8. *For $n \geq 1$ and $0 \leq c < n$,*

$$f(n,c,x) = \sum_{i=1}^{n} x^{\Delta(n,c,i)} f(i-1,c,x) f(n-i,c+1,x),$$

where

$$\Delta(n,c,i) = \min\{c,\ i-1,\ n-c,\ n-i+1\}.$$

The initial and boundary conditions are

$$f(0,c,x) = 1$$

and

$$f(n,c,x) = C_n, \qquad c \geq n,$$

where C_n is the nth Catalan number. In particular,

$$f_{231}(n,x) = f(n,0,x).$$

Proof. For $0 \leq c < n$, the recurrence is an immediate consequence of the preceding lemma. If $c \geq n$, then $j + c > n$ for every local position $j \geq 1$, so no shifted double deficiency is possible. Hence, every permutation contributes weight 1, giving

$$f(n,c,x) = |\operatorname{Av}_n(231)| = C_n.$$

□

Using the above recurrence, we compute the probability generating function for $n = 120$. The resulting probability distribution is shown in Figure 7.

Empirically, Av(231) has the largest average number of double deficiencies among the six pattern-avoiding classes of length three. This appears to be related to the interaction term $\Delta(n,c,i)$, which may contribute several new shifted double deficiencies when the two recursive blocks are joined.

Figure 8 shows the mean number of double deficiencies as a function of n.

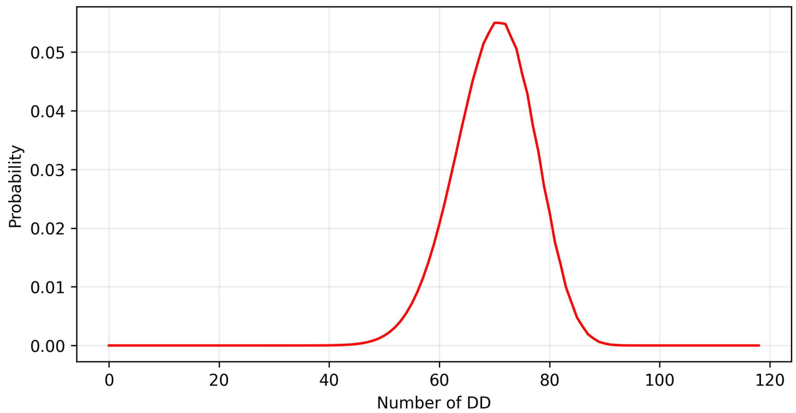


Figure 7: Probability distribution of $f(120, 0)$.

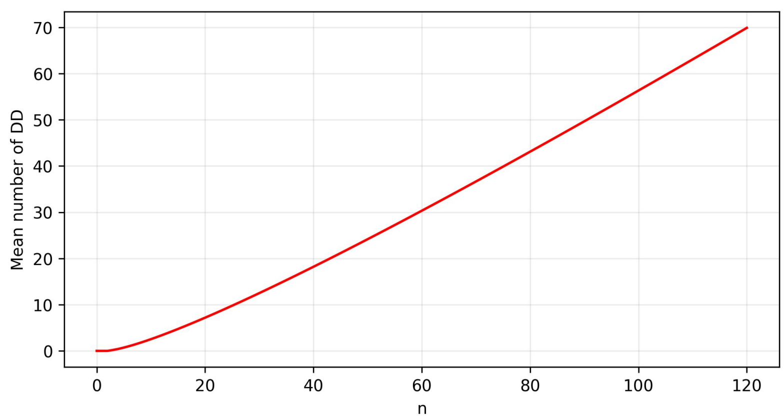


Figure 8: Mean number of double deficiencies in 231-avoiding permutations of length n, for $0 \leq n \leq 120$.

3.6 The class Av(123)

The class Av(123) does not appear to admit a decomposition analogous to those used for Av(132), Av(213), or Av(231). In particular, the interaction of double deficiencies across the recursive blocks is more difficult to describe, and we have not obtained a closed recurrence for $f_{123}(n, x)$.

Determining the generating polynomials

$$f_{123}(n, x) = \sum_{\pi \in \mathrm{Av}_n(123)} x^{\mathrm{DD}(\pi)}$$

therefore remains an open problem.

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