A Proof of the Celebrated Goldbach's Theorem

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Theorem (Goldbach [3], see also [1][2][4])

$$\sum_{m,n \geq 2}{}' \frac{1}{m^n - 1} = 1 \quad ,$$

where $\sum'$ means that every term only occurs once (for example $1/15 = 1/(16-1)$ is only added once even though $16 = 4^2 = 2^4$).

Proof: Let R denote the set of all integers larger than 1 that are *not* perfect powers: $R = \{2, 3, 5, 6, 7, 10, 11, 12, 13, 14, 15, \ldots\}$. Since every perfect power can be written uniquely as r^s ($r \in R$, $s \geq 2$) and every integer ≥ 2 can be written uniquely as r^s ($r \in R$, $s \geq 1$), we have

$$\sum_{m,n \geq 2}{}' \frac{1}{m^n - 1} = \sum_{r \in R} \sum_{s=2}^{\infty} \frac{1}{r^s - 1} = \sum_{r \in R} \sum_{s=2}^{\infty} \sum_{i=1}^{\infty} \frac{1}{r^{si}} = \sum_{r \in R} \sum_{i=1}^{\infty} \sum_{s=2}^{\infty} (\frac{1}{r^i})^s =$$

$$\sum_{r \in R} \sum_{i=1}^{\infty} \frac{(\frac{1}{r^i})^2}{1 - \frac{1}{r^i}} = \sum_{r \in R} \sum_{i=1}^{\infty} \frac{1}{r^i(r^i - 1)} = \sum_{m=2}^{\infty} \frac{1}{m(m-1)} = \sum_{m=2}^{\infty} \left(\frac{1}{m-1} - \frac{1}{m} \right) = 1 \quad . \quad \square$$

References

1. E. Catalan, *Note sur la sommation de quelques séries*, Journal de Mathématiques Pures et Appliquées **7** (1842), 1-12.

2. G. Chrystal, "*Algebra* ", Part II, reprinted by Chelsea, N.Y. 1964, [p. 422].

3. C. Goldbach, *Letter to L. Euler*, 1737.

4. R. L. Graham, O. Patashnik and D.E. Knuth, "*Concrete Mathematics*", Addison Wesley, 1989, [p. 66].

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