RUTGERS EXPERIMENTAL MATHEMATICS SEMINAR

December 14, 2023

Nontrivial zeros of the Riemann zeta function know a lot

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Plan of the talk

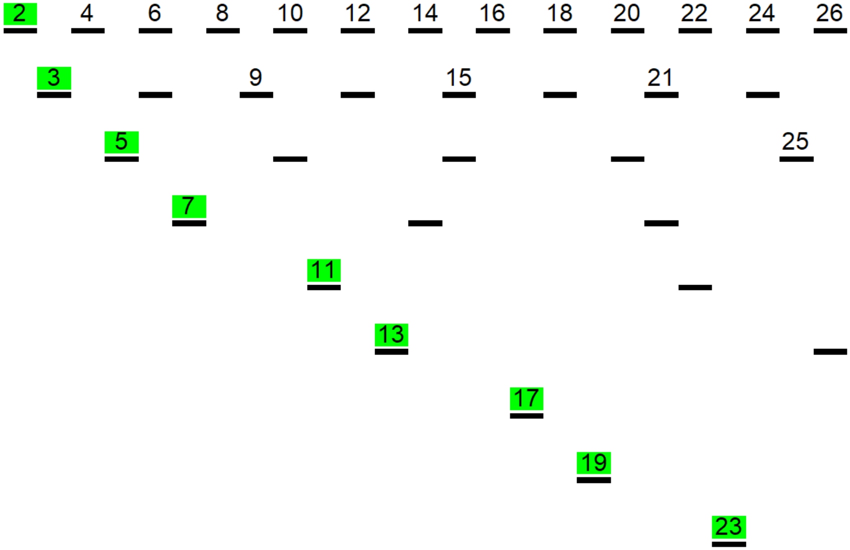
Part I. Some facts from Number Theory

Part II. Numerical experiments and discoveries

Part I

Some facts from Number Theory

Sieve of Eratosthenes (276–194 B. C.)



Georg Friedrich Bernhard Riemann (1826–1866)

Riemann's zeta function:

$\zeta(s) = 1^{-s} + 2^{-s} + \ldots + n^{-s} + \ldots$

The series converges in the half-plane $\mathrm{Re}(s) > 1$ and defines a function that can be analytically extended to the entire complex plane except for the point $s = 1$, its only (and simple) pole.

Leonhard Euler (1707–1783)

Theorem (Euler Identity)

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p \text{ is prime}} \frac{1}{1-p^{-s}}$$

Euler identity

Theorem (Euler). $\sum_{n=1}^{\infty} n^{-s} = \zeta(s) = \prod_{p \text{ is prime}} (1 - p^{-s})^{-1}$

Proof (Eratosthenes).

$$\begin{aligned}
\zeta(s) &= 1^{-s} + 2^{-s} + 3^{-s} + 4^{-s} + 5^{-s} + 6^{-s} + 7^{-s} + 8^{-s} + 9^{-s} + \ldots \\
\zeta(s) \times -2^{-s} &= \quad - 2^{-s} \quad - 4^{-s} \quad - 6^{-s} \quad - 8^{-s} \quad - \ldots \\
\zeta(s)(1 - 2^{-s}) &= 1^{-s} \quad + 3^{-s} \quad + 5^{-s} \quad + 7^{-s} \quad + 9^{-s} + \ldots
\end{aligned}$$

$$\begin{aligned}
\zeta(s)(1 - 2^{-s}) &= 1^{-s} + 3^{-s} + 5^{-s} + 7^{-s} + 9^{-s} + 11^{-s} + 13^{-s} + 15^{-s} + . \\
\zeta(s)(1 - 2^{-s}) \times -3^{-s} &= \quad - 3^{-s} + \quad - 9^{-s} \quad - 15^{-s} + . \\
\zeta(s)(1 - 2^{-s})(1 - 3^{-s}) &= 1^{-s} \quad + 5^{-s} + 7^{-s} \quad + 11^{-s} + 13^{-s} + \ldots
\end{aligned}$$

....

$\zeta(s) \prod_{p \text{ is prime}} (1 - p^{-s}) = 1^{-s} = 1$

The infinitude of prime numbers

Euler identity

$$1^{-s} + 2^{-s} + \cdots + n^{-s} + \ldots \quad = \quad \prod_{p \text{ is prime}} \frac{1}{1 - p^{-s}}$$

Theorem (Euclid). *There are infinitely many prime numbers.*

Proof (Euler). If the number of primes would be finite, then the (divergent) harmonic series would have finite value:

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots = \prod_{p \text{ is prime}} \frac{1}{1 - \frac{1}{p}}$$

Zeros of the zeta function

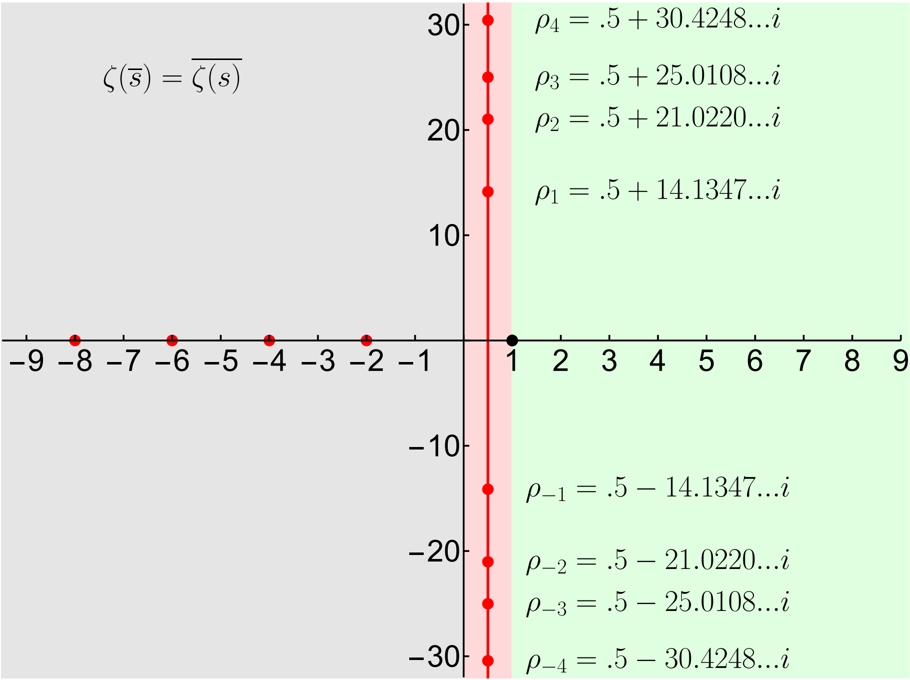
Euler: $0 = \zeta(-2) = \zeta(-4) = \cdots = \zeta(-2m) = \ldots$

Negative even integer called the trivial zeros of the zeta function

Riemann (1859): *All other (called non-trivial) zeros of the zeta function are non-real and are located inside the critical strip*

$$0 \leq \mathrm{Re}(s) \leq 1$$

The Riemann Hypothesis: *All non-trivial zeros of the zeta function are located on the critical line* $\mathrm{Re}(s) = 1/2$.



$\zeta(\overline{s}) = \overline{\zeta(s)}$
$\rho_4 = .5 + 30.4248...i$
$\rho_3 = .5 + 25.0108...i$
$\rho_2 = .5 + 21.0220...i$
$\rho_1 = .5 + 14.1347...i$
$\rho_{-1} = .5 - 14.1347...i$
$\rho_{-2} = .5 - 21.0220...i$
$\rho_{-3} = .5 - 25.0108...i$
$\rho_{-4} = .5 - 30.4248...i$
30
20
10
−10
−20
−30
−9 −8 −7 −6 −5 −4 −3 −2 −1 1 2 3 4 5 6 7 8 9

Leonhard Euler

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

Alternating zeta function:

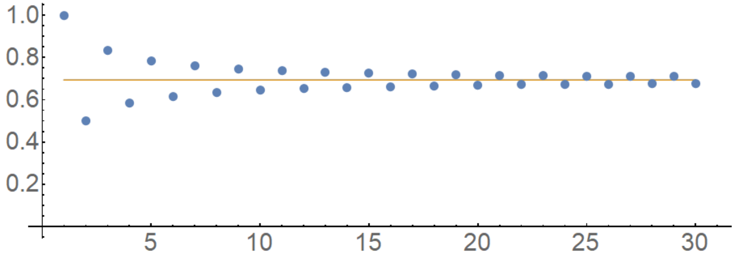
$$\eta(s) = \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s} \qquad (*)$$
$$= (1 - 2 \times 2^{-s})\zeta(s)$$

1) Series $(*)$ converges in the larger half-plane $\mathrm{Re}(s) > 0$
2) $\eta(s)$ is an entire function

Sample calculations

$$\eta(s) = \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s} \qquad \eta(1) = \ln(2) = 0.693147180...$$

$$\eta_N(s) = \sum_{n=1}^{N} (-1)^{n+1} n^{-s} \qquad \eta_{1000}(1) = 0.69264...$$



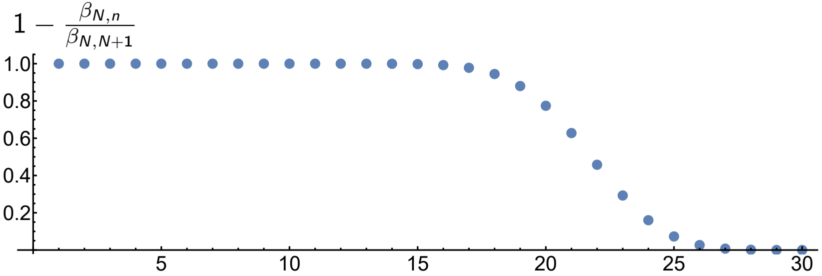
$$\eta_N(s) = \sum_{n=1}^{N-1} (-1)^{n+1} n^{-s} + \frac{1}{2}(-1)^{N+1} N^{-s} \qquad \eta_{1000}(1) = 0.693147430...$$

Approximations proposed by Peter Borwein (1953 – 2020)

$$\eta(s) = \sum_{n=1}^{\infty} (-1)^{n+1} n^s \qquad \eta_N(s) = \sum_{n=1}^{N} \alpha_{N,n} n^{-s}$$

$$\alpha_{N,n} = (-1)^{n+1}\left(1 - \frac{\beta_{N,n}}{\beta_{N,N+1}}\right) \qquad \beta_{N,n} = N \sum_{i=1}^{n} \frac{4^{i-1}(N+i-2)!}{(N-i+1)!(2i-2)!}$$

$$\begin{aligned} \eta_{30}(1) &= 0.69314718055994531125\ldots \\ \eta(1) = \ln(2) &= 0.69314718055994530941 7\ldots \end{aligned}$$



Part II

Experiments and discoveries

Our approximation by a Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \qquad (*)$$

$$\Omega_N(s) = \sum_{n=1}^{N} \delta_{N,n} n^{-s} \qquad (**)$$

Let us define numbers $\delta_{N,n}$ by the following conditions:

- the finite sum $(**)$ has $N-1$ common zeros with the infinite sum $(*)$
- $\delta_{N,1} = 1$

Formal definition of our approximation $\Omega_N(s)$

$$N = 2M + 1$$

$$\Omega_N(s) = 1 + \sum_{n=2}^{N} \delta_{N,n} n^{-s}$$

$\delta_{N,n}$ are defined by condition

$$\Omega_N(\overline{\rho_M}) = \cdots = \Omega_N(\overline{\rho_1}) = 0 = \Omega_N(\rho_1) = \cdots = \eta_N(\rho_M) \qquad (*)$$

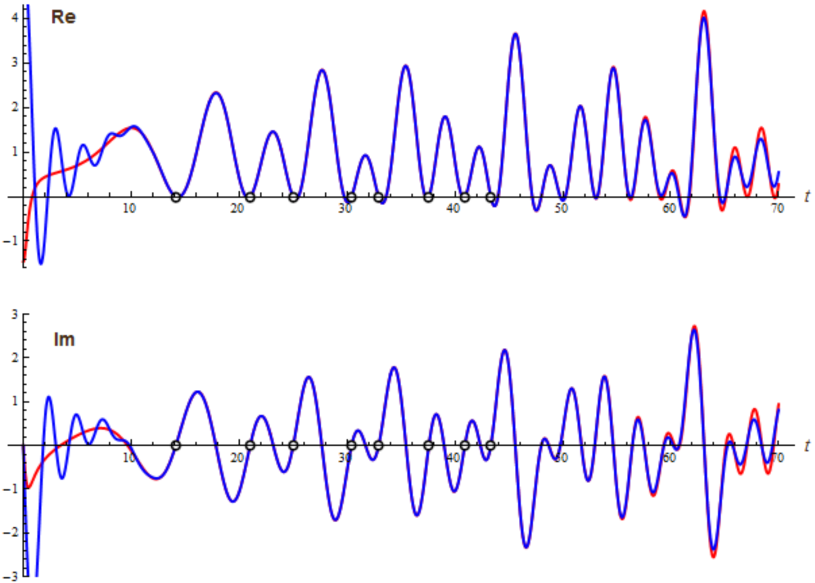
where $\rho_1, \ldots, \rho_M$ are the initial zeroes of the zeta function in the upper half-plane:

$$\zeta(\rho_1) = \cdots = \zeta(\rho_M) = 0$$

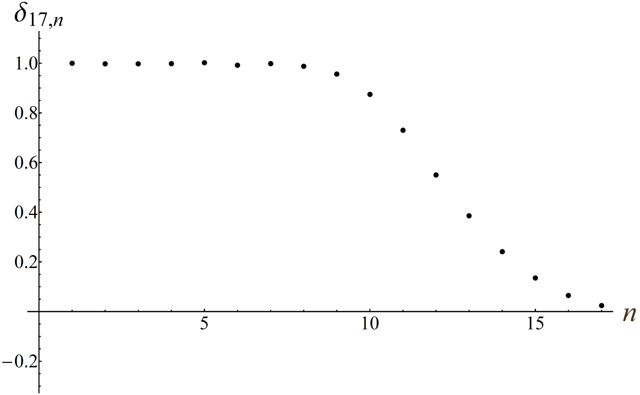
$$0 < \mathrm{Im}(\rho_1) \leq \cdots \leq \mathrm{Im}(\rho_M)$$

In order to find $\delta_{N,2}, \ldots, \delta_{N,N}$ we should solve linear system $(*)$; $\delta_{N,1} = 1$

$N = 17$: $\zeta(\frac{1}{2} + \mathrm{i}t)$ and $\Omega_{17}(\frac{1}{2} + \mathrm{i}t) = \sum_{n=1}^{17} \delta_{17,n} n^{-\frac{1}{2} - \mathrm{i}t}$



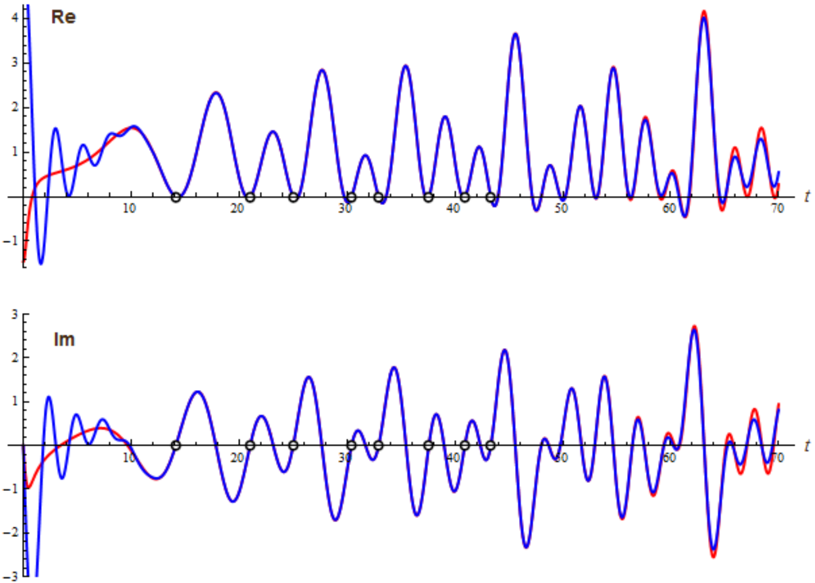
Case $M = 8, \;\; N = 2M + 1 = 17$



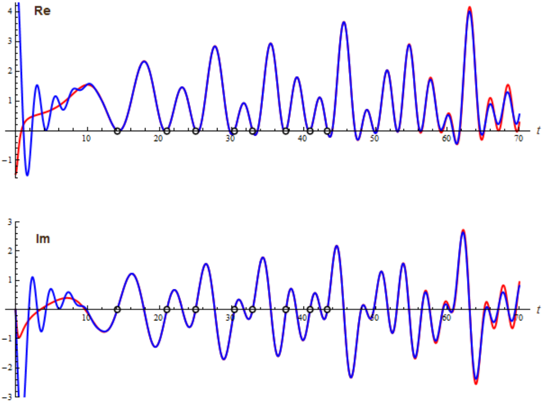
Non-trivial zeta zeros know a lot, in particular

- they know that (at least, initial) coefficients of the Dirichlet series for the zeta function are (approximately) equal to 1
- they know a smooth truncation giving good approximation to $\zeta(s)$ on (at least, part of) the critical line, that it outside the half-plane of convergence of the Dirichlet series for the zeta function

$N = 17$: $\zeta(\frac{1}{2} + \mathrm{i}t)$ and $\Omega_{17}(\frac{1}{2} + \mathrm{i}t) = \sum_{n=1}^{17} \delta_{17,n} n^{-\frac{1}{2} - \mathrm{i}t}$



$N = 17$: $\zeta(\frac{1}{2} + \mathrm{i}t)$ and $\Omega_{17}(\frac{1}{2} + \mathrm{i}t) = \sum_{n=1}^{17} \delta_{17,n} n^{-\frac{1}{2}-\mathrm{i}t}$



$$
\begin{aligned}
0 &= \Omega_{17}(\rho_9 - 4.396\ldots\cdot 10^{-3} + 5.711\ldots\cdot 10^{-3}\mathrm{i}) \\
0 &= \Omega_{17}(\rho_{10} - 1.141\ldots\cdot 10^{-2} - 3.345\ldots\cdot 10^{-3}\mathrm{i}) \\
0 &= \Omega_{17}(\rho_{11} - 1.498\ldots\cdot 10^{-2} + 1.762\ldots\cdot 10^{-3}\mathrm{i}) \\
0 &= \Omega_{17}(\rho_{12} - 1.158\ldots\cdot 10^{-2} + 2.264\ldots\cdot 10^{-2}\mathrm{i})
\end{aligned}
$$

Case $M = 50$, $N = 2M + 1 = 101$: zeroes of $\Omega_{101}(s)$

$$
\begin{aligned}
0 &= \Omega_{101}(\rho_{51} + 3.469\ldots \cdot 10^{-15} - 1.283\ldots \cdot 10^{-15}\mathrm{i}) \\
0 &= \Omega_{101}(\rho_{52} + 1.472\ldots \cdot 10^{-14} - 4.170\ldots \cdot 10^{-15}\mathrm{i}) \\
0 &= \Omega_{101}(\rho_{53} - 3.949\ldots \cdot 10^{-13} + 1.223\ldots \cdot 10^{-14}\mathrm{i}) \\
0 &= \Omega_{101}(\rho_{54} - 4.684\ldots \cdot 10^{-13} - 9.387\ldots \cdot 10^{-13}\mathrm{i}) \\
0 &= \Omega_{101}(\rho_{55} - 5.303\ldots \cdot 10^{-12} + 2.129\ldots \cdot 10^{-12}\mathrm{i}) \\
0 &= \Omega_{101}(\rho_{56} + 2.104\ldots \cdot 10^{-11} + 4.691\ldots \cdot 10^{-11}\mathrm{i}) \\
0 &= \Omega_{101}(\rho_{57} + 1.054\ldots \cdot 10^{-10} + 1.430\ldots \cdot 10^{-10}\mathrm{i})
\end{aligned}
$$

Non-trivial zeta zeros know a lot, in particular

- they know (approximate values of several) next non-trivial zeros

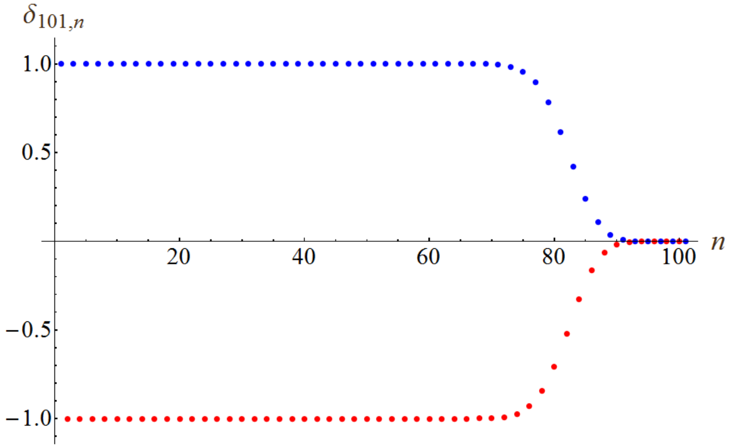
Case $M = 1550$, $N = 2M + 1 = 3101$: zeroes of $\Omega_{3101}(s)$

$$
\begin{aligned}
0 &= \Omega_N(-2 - 1.884\ldots \cdot 10^{-1510}) \\
0 &= \Omega_N(-4 + 2.013\ldots \cdot 10^{-1504}) \\
0 &= \Omega_N(-6 - 1.158\ldots \cdot 10^{-1498}) \\
0 &= \Omega_N(-8 + 4.508\ldots \cdot 10^{-1493}) \\
0 &= \Omega_N(-10 - 1.316\ldots \cdot 10^{-1487}) \\
0 &= \Omega_N(-12 + 3.066\ldots \cdot 10^{-1482}) \\
0 &= \Omega_N(-14 - 5.931\ldots \cdot 10^{-1477}) \\
0 &= \Omega_N(-16 + 9.796\ldots \cdot 10^{-1472})
\end{aligned}
$$

Non-trivial zeta zeros know a lot, in particular

- they know (approximate values of several) initial trivial zeros

$N = 101$. Coefficients $\delta_{101,n}$, red for even n, blue for odd n



Non-trivial zeta zeros know a lot, in particular

- they know about the alternating zeta function, that is, the eta function $\eta(s)$

Why $\eta(s)$?

$$\Omega_N(s) = 1 + \sum_{n=2}^{N} \delta_{N,n} n^{-s}$$

$$\Omega_N(\overline{\rho_M}) = \cdots = \Omega_N(\overline{\rho_1}) = 0 = \Omega_N(\rho_1) = \cdots = \Omega_N(\rho_M)$$

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

$$\zeta_N(\overline{\rho_M}) = \cdots = \zeta_N(\overline{\rho_1}) = 0 = \zeta_N(\rho_1) = \cdots = \zeta_N(\rho_M)$$

$$\sum_{n=1}^{\infty}(-1)^{n+1} n^{-s} = \eta(s) = (1 - 2 \cdot 2^{-s})\zeta(s)$$

$$\eta_N(\overline{\rho_M}) = \cdots = \eta_N(\overline{\rho_1}) = 0 = \eta_N(\rho_1) = \cdots = \eta_N(\rho_M)$$

Case $M = 1550$, $N = 2M + 1 = 3101$: extra eta zeroes

$$\eta(s) = (1 - 2 \cdot 2^{-s})\zeta(s)$$

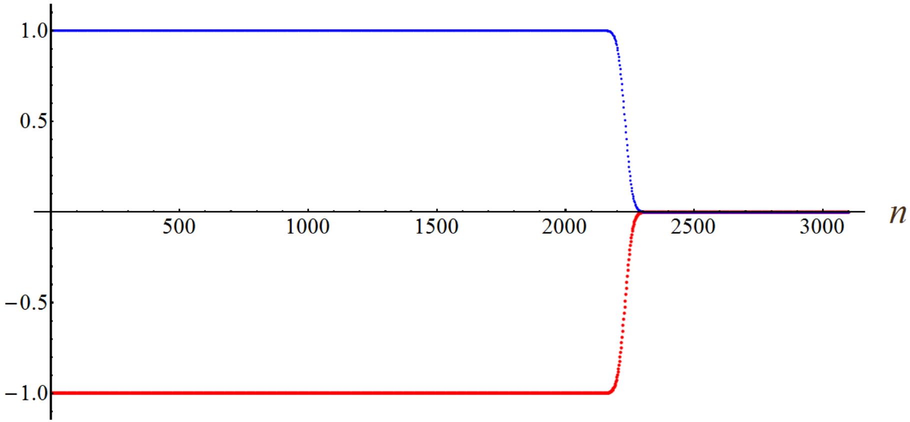
$$1 - 2 \cdot 2^{-s} = 0 \iff s = s_k = 1 + \frac{2\pi k}{\ln(2)}\mathrm{i}, \quad k = 0, \pm 1, \pm 2, \ldots$$

$$\begin{aligned}
0 &= \Omega_N(s_{50} - 5.481\ldots \cdot 10^{-133} - 5.546\ldots \cdot 10^{-133}\mathrm{i}) \\
0 &= \Omega_N(s_{100} - 1.109\ldots \cdot 10^{-132} - 1.306\ldots \cdot 10^{-134}\mathrm{i}) \\
0 &= \Omega_N(s_{150} - 5.743\ldots \cdot 10^{-133} + 5.543\ldots \cdot 10^{-133}\mathrm{i}) \\
0 &= \Omega_N(s_{200} - 6.157\ldots \cdot 10^{-136} + 2.613\ldots \cdot 10^{-134}\mathrm{i})
\end{aligned}$$

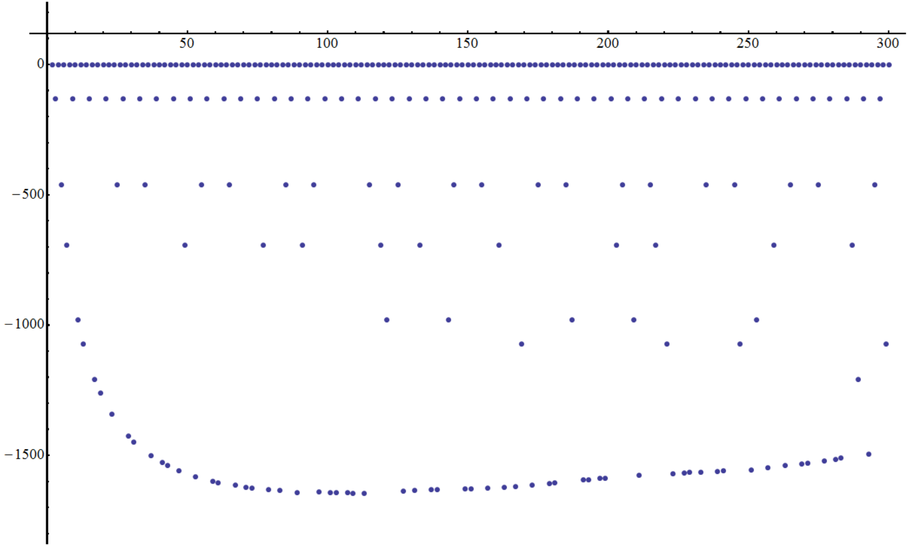
Non-trivial ZETA zeros know a lot, in particular

- they know (approximate values of several) initial extra zeros of the ETA function

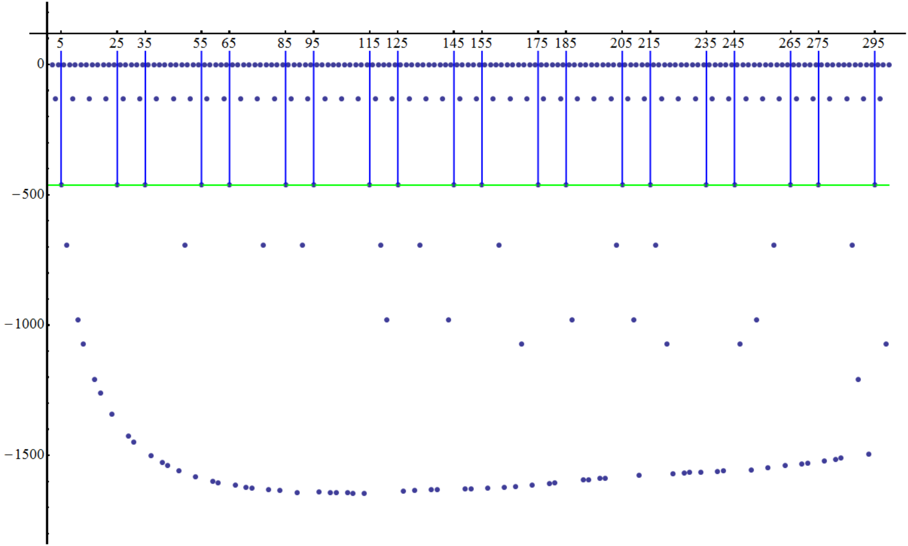
$N = 3101$, coefficients $\delta_{3101,n}$, red for even n, blue for odd n



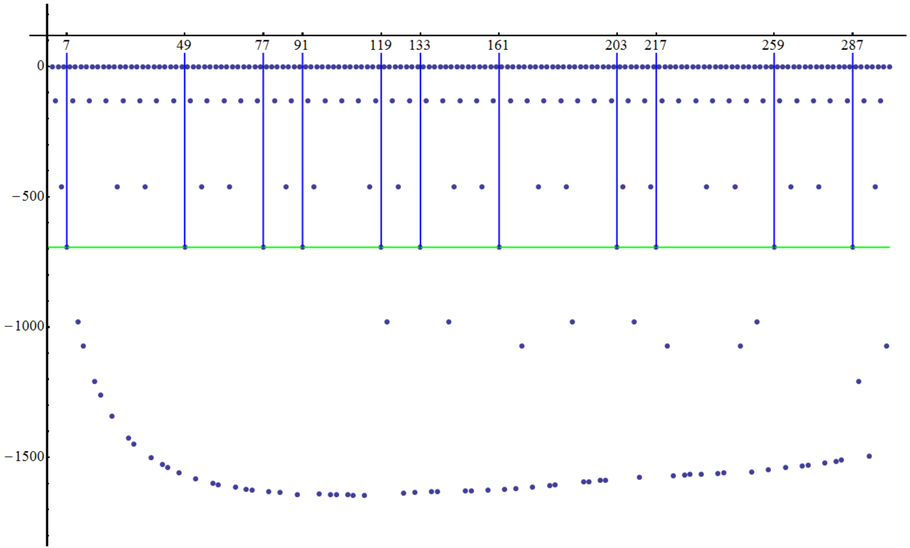
Plot of $\log_{10}|\delta_{3101,n} - 1|$



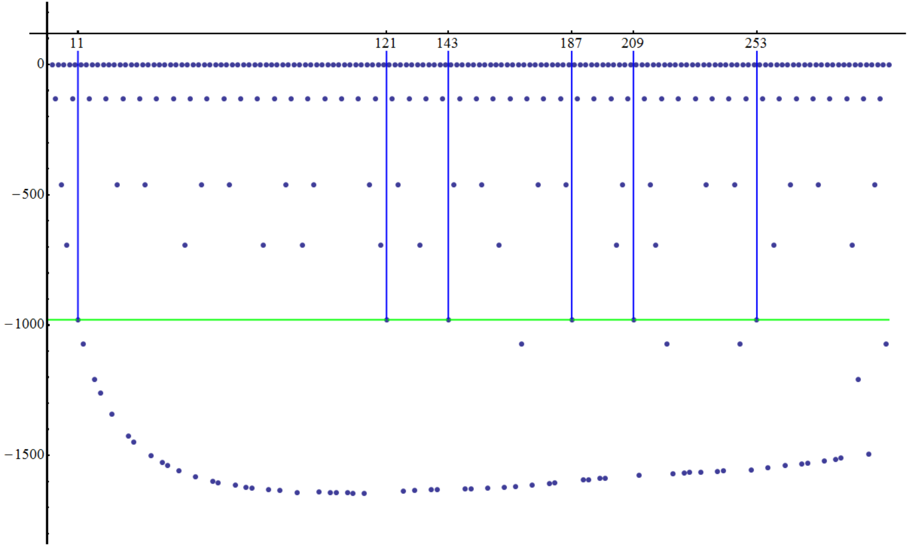
Plot of $\log_{10} |\delta_{3101,n} - 1|$



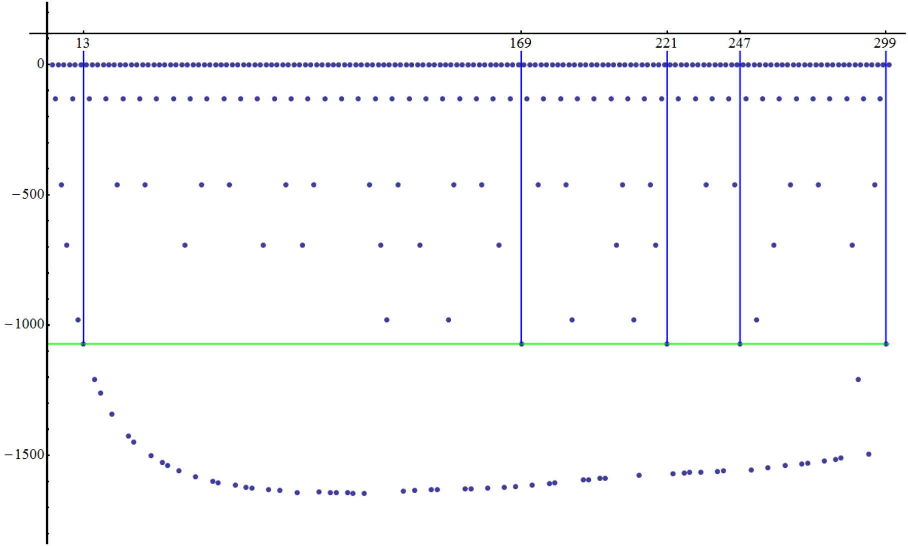
Plot of $\log_{10}|\delta_{3101,n} - 1|$



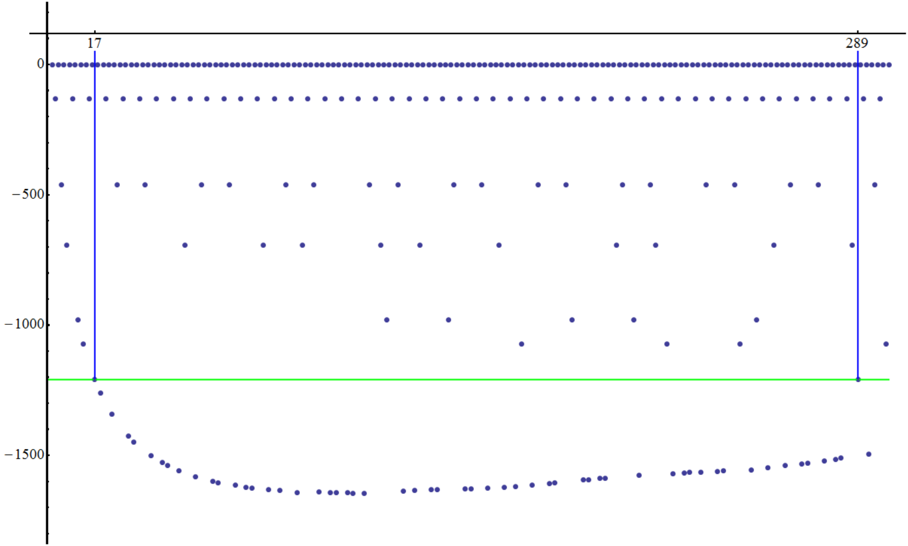
Plot of $\log_{10} |\delta_{3101,n} - 1|$



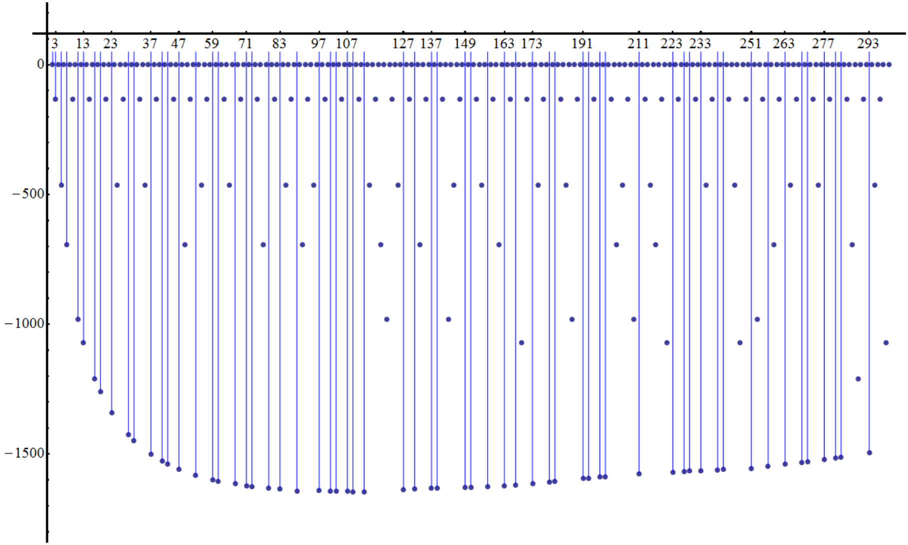
Plot of $\log_{10} |\delta_{3101,n} - 1|$



Plot of $\log_{10}|\delta_{3101,n} - 1|$



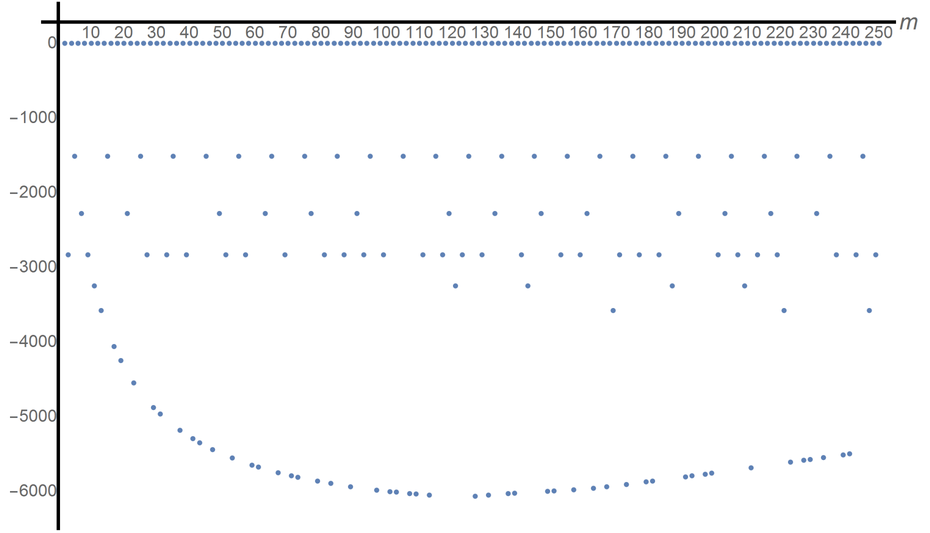
Plot of $\log_{10}|\delta_{3101,n} - 1|$ = Sieve of Eratosthenes



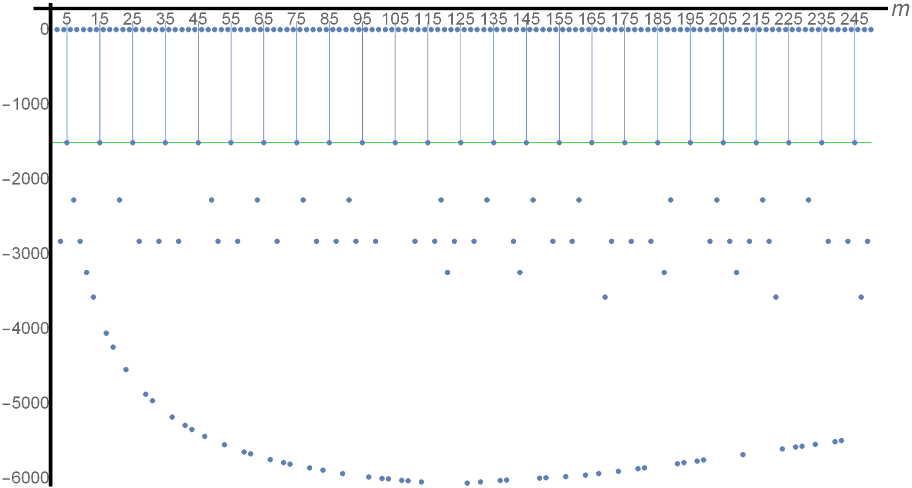
Non-trivial zeta zeros know a lot, in particular

- they know the Sieve of Eratosthenes

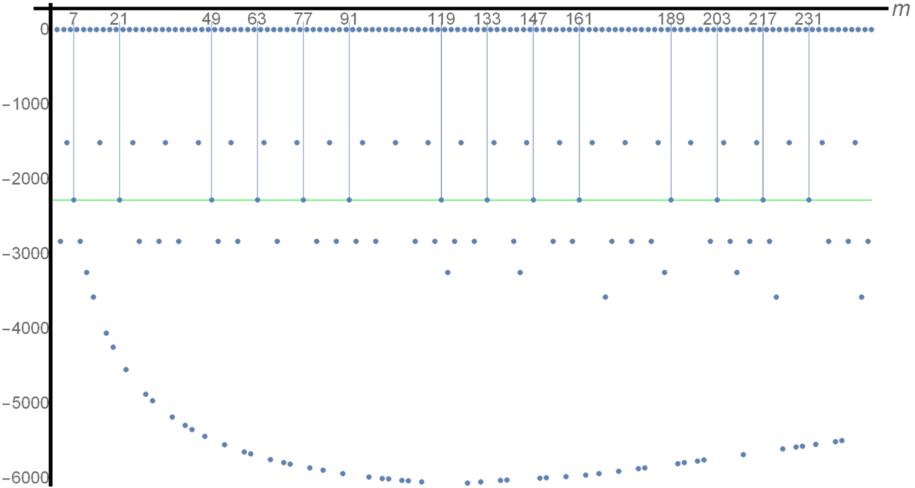
Finer Structure: Plot of $\log_{10}|\delta_{10001,3m} - \delta_{10001,3}|$



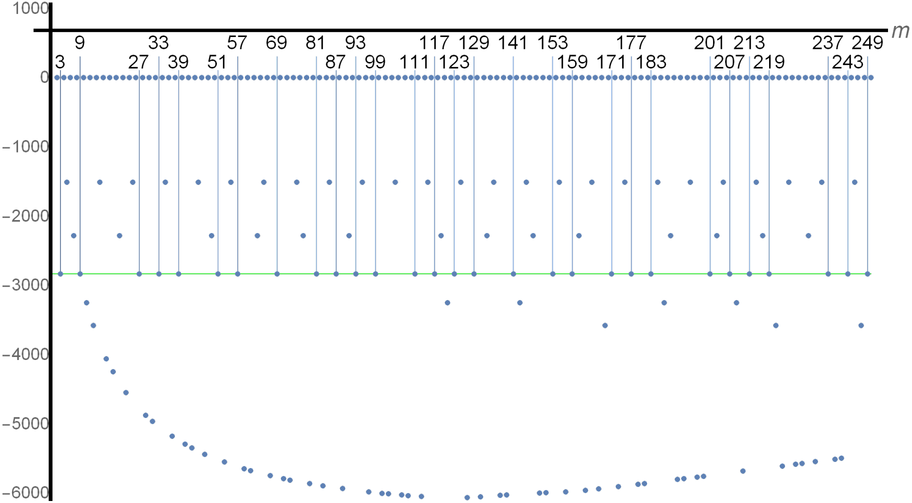
Finer Structure: Plot of $\log_{10}|\delta_{10001,3m} - \delta_{10001,3}|$



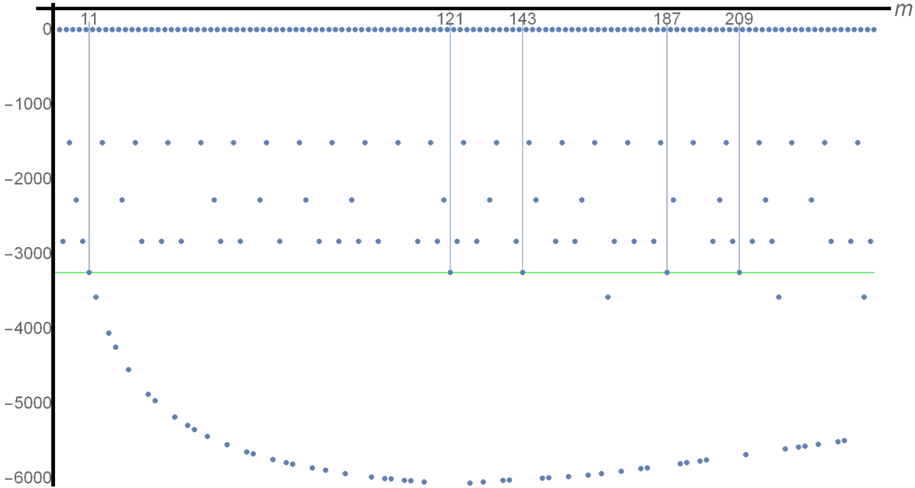
Finer Structure: Plot of $\log_{10} |\delta_{10001,3m} - \delta_{10001,3}|$



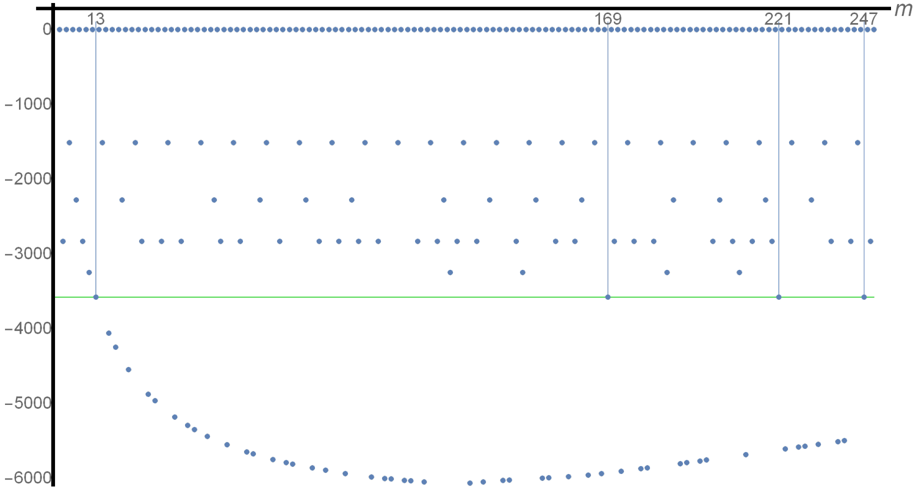
Finer Structure: Plot of $\log_{10}|\delta_{10001,3m} - \delta_{10001,3}|$



Finer Structure: Plot of $\log_{10}|\delta_{10001,3m} - \delta_{10001,3}|$

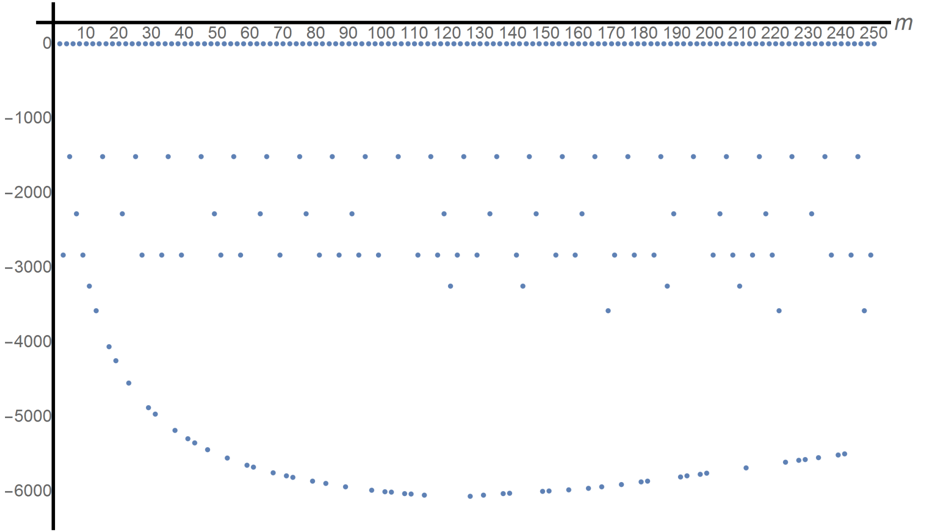


Finer Structure: Plot of $\log_{10}|\delta_{10001,3m} - \delta_{10001,3}|$



Finer Structure: Plot of $\log_{10} |\delta_{10001,3m} - \delta_{10001,3}|$

= Eratosthenes Sieve with primes order 2, 5, 7, 3, 11, 13, . . .



Expected Fractal Structure

Let n range over an arithmetical progression $d, 2d, \dots, md, \dots$ with

$$d = 2^{k_2} 3^{k_3} 5^{k_5} \dots$$

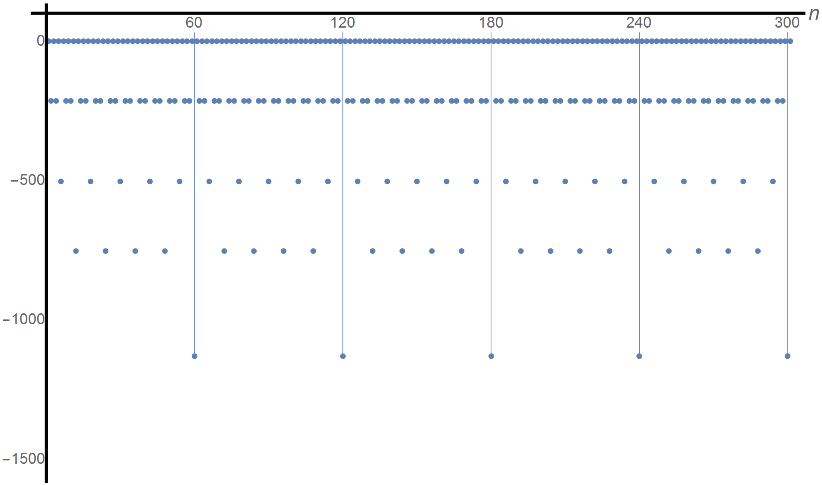
Corresponding Eratosthenes sublevel splits according to the divisibility of m by q_1, q_2, …where these prime numbers are ordered in such a way that

$$q_1^{k_{q_1}+1} < q_2^{k_{q_2}+1} < \dots < q_j^{k_{q_j}+1} < \dots$$

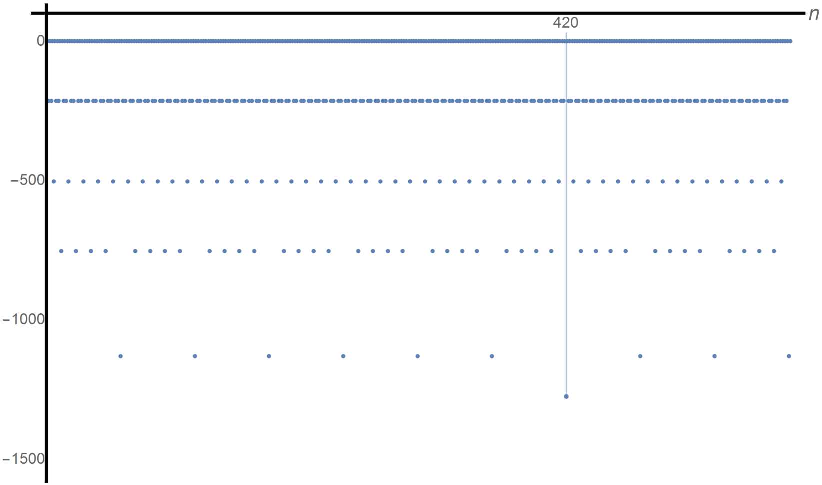
In the previous example $m = 3$, hence $k_2 = 0$, $k_3 = 1$, $k_5 = k_7 = \dots = 0$ and $q_1 = 2,\ q_2 = 5,\ q_3 = 7,\ q_4 = 3,\ q_5 = 11,\ q_6 = 13,\ \dots$ according to

$$2^1 < 5^1 < 7^1 < 3^2 < 11^1 < 13^1 < \dots$$

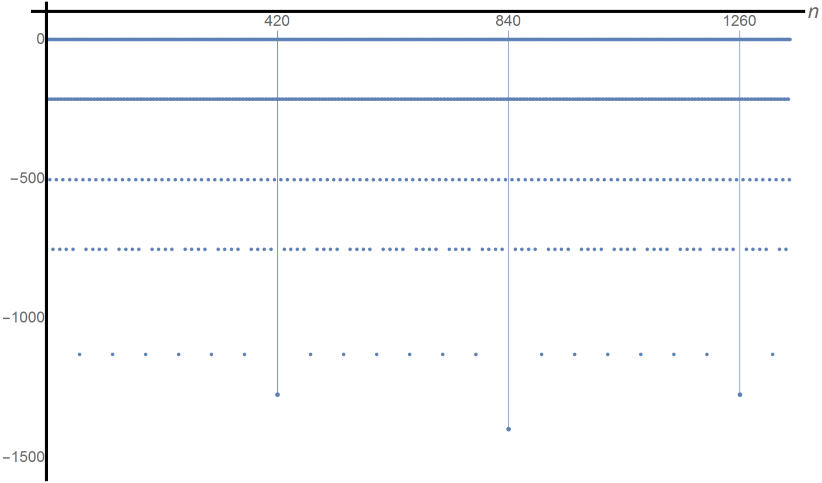
Dual sieve: Plot of $\log_{10}\left|\sum_{n=1}^{m}\delta_{N,n}\right|$ при $N = 5001$



Dual sieve: Plot of $\log_{10}|\sum_{n=1}^{m}\delta_{N,n}|$ при $N = 5001$



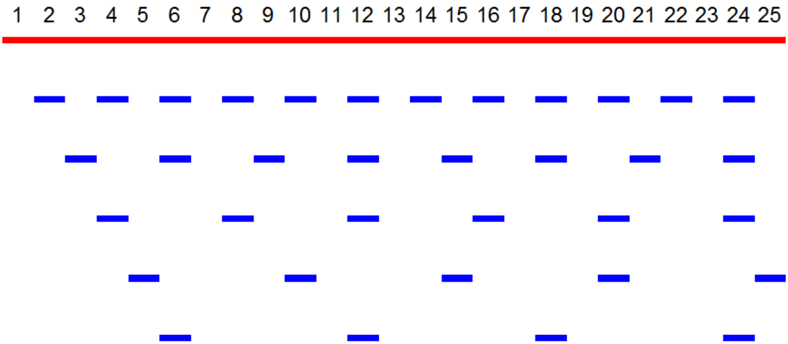
Dual sieve: Plot of $\log_{10}\left|\sum_{n=1}^{m}\delta_{N,n}\right|$ при $N = 5001$



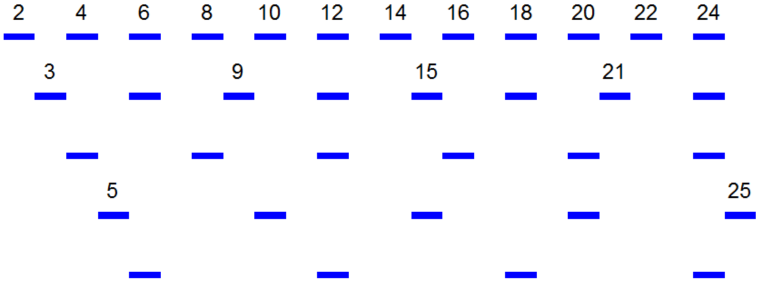
Inheritable divisor: $k_{\leq}|m \iff 1|m \ \& \ 2|m \ \& \ 3|m \ \& \ \ldots \ \& \ k|m$

Maximal inheritable divisor: $k_{\leq}||m \iff k_{\leq}|m \ \& \ (k+1)\nmid m$

Sieve of Eratosthenes

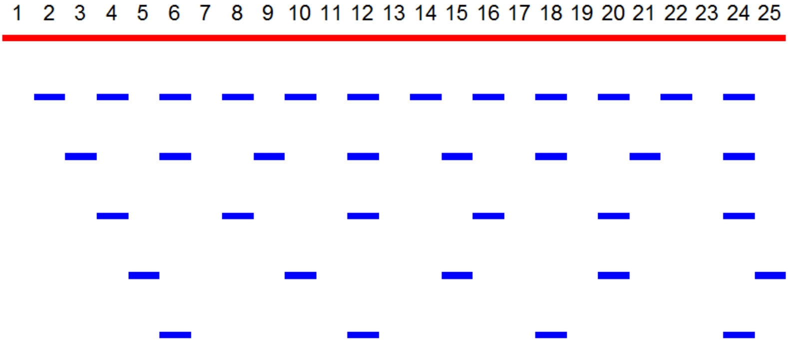


Sieve of Eratosthenes

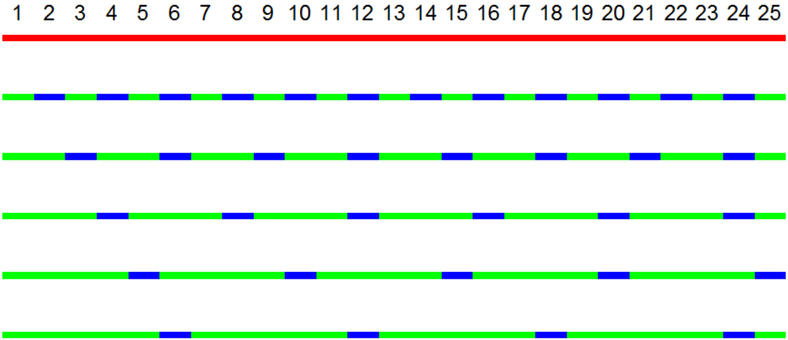


$$\log |\delta_{N,n} - 1|$$

Sieve of Eratosthenes (repeated)



Sieve of Eratosthenes vs dual sieve



Dual sieve

Dual sieve

$$\log \left| \sum_{n=1}^{m} \delta_{N,n} \right|$$

Davenport–Heilbronn function

$$f(s) = \sum_{n=1}^{\infty} d(n) n^{-s}$$

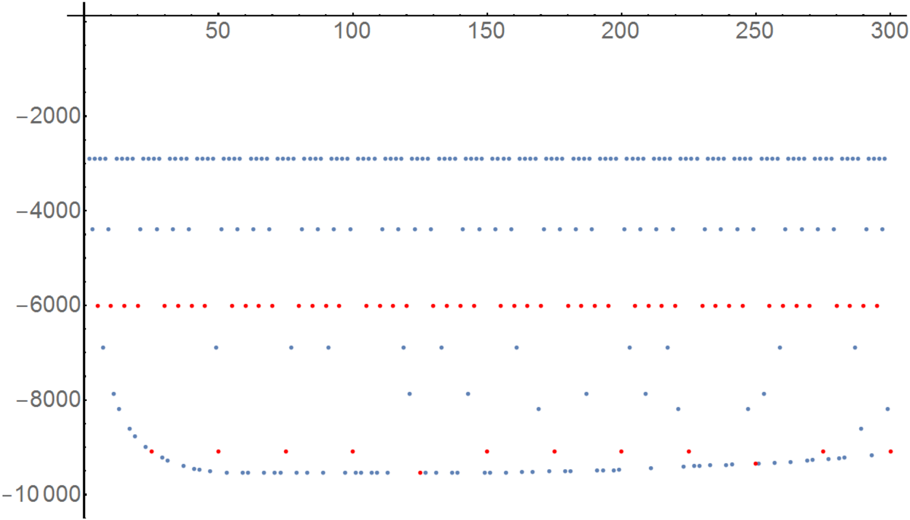
where

$$d(n) = \begin{cases} 0, & \text{if } n \equiv 0 \pmod 5 \\ 1, & \text{if } n \equiv 1 \pmod 5 \\ \tau, & \text{if } n \equiv 2 \pmod 5 \\ -\tau, & \text{if } n \equiv 3 \pmod 5 \\ -1, & \text{if } n \equiv 4 \pmod 5 \end{cases}$$

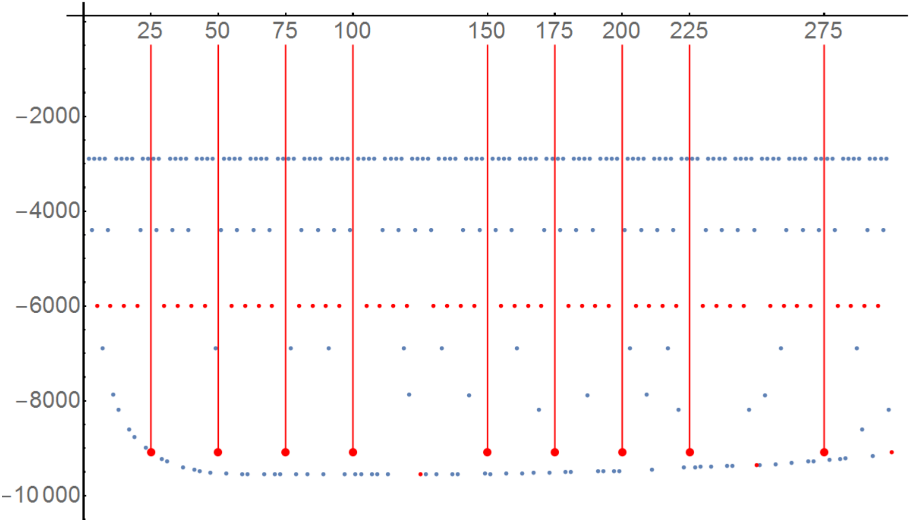
and

$$\tau = \frac{-2 + \sqrt{10 - 2\sqrt{5}}}{-1 + \sqrt{5}}$$

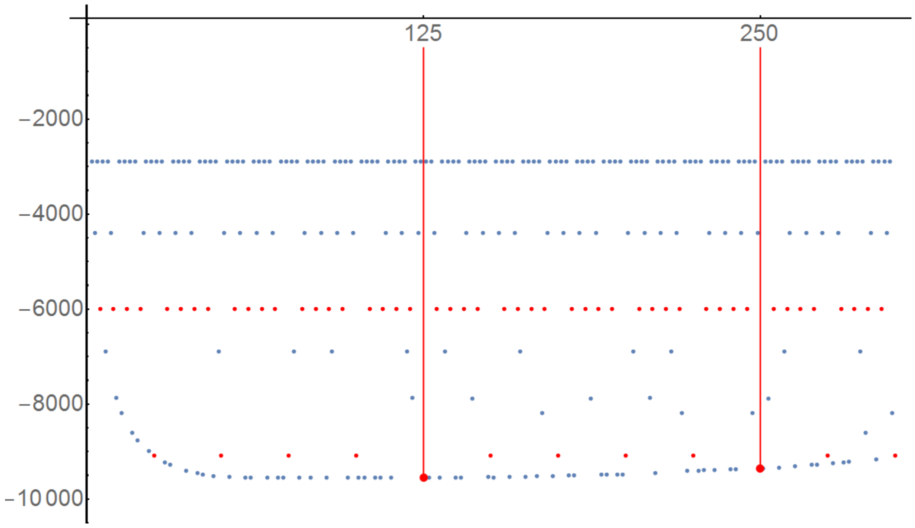
Sieve of Eratosthenes for $f(s)$ (N=7999)



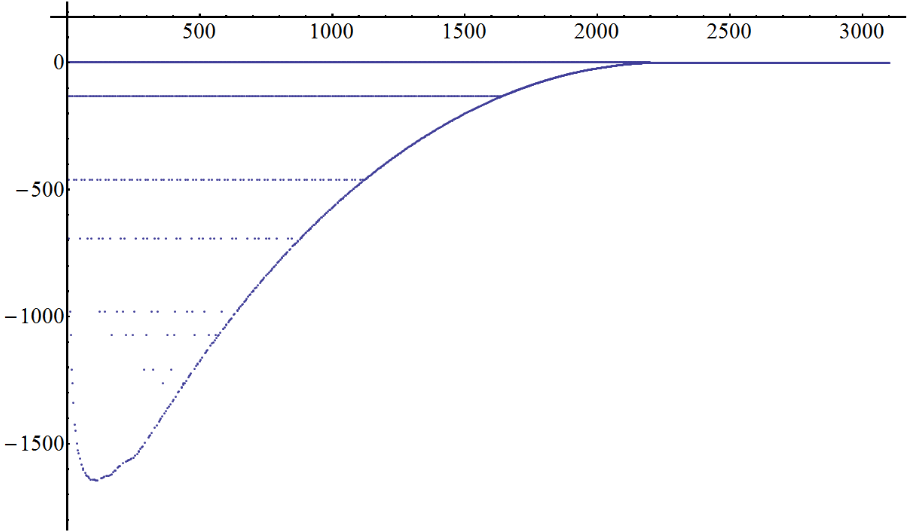
Sieve of Eratosthenes for $f(s)$ (N=7999)



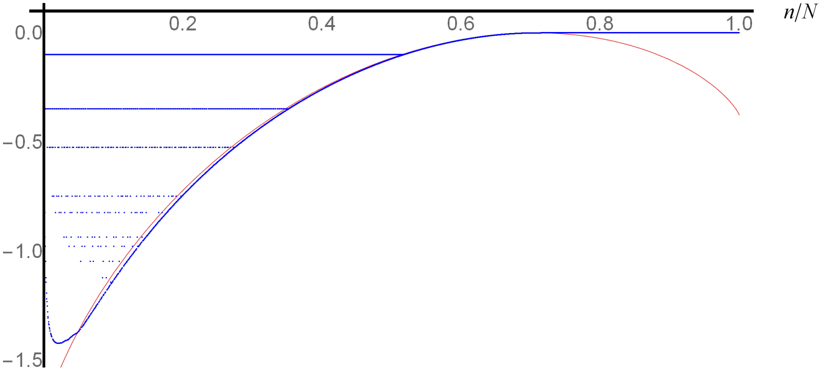
Sieve of Eratosthenes for $f(s)$ (N=7999)



Total plot of $\log_{10} |\delta_{3101,n} - 1|$

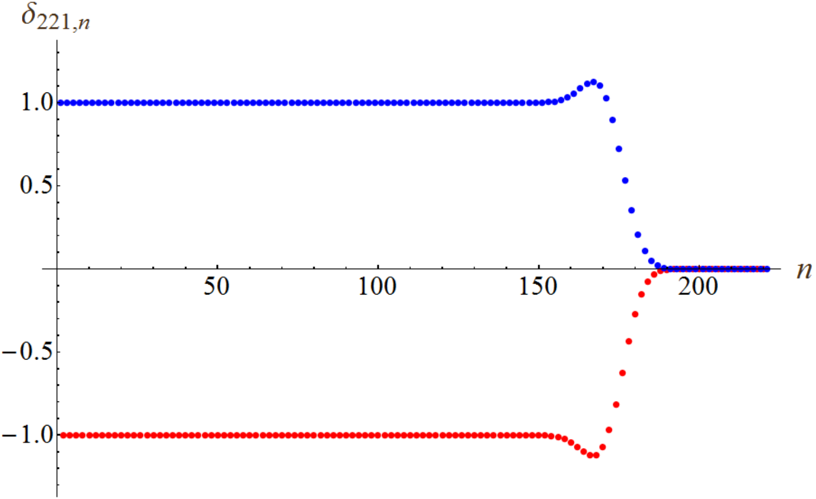


Scaled plot of $(\ln|\delta_{N,n}+(-1)^n|)/N$ for $N=10001$

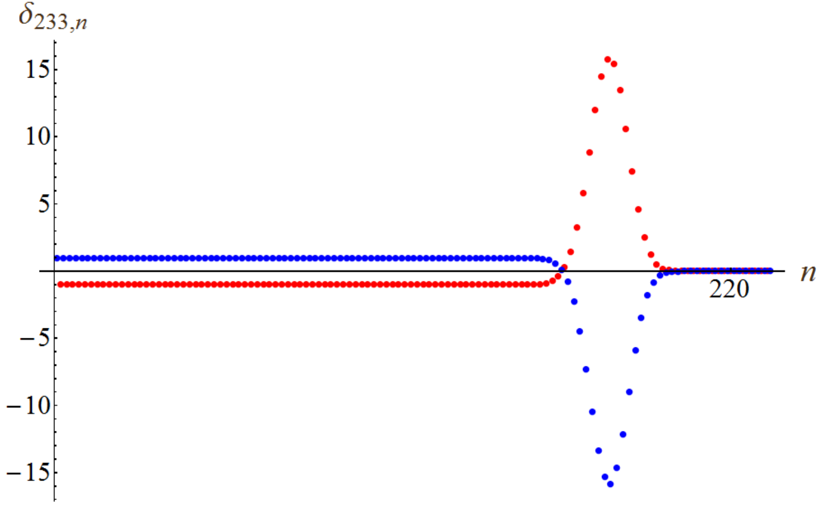


$$\ell(\tfrac{n}{N}) = (\tfrac{n}{N}-1)\ln(1-\tfrac{n}{N}) - 2\tfrac{n}{N}\ln(\tfrac{n}{N}) + (\tfrac{n}{N}+1)\ln(\tfrac{n}{N}+1) - \ln(2\sqrt{2}+3)$$

$N = 221$, coefficients $\delta_{221,n}$, red for even n, blue for odd n



$N = 233$, coefficients $\delta_{233,n}$, red for even n, blue for odd n



Where to look for

https://logic.pdmi.ras.ru/~yumat

https://logic.pdmi.ras.ru/~yumat/publications/publications.php?istate=state_show_paper&imykey=94&ilang=eng

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