RUTGERS EXPERIMENTAL MATHEMATICS SEMINAR

December 14, 2023

Nontrivial zeros of the Riemann zeta function know a lot

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Plan of the talk

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Part I. Some facts from Number Theory

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Part II. Numerical experiments and discoveries

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Part I

Some facts from Number Theory

Sieve of Eratosthenes (276–194 B. C.)

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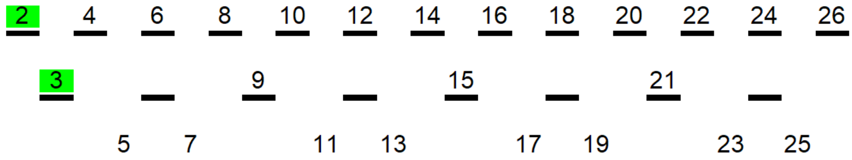

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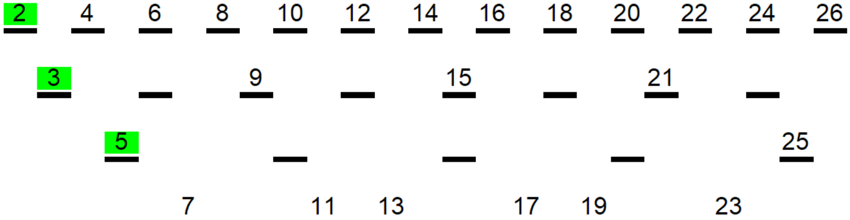

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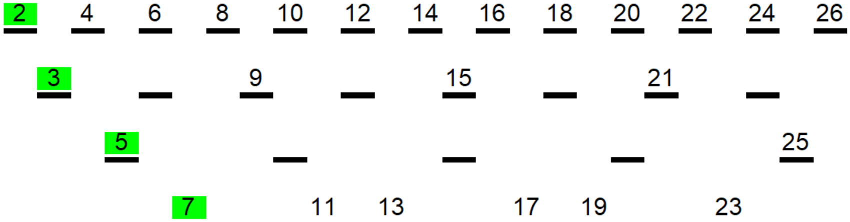

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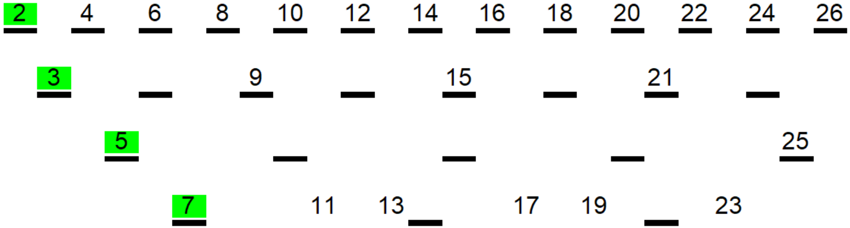

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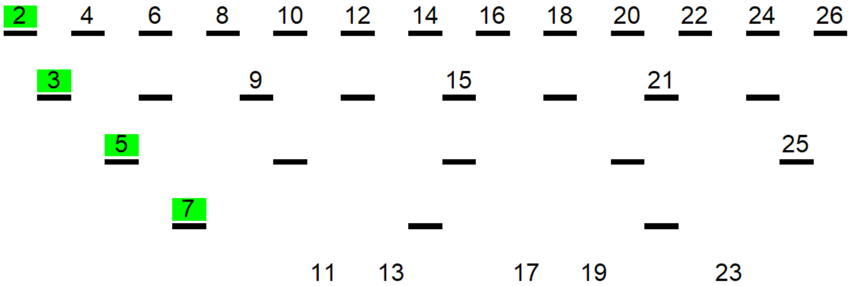

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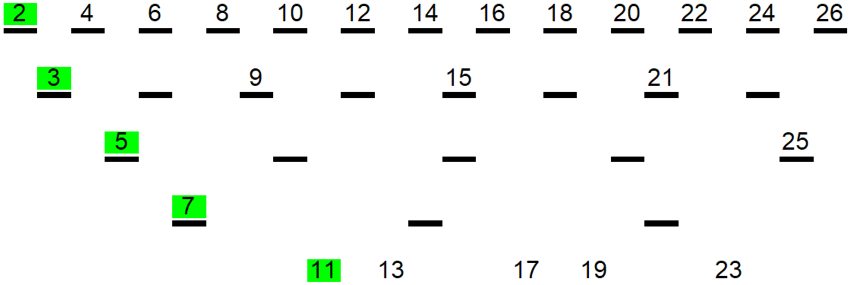

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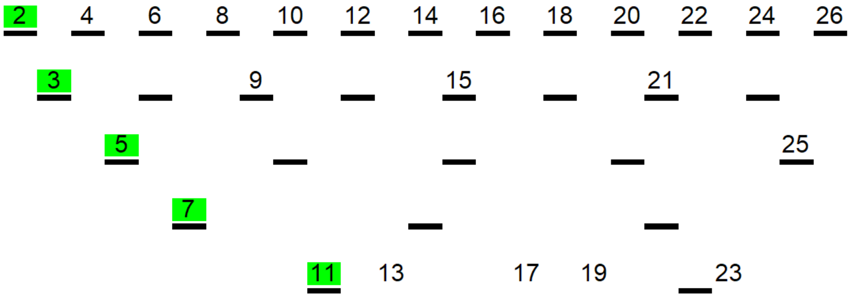

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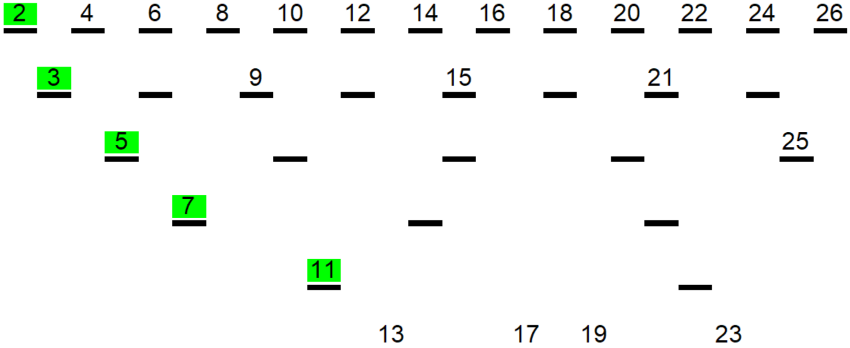

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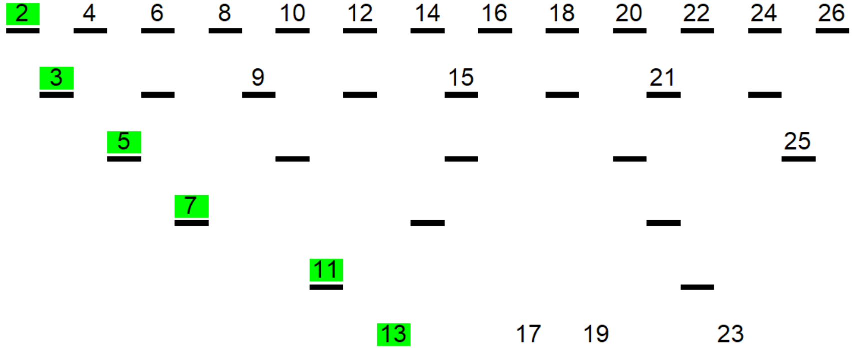

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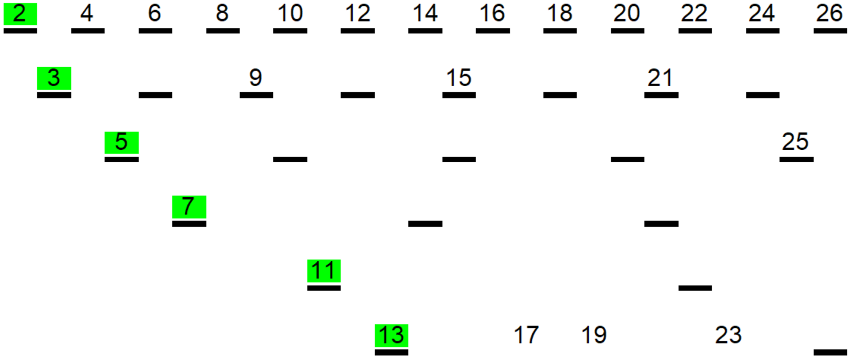

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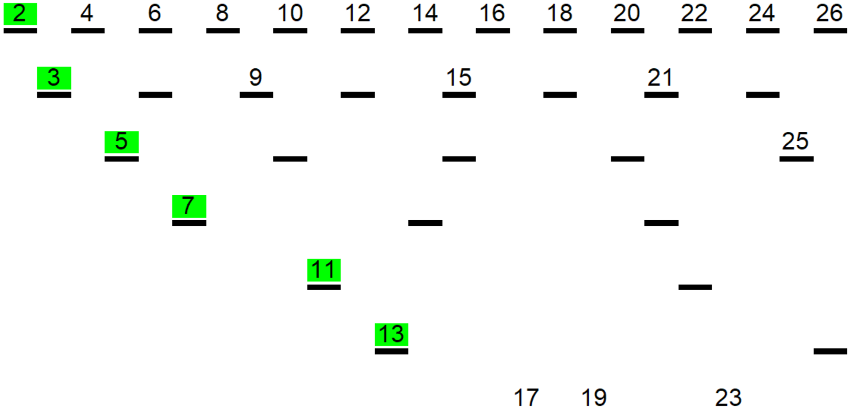

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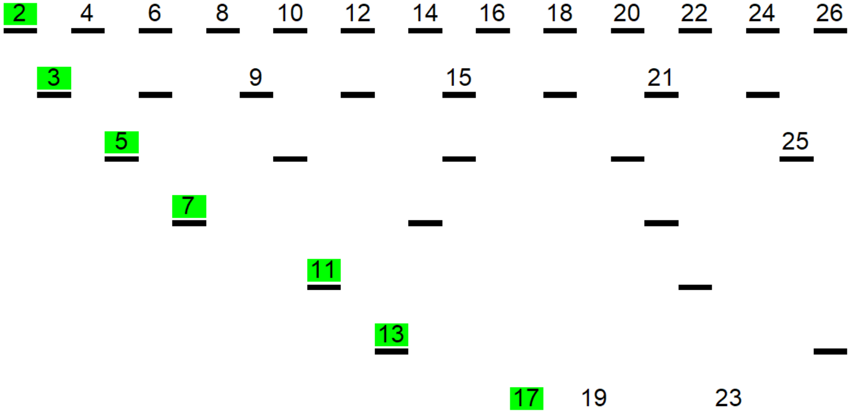

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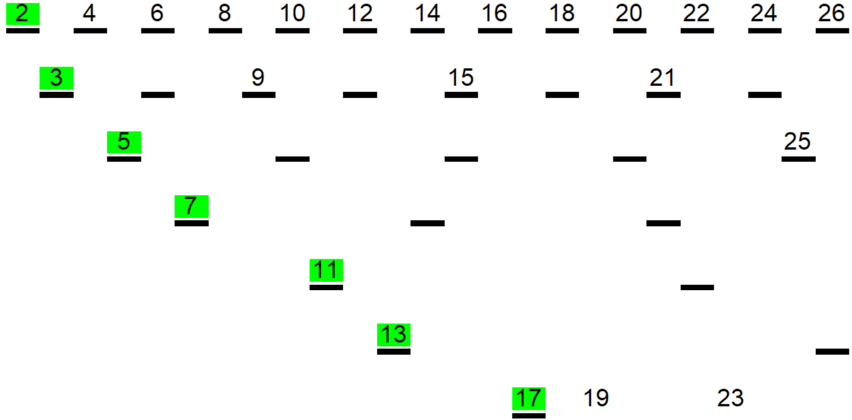

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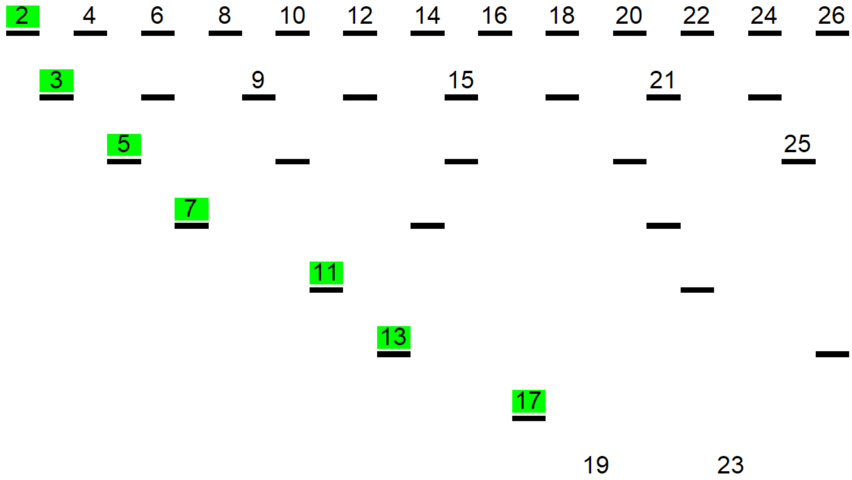

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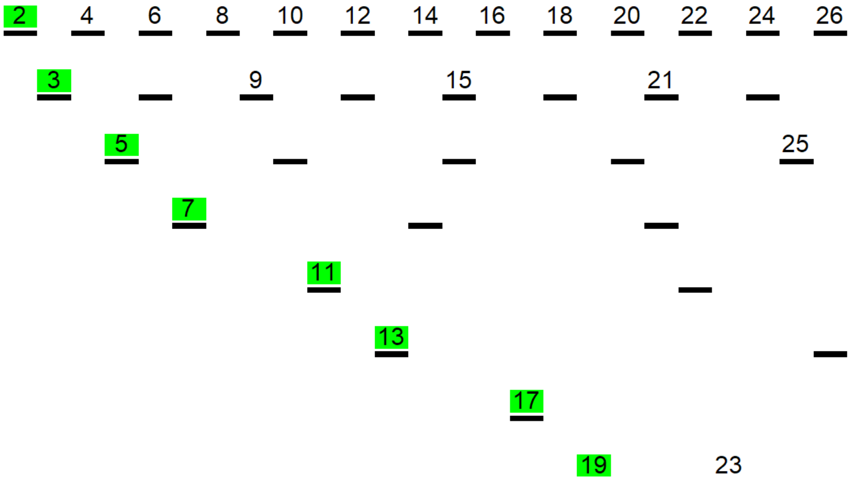

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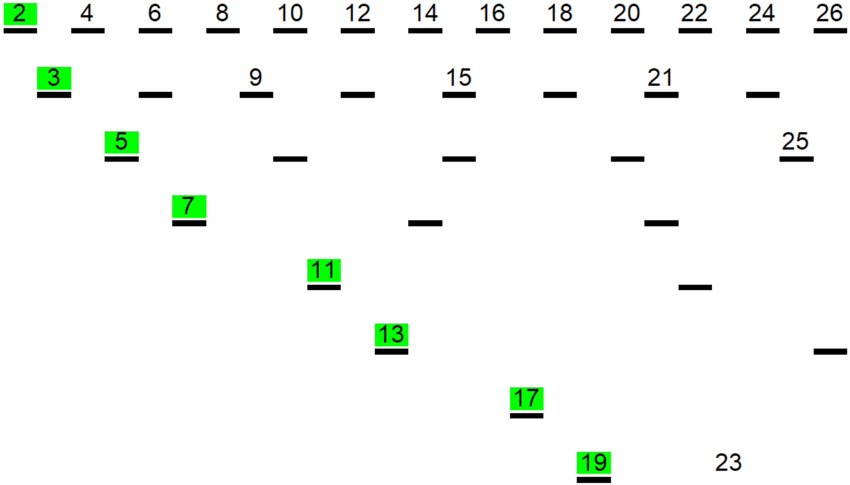

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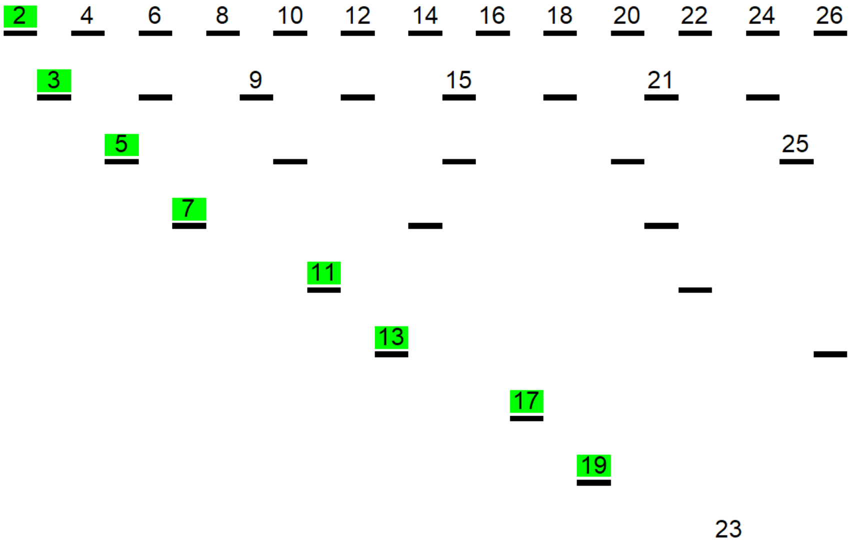

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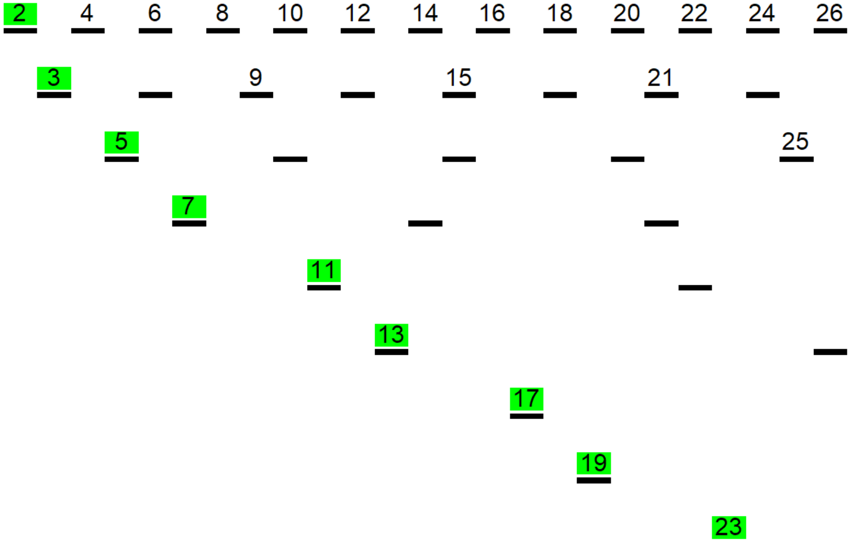

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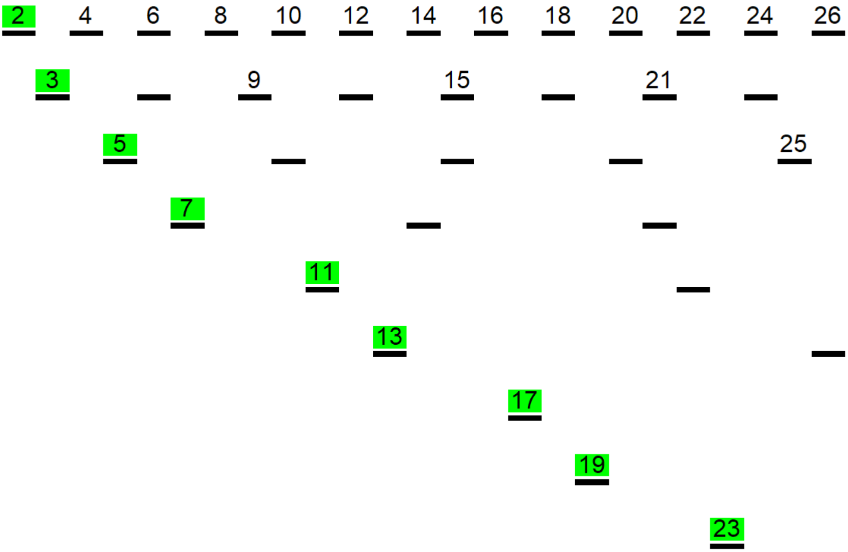

Georg Friedrich Bernhard Riemann (1826–1866)

Riemann's zeta function:

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The series converges in the half-plane $\mathrm{Re}(s) > 1$ and defines a function that can be analytically extended to the entire complex plane except for the point $s = 1$, its only (and simple) pole.

Leonhard Euler (1707–1783)

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Theorem (Euler Identity)

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p \text{ is prime}} \frac{1}{1-p^{-s}}$$

Euler identity

Theorem (Euler). $\sum_{n=1}^{\infty} n^{-s} = \zeta(s) = \prod_{p \text{ is prime}} (1 - p^{-s})^{-1}$

Proof

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$\zeta(s) \prod_{p \text{ is prime}} (1-p^{-s}) = 1^{-s} = 1$

The infinitude of prime numbers

Euler identity

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Theorem (Euclid). *There are infinitely many prime numbers.*

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Theorem (Euclid). *There are infinitely many prime numbers.*

Proof (Euler). If the number of primes would be finite, then the (divergent) harmonic series would have finite value:

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots = \prod_{p \text{ is prime}} \frac{1}{1 - \frac{1}{p}}$$

Zeros of the zeta function

Euler: $0 = \zeta(-2) = \zeta(-4) = \dots = \zeta(-2m) = \dots$

Negative even integer called the trivial zeros of the zeta function

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Riemann (1859): *All other (called non-trivial) zeros of the zeta function are non-real and are located inside the critical strip*

$$0 \leq \mathrm{Re}(s) \leq 1$$

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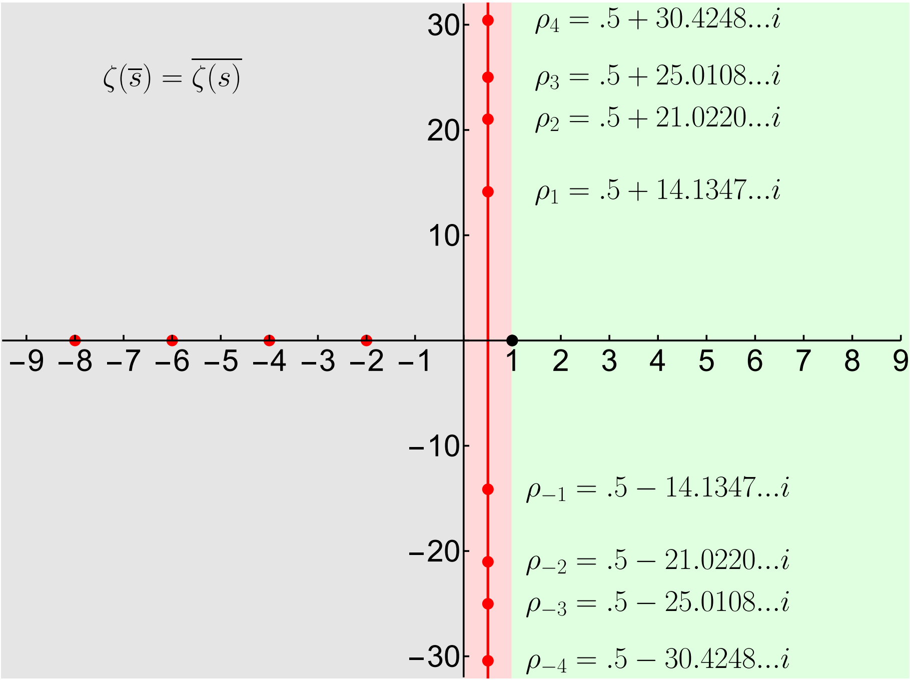
The Riemann Hypothesis: *All non-trivial zeros of the zeta function are located on the critical line* $\mathrm{Re}(s) = 1/2$.


$\zeta(\overline{s}) = \overline{\zeta(s)}$
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30
20
10
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Alternating zeta function:

$$\eta(s) = \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s} \qquad (*)$$

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$$= (1 - 2 \times 2^{-s})\zeta(s)$$

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1) Series $(*)$ converges in the larger half-plane $\mathrm{Re}(s) > 0$
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Sample calculations

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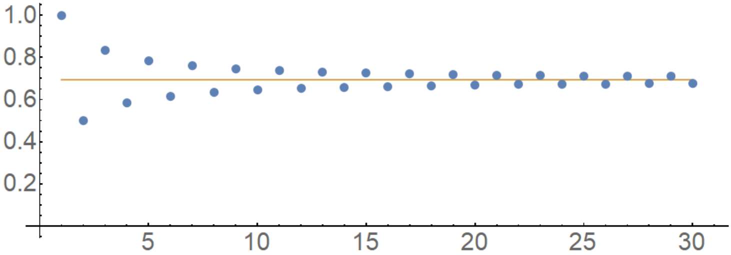

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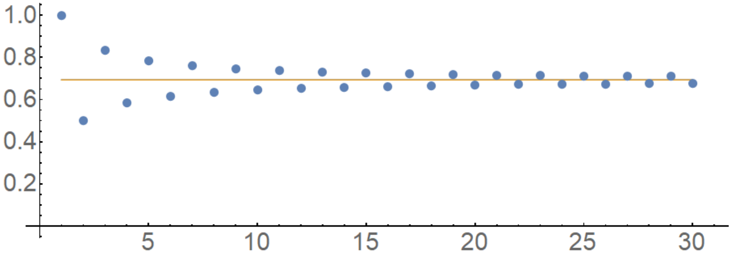

$$\eta_N(s) = \sum_{n=1}^{N-1} (-1)^{n+1} n^{-s} + \frac{1}{2}(-1)^{N+1} N^{-s}$$

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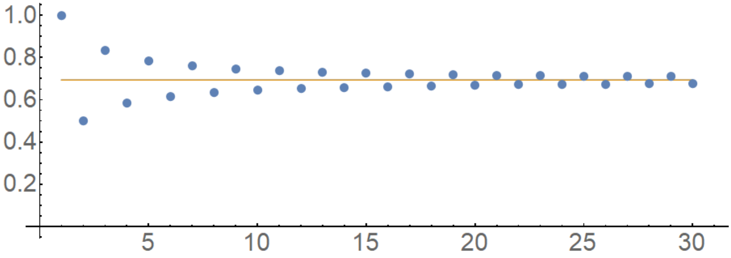

$$\eta_N(s) = \sum_{n=1}^{N-1} (-1)^{n+1} n^{-s} + \frac{1}{2}(-1)^{N+1} N^{-s} \qquad \eta_{1000}(1) = 0.693147430...$$

Approximations proposed by Peter Borwein (1953 – 2020)

$$\eta(s) = \sum_{n=1}^{\infty} (-1)^{n+1} n^{s} \qquad \eta_N(s) = \sum_{n=1}^{N} \alpha_{N,n} n^{-s}$$

$$\alpha_{N,n} = (-1)^{n+1} \left(1 - \frac{\beta_{N,n}}{\beta_{N,N+1}} \right) \qquad \beta_{N,n} = N \sum_{i=1}^{n} \frac{4^{i-1}(N+i-2)!}{(N-i+1)!(2i-2)!}$$

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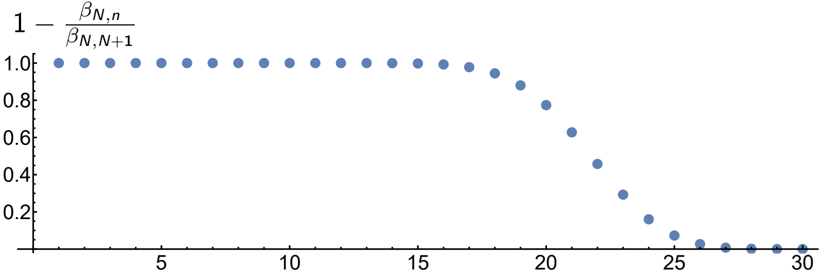

Part II

Experiments and discoveries

Our approximation by a Dirichlet series

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Let us define numbers $\delta_{N,n}$ by the following conditions:

- the finite sum $(**)$ has $N - 1$ common zeros with the infinite sum $(*)$

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- $\delta_{N,1} = 1$

Formal definition of our approximation $\Omega_N(s)$

$$N = 2M + 1$$

$$\Omega_N(s) = 1 + \sum_{n=2}^{N} \delta_{N,n} n^{-s}$$

$\delta_{N,n}$ are defined by condition

$$\Omega_N(\overline{\rho_M}) = \cdots = \Omega_N(\overline{\rho_1}) = 0 = \Omega_N(\rho_1) = \cdots = \eta_N(\rho_M) \qquad (*)$$

where $\rho_1, \ldots, \rho_M$ are the initial zeroes of the zeta function in the upper half-plane:

$$\zeta(\rho_1) = \cdots = \zeta(\rho_M) = 0$$

$$0 < \mathrm{Im}(\rho_1) \leq \cdots \leq \mathrm{Im}(\rho_M)$$

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$N = 17$: $\zeta(\frac{1}{2} + \mathrm{i}t)$ and $\Omega_{17}(\frac{1}{2} + \mathrm{i}t) = \sum_{n=1}^{17} \delta_{17,n} n^{-\frac{1}{2} - \mathrm{i}t}$

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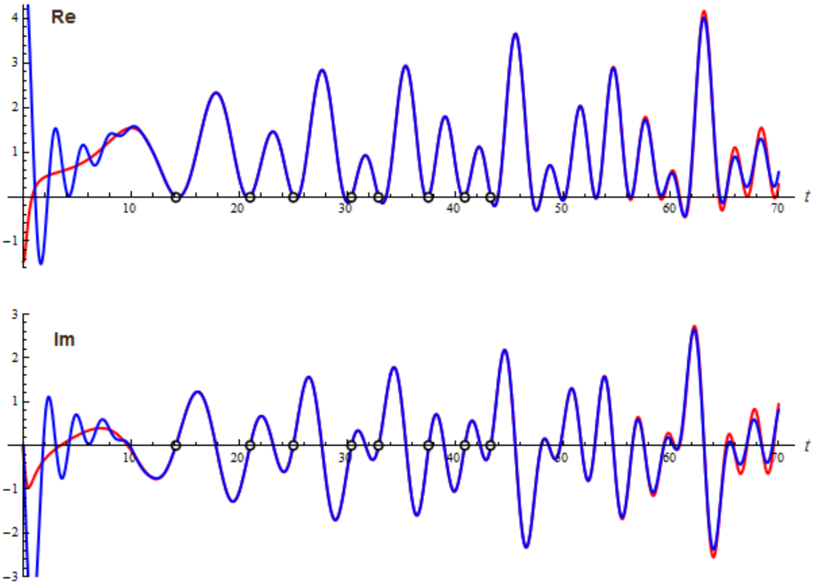
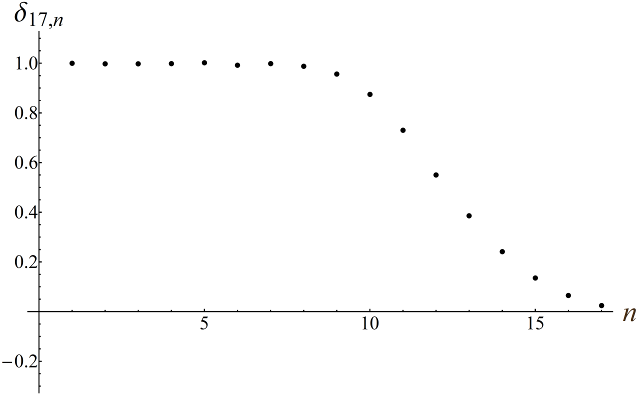


Case $M = 8, \ \ N = 2M + 1 = 17$



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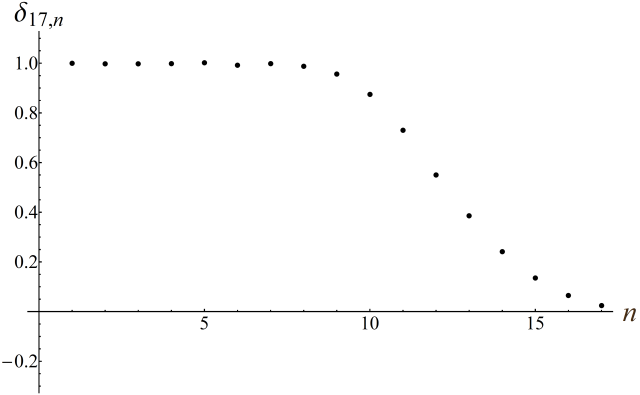

Non-trivial zeta zeros know a lot, in particular

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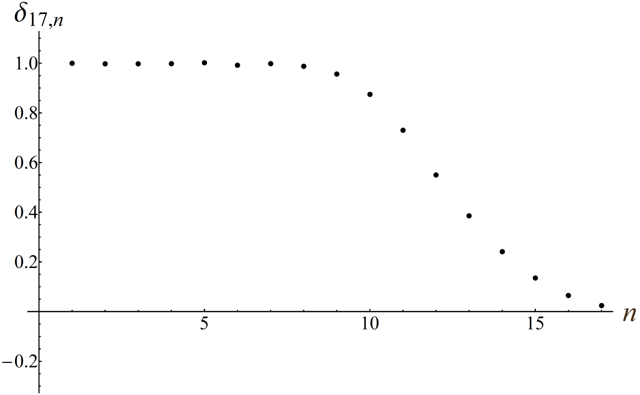

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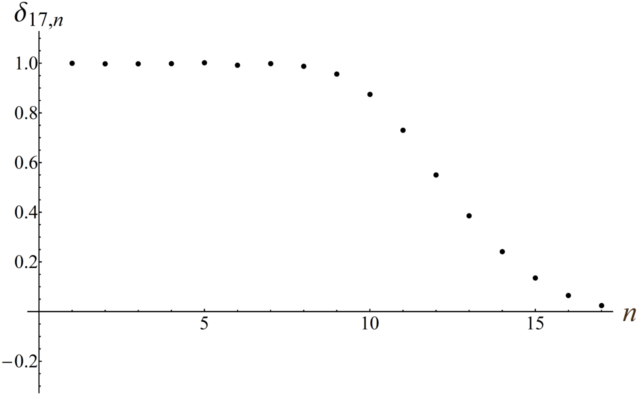

Non-trivial zeta zeros know a lot, in particular

- they know that (at least, initial) coefficients of the Dirichlet series for the zeta function are (approximately) equal to 1
- they know a smooth truncation giving good approximation to $\zeta(s)$ on (at least, part of) the critical line, that it outside the half-plane of convergence of the Dirichlet series for the zeta function

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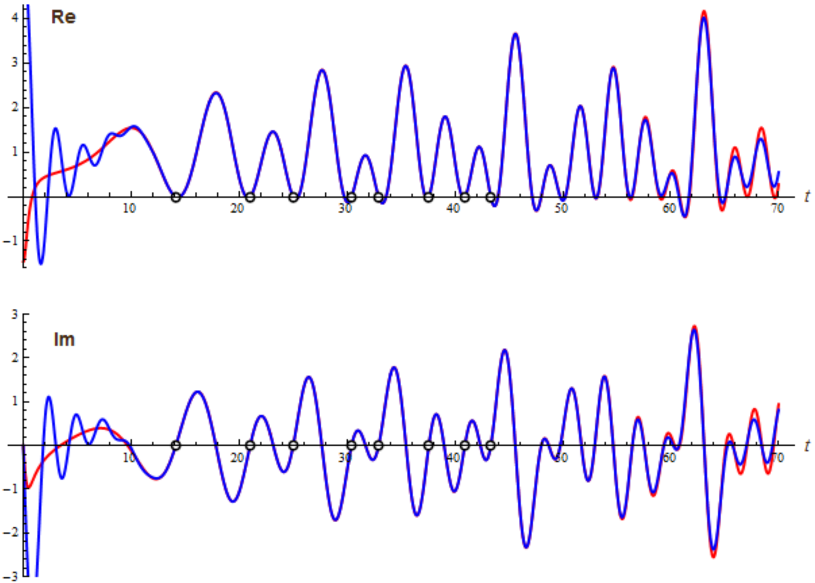

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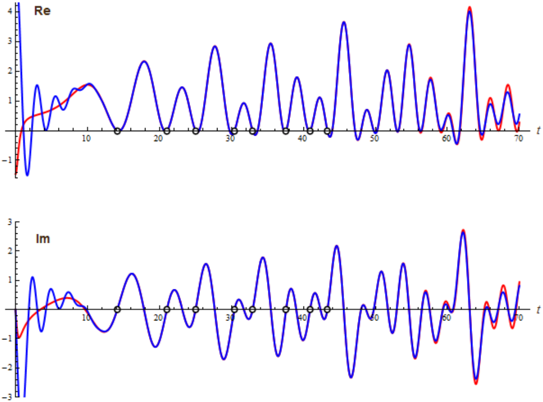

$$
\begin{aligned}
0 &= \Omega_{17}(\rho_9 - 4.396\ldots \cdot 10^{-3} + 5.711\ldots \cdot 10^{-3}\mathrm{i}) \\
0 &= \Omega_{17}(\rho_{10} - 1.141\ldots \cdot 10^{-2} - 3.345\ldots \cdot 10^{-3}\mathrm{i}) \\
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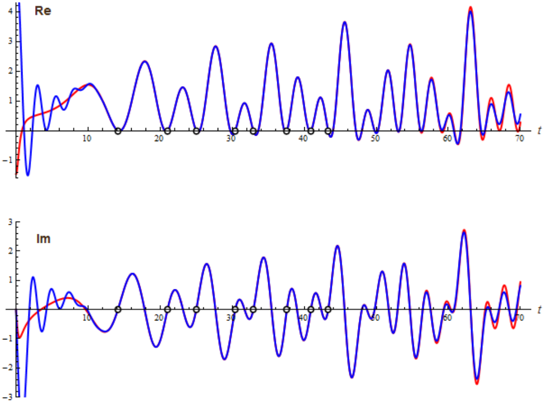

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- they know (approximate values of several) next non-trivial zeros

Case $M = 1550$, $N = 2M + 1 = 3101$: zeroes of $\Omega_{3101}(s)$

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0 &= \Omega_N(-4 + 2.013\ldots \cdot 10^{-1504}) \\
0 &= \Omega_N(-6 - 1.158\ldots \cdot 10^{-1498}) \\
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0 &= \Omega_N(-10 - 1.316\ldots \cdot 10^{-1487}) \\
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$N = 101$. Coefficients $\delta_{101,n}$, red for even n, blue for odd n

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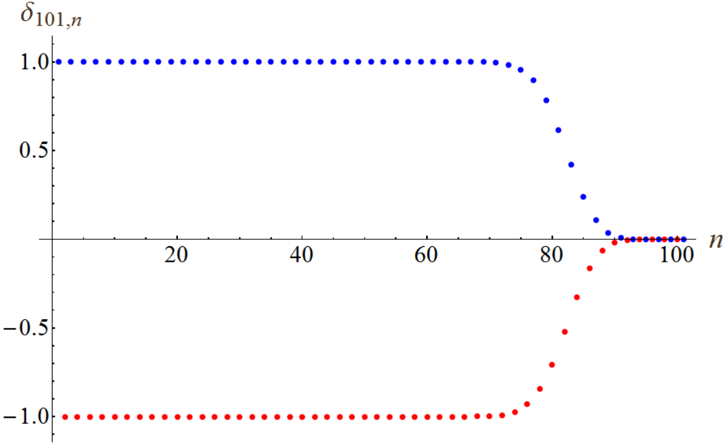

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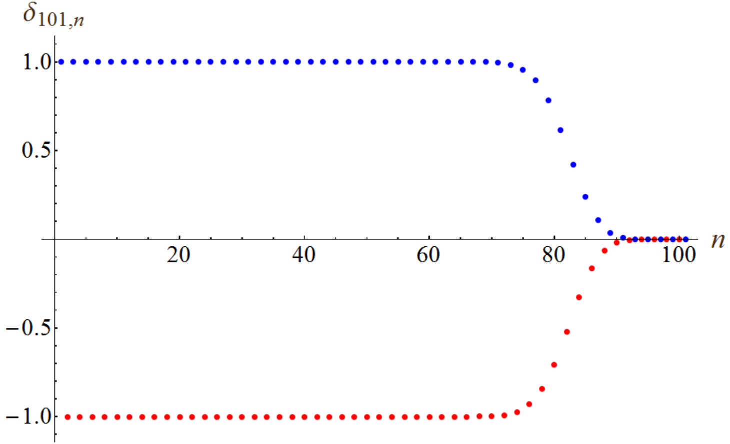

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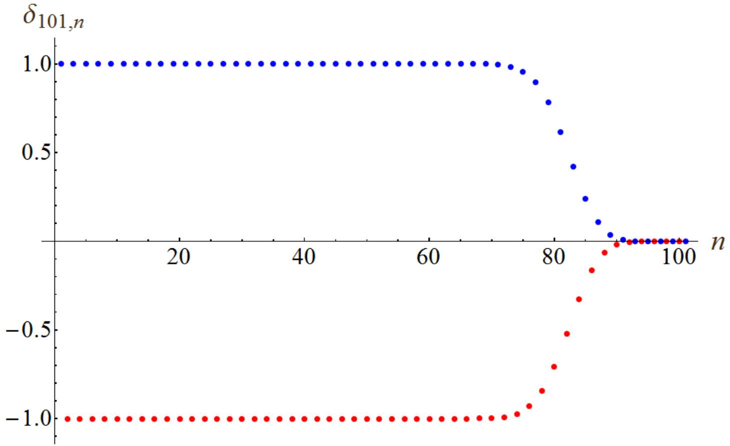

Non-trivial zeta zeros know a lot, in particular

- they know about the alternating zeta function, that is, the eta function $\eta(s)$

Why $\eta(s)$?

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$$\Omega_N(s) = 1 + \sum_{n=2}^{N} \delta_{N,n} n^{-s}$$

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$$\sum_{n=1}^{\infty} (-1)^{n+1} n^{-s} = \eta(s) = (1 - 2 \cdot 2^{-s})\zeta(s)$$

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0 &= \Omega_N(s_{50} - 5.481\ldots \cdot 10^{-133} - 5.546\ldots \cdot 10^{-133}\mathrm{i}) \\
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Non-trivial ZETA zeros know a lot, in particular

- they know (approximate values of several) initial extra zeros of the ETA function

$N = 3101$, coefficients $\delta_{3101,n}$, red for even n, blue for odd n

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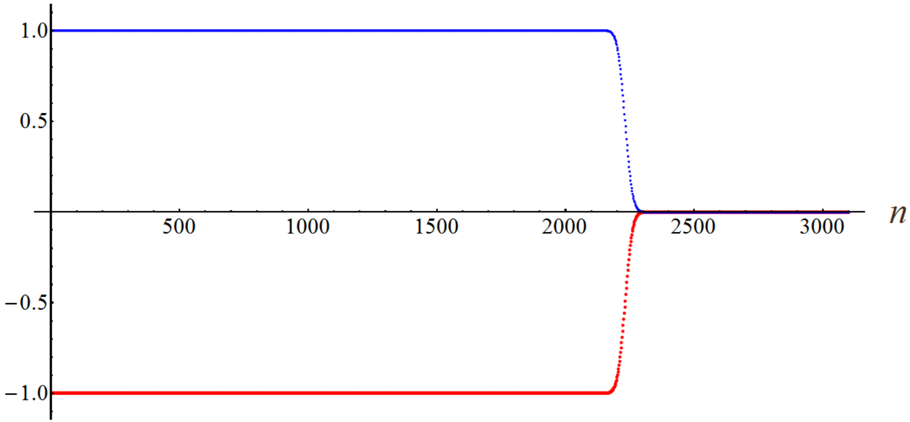

$\delta_{3101,n}$

$$\delta_{3101,n} - 1$$

$$|\delta_{3101,n} - 1|$$

$\log_{10}|\delta_{3101,n} - 1|$

Plot of $\log_{10} |\delta_{3101,n} - 1|$

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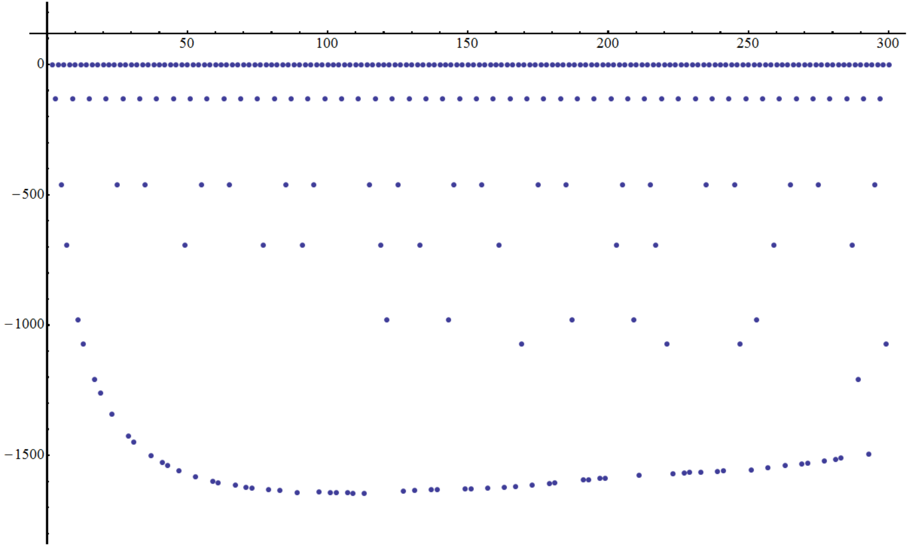
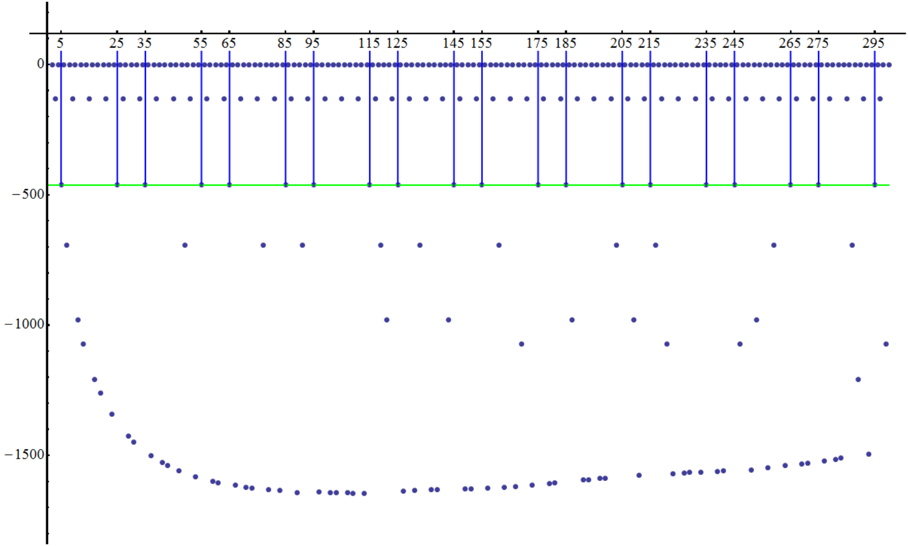
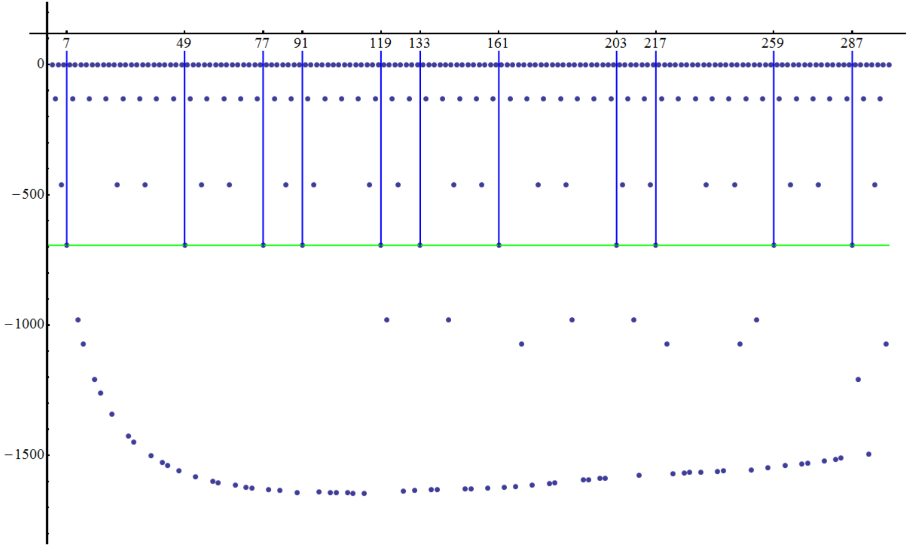


Plot of $\log_{10}|\delta_{3101,n} - 1|$



Plot of $\log_{10} |\delta_{3101,n} - 1|$



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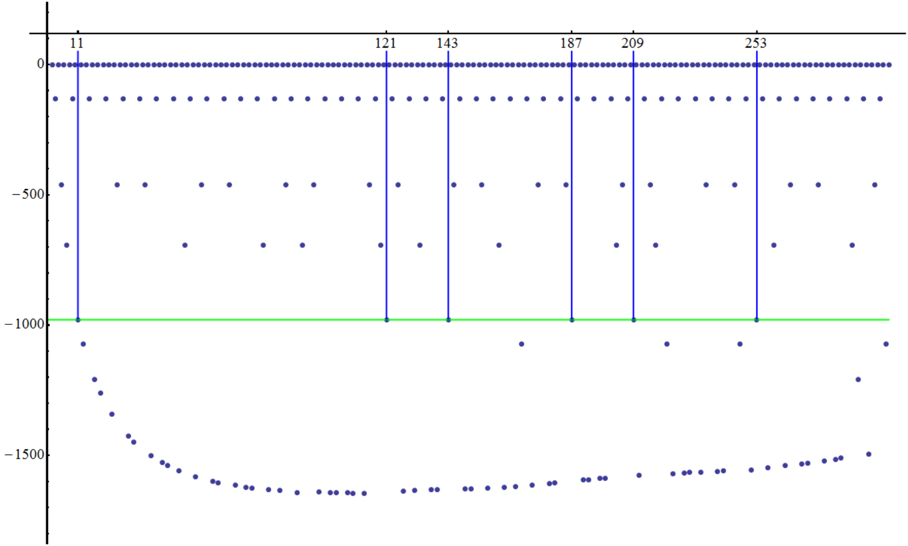

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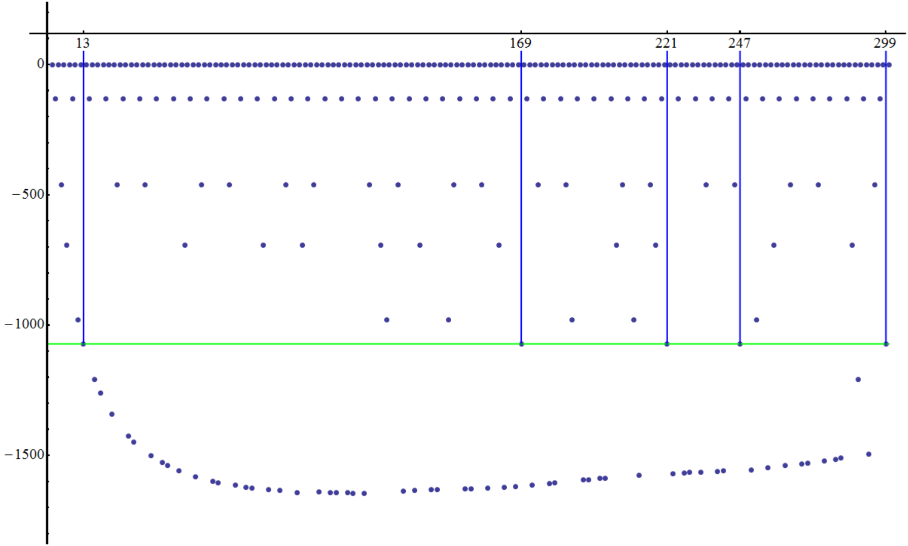

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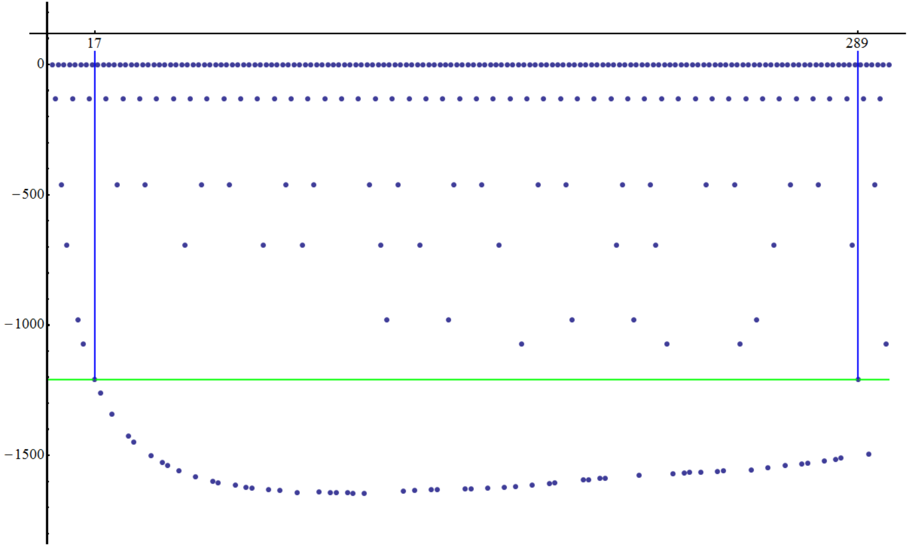

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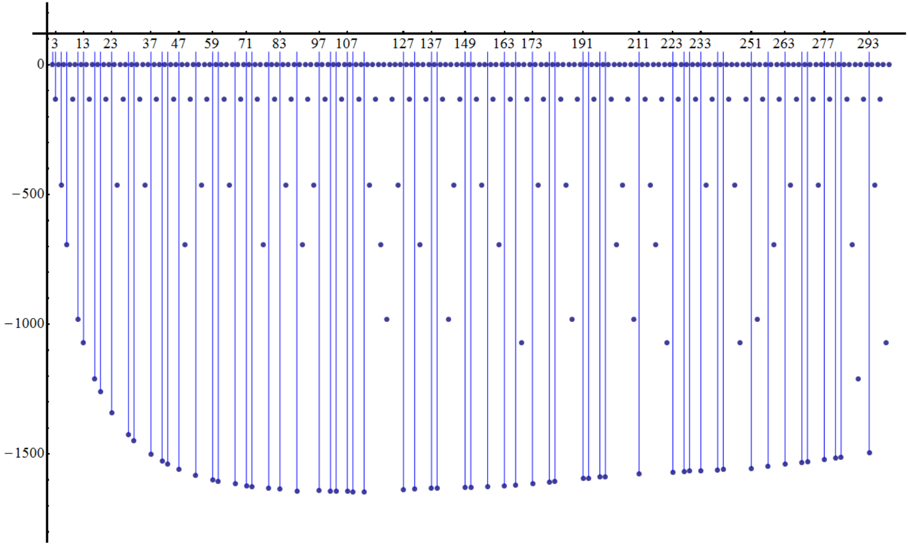
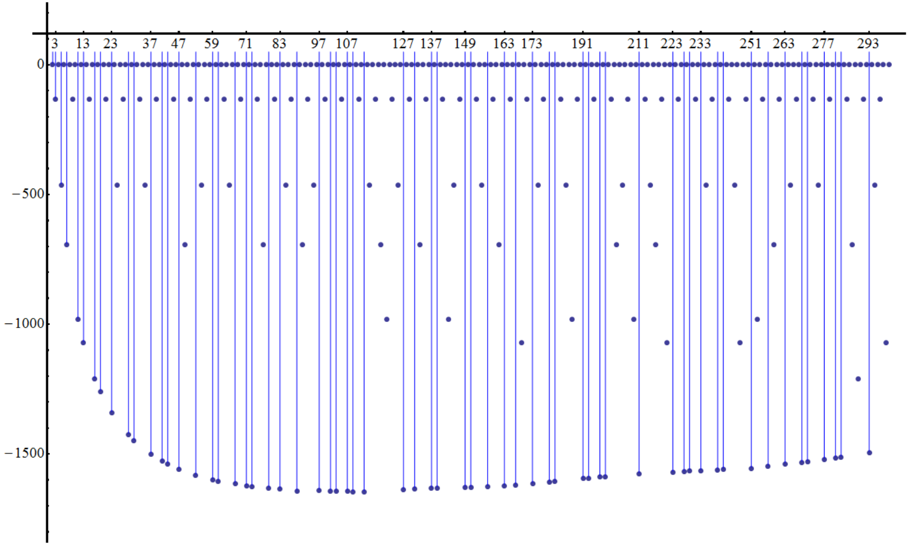


Plot of $\log_{10} |\delta_{3101,n} - 1| =$ Sieve of Eratosthenes



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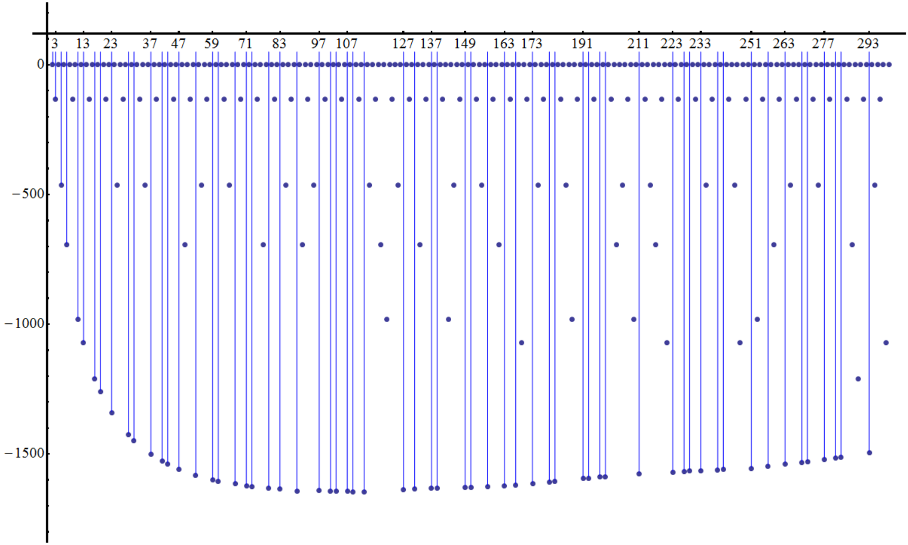

Non-trivial zeta zeros know a lot, in particular

- they know the Sieve of Eratosthenes

Finer Structure: Plot of $\log_{10}|\delta_{10001,3m} - \delta_{10001,3}|$

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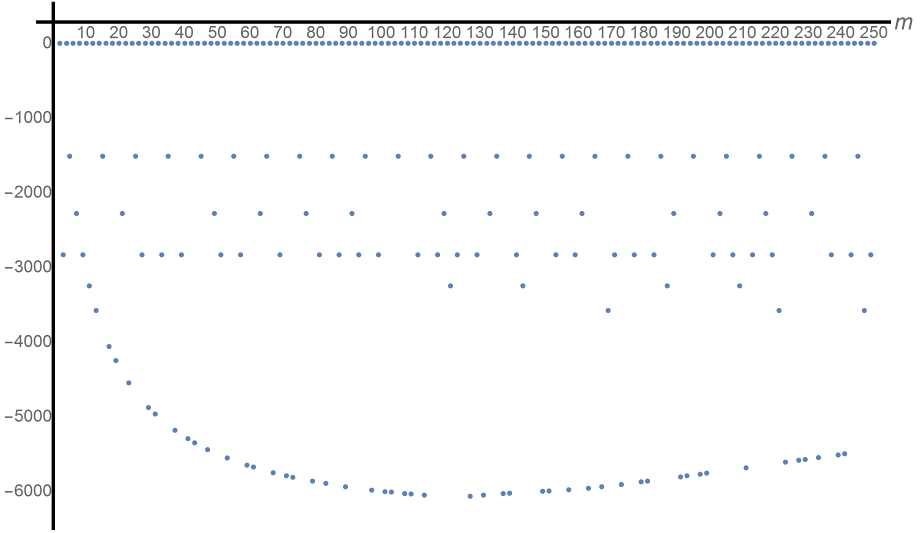

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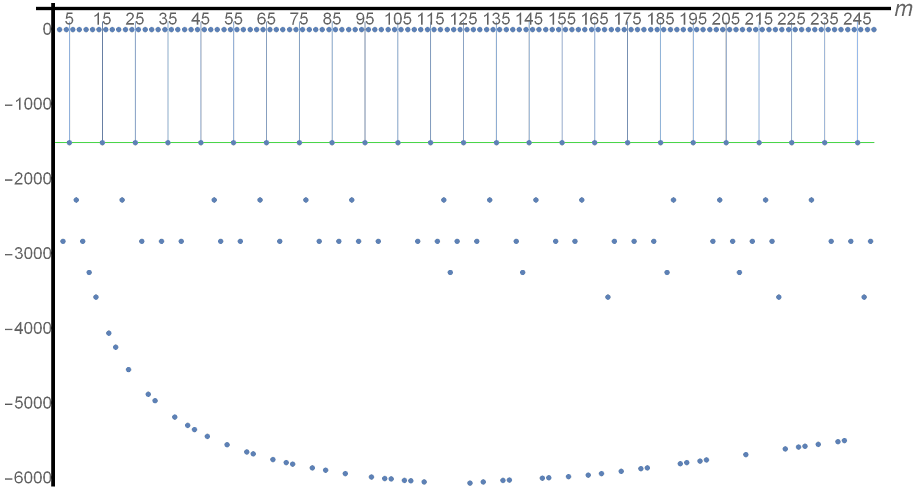

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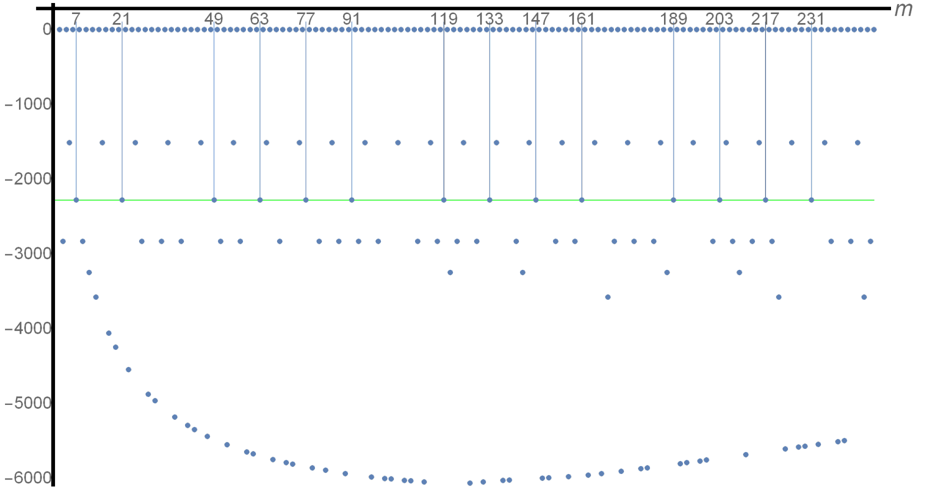

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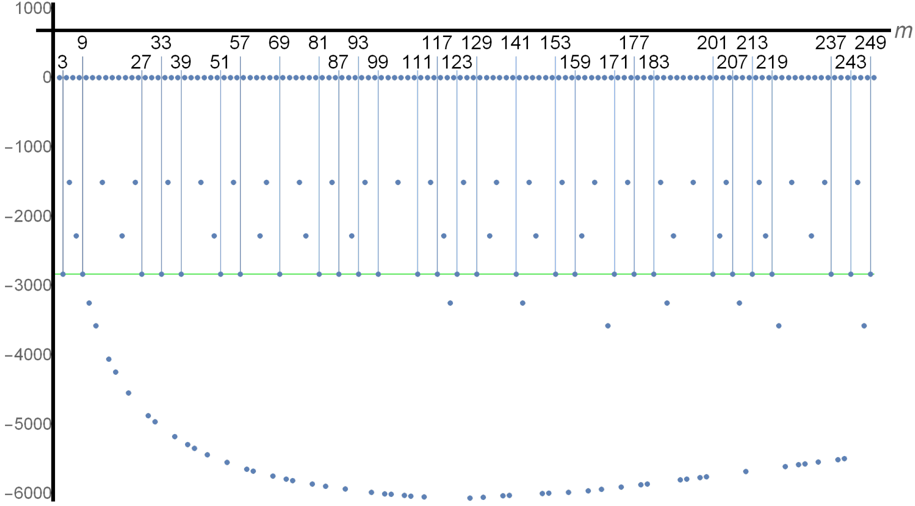

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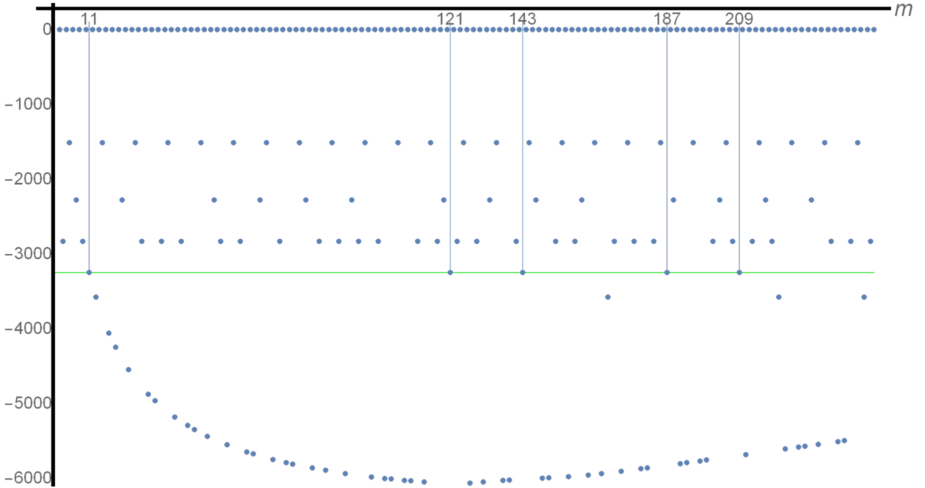

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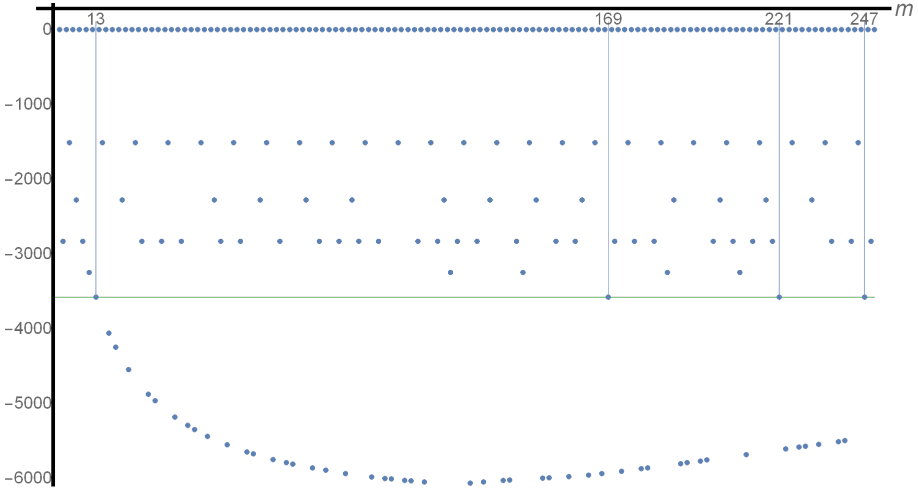

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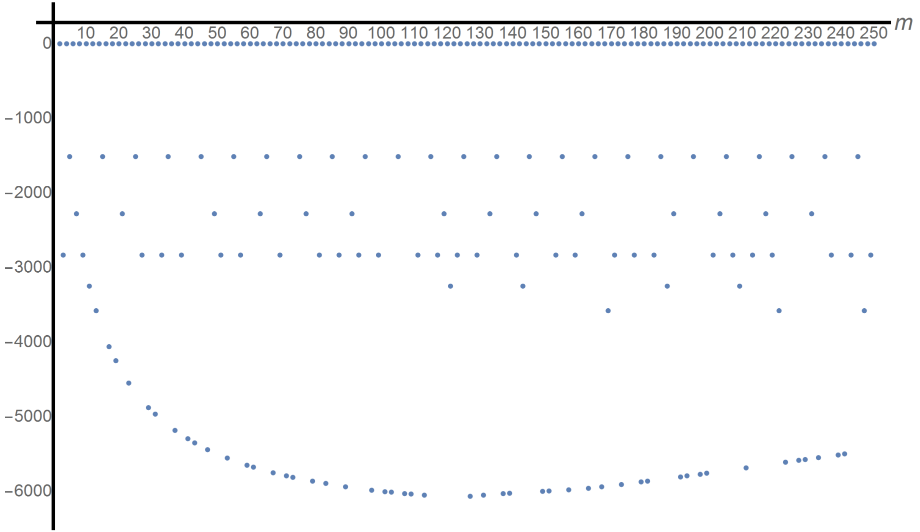
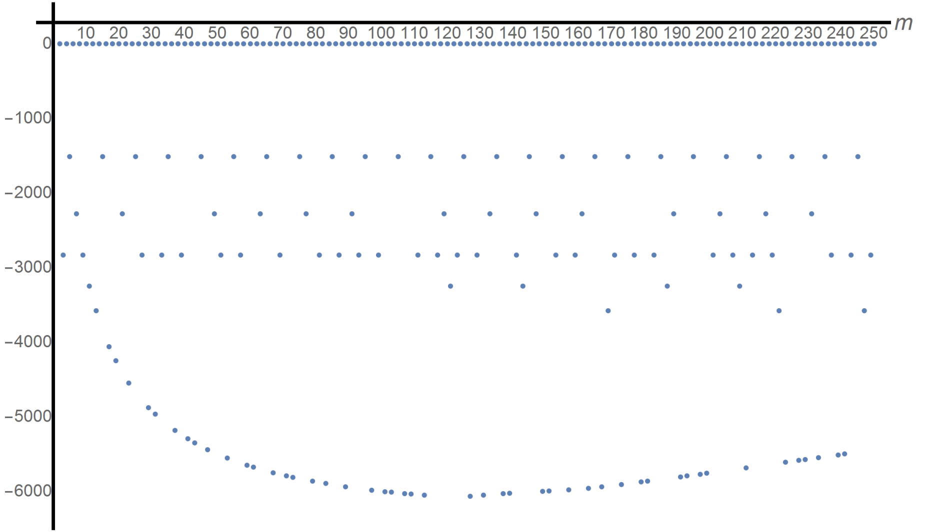


Finer Structure: Plot of $\log_{10} |\delta_{10001,3m} - \delta_{10001,3}|$

$= $ Eratosthenes Sieve with primes order 2, 5, 7, 3, 11, 13, ...



Expected Fractal Structure

Let n range over an arithmetical progression $d, 2d, \ldots, md, \ldots$ with

$$d = 2^{k_2} 3^{k_3} 5^{k_5} \ldots$$

Corresponding Eratosthenes sublevel splits according to the divisibility of m by q_1, q_2, $\ldots$ where these prime numbers are ordered in such a way that

$$q_1^{k_{q_1}+1} < q_2^{k_{q_2}+1} < \cdots < q_j^{k_{q_j}+1} < \ldots$$

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In the previous example $m = 3$, hence $k_2 = 0$, $k_3 = 1$, $k_5 = k_7 = \dots = 0$ and $q_1 = 2,\ q_2 = 5,\ q_3 = 7,\ q_4 = 3,\ q_5 = 11,\ q_6 = 13,\ \dots$ according to

$$2^1 < 5^1 < 7^1 < 3^2 < 11^1 < 13^1 < \dots$$

Dual sieve

Dual sieve: Plot of $\log_{10}|\sum_{n=1}^{m}\delta_{N,n}|$ при $N = 5001$

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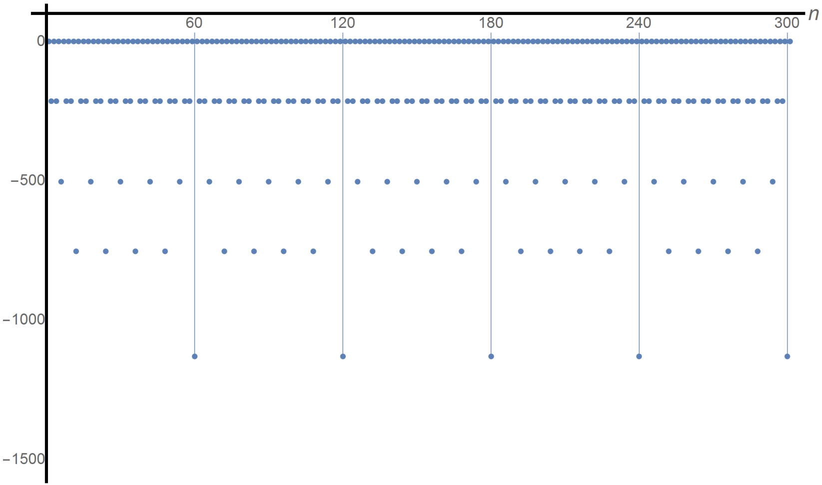

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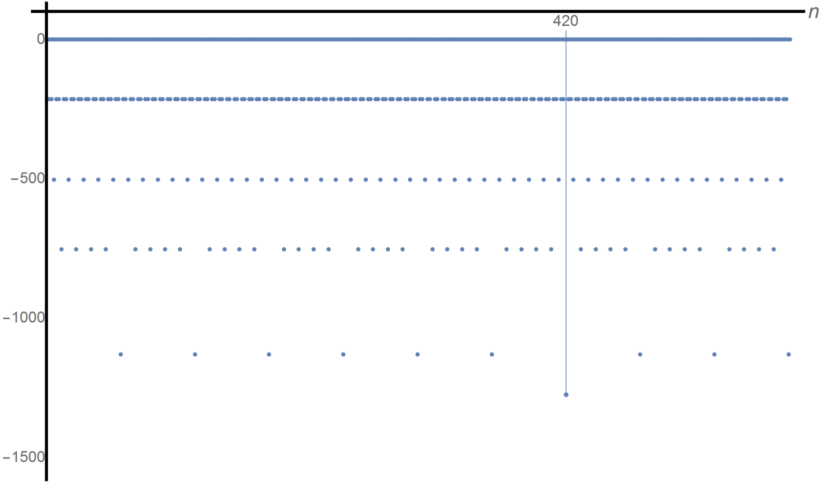

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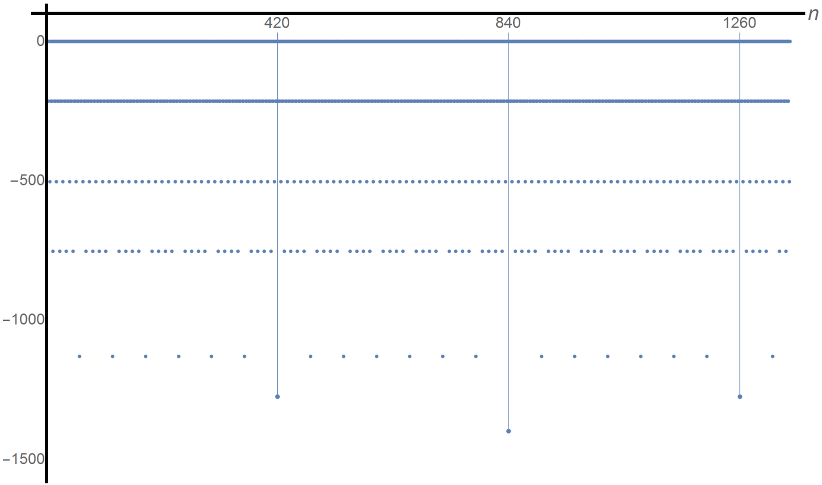

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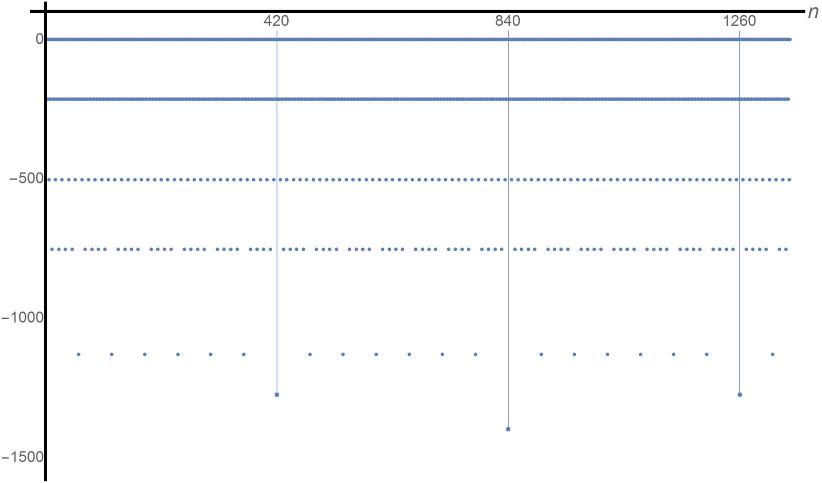
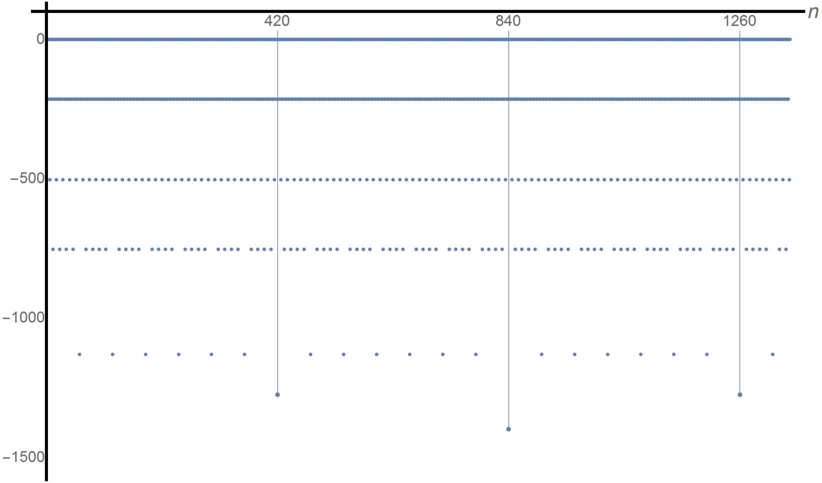


Inheritable divisor: $k_{\leq} | m$

Dual sieve: Plot of $\log_{10}\left|\sum_{n=1}^{m}\delta_{N,n}\right|$ при $N=5001$



Inheritable divisor: $k_{\leq}|m \iff 1|m \;\&\; 2|m \;\&\; 3|m \;\&\; \ldots \;\&\; k|m$

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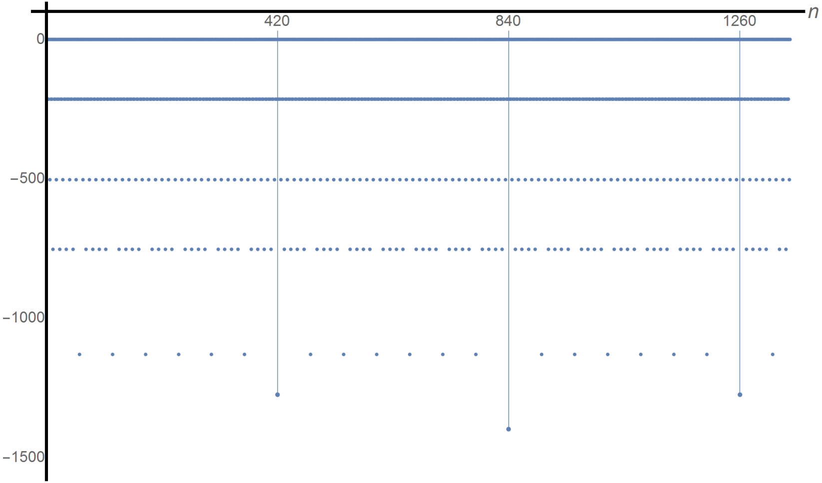

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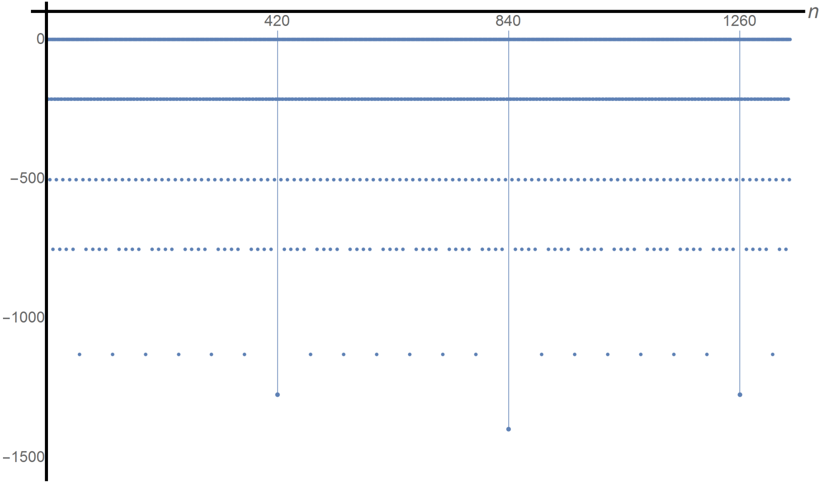

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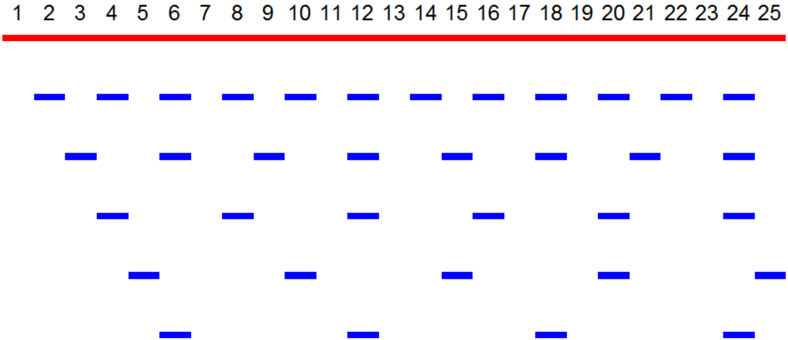
Sieve of Eratosthenes



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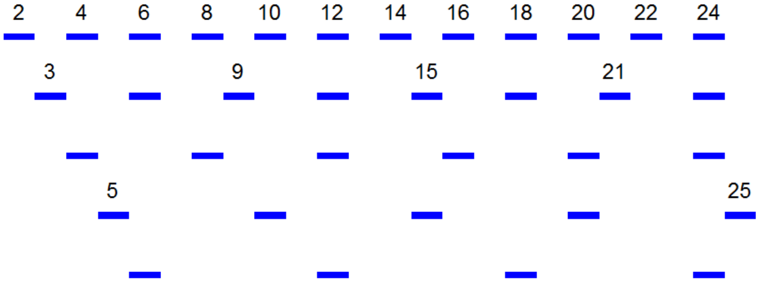

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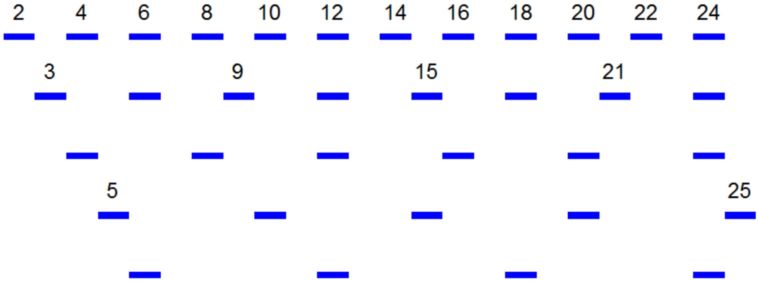
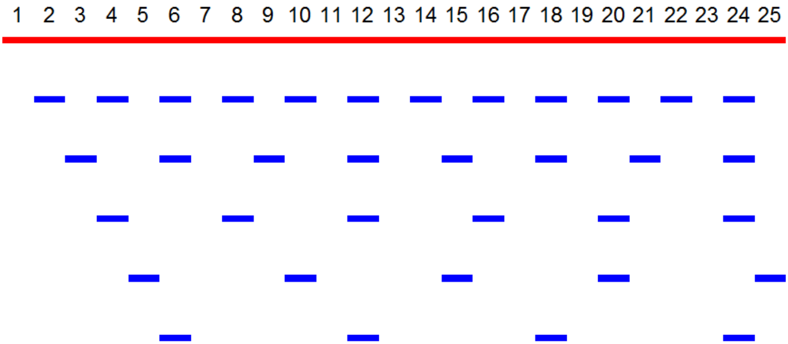
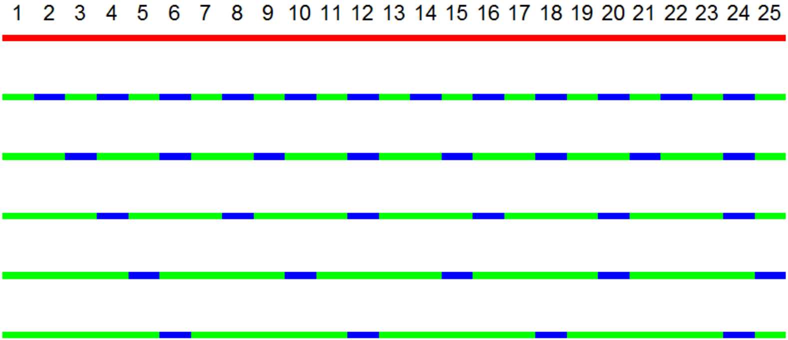


$$\log|\delta_{N,n} - 1|$$

Sieve of Eratosthenes (repeated)



Sieve of Eratosthenes vs dual sieve



Dual sieve

Dual sieve

Dual sieve

$$\log \left| \sum_{n=1}^{m} \delta_{N,n} \right|$$

Davenport–Heilbronn function

$$f(s) = \sum_{n=1}^{\infty} d(n) n^{-s}$$

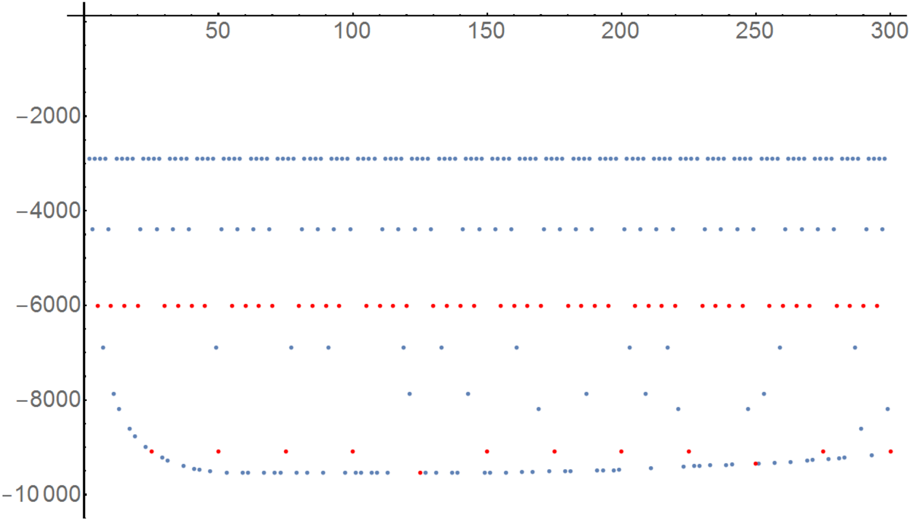
where

$$d(n) = \begin{cases} 0, & \text{if } n \equiv 0 \pmod 5 \\ 1, & \text{if } n \equiv 1 \pmod 5 \\ \tau, & \text{if } n \equiv 2 \pmod 5 \\ -\tau, & \text{if } n \equiv 3 \pmod 5 \\ -1, & \text{if } n \equiv 4 \pmod 5 \end{cases}$$

and

$$\tau = \frac{-2 + \sqrt{10 - 2\sqrt{5}}}{-1 + \sqrt{5}}$$

Sieve of Eratosthenes for $f(s)$ (N=7999)



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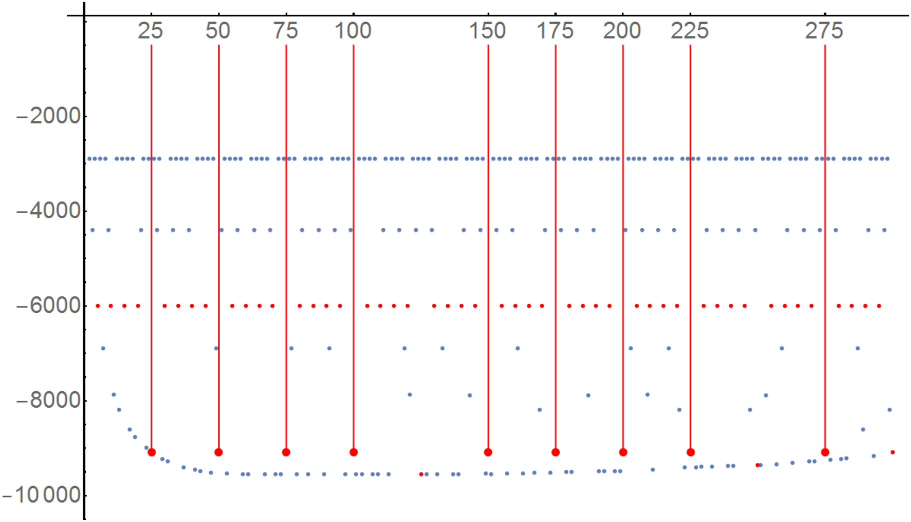

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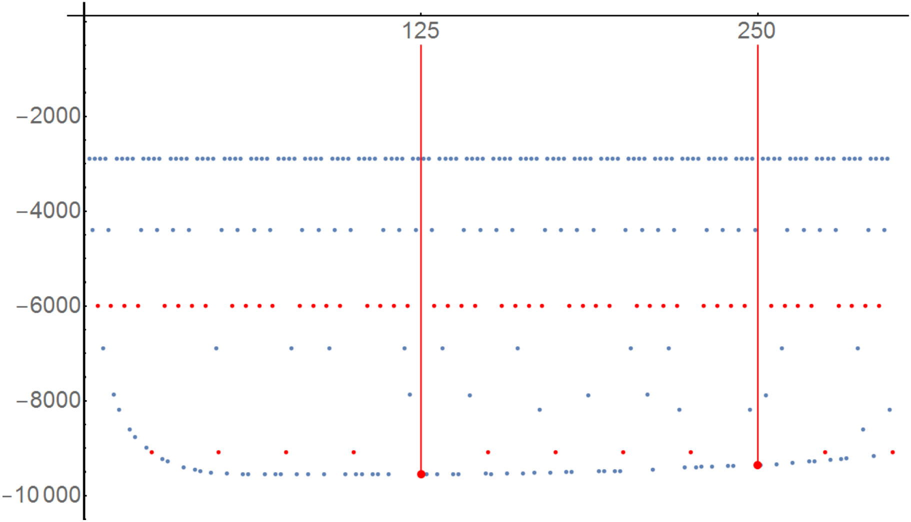
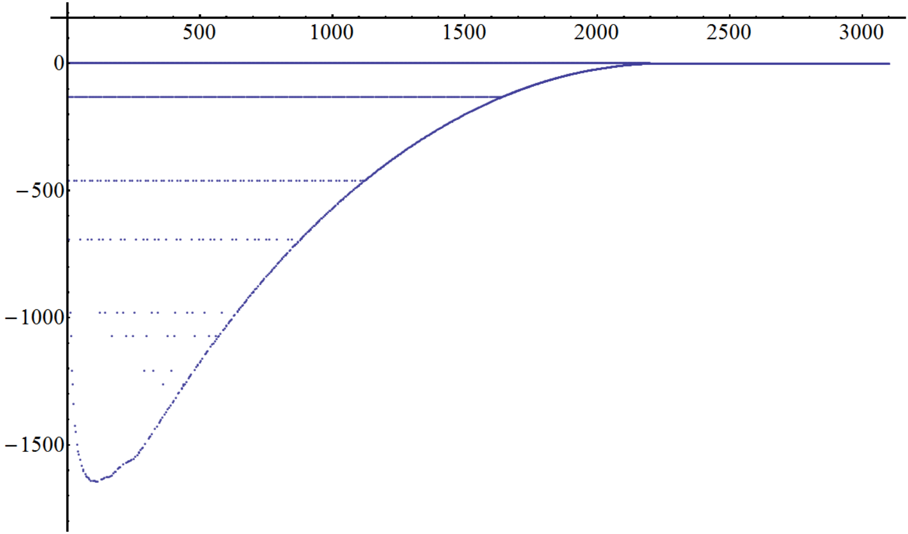
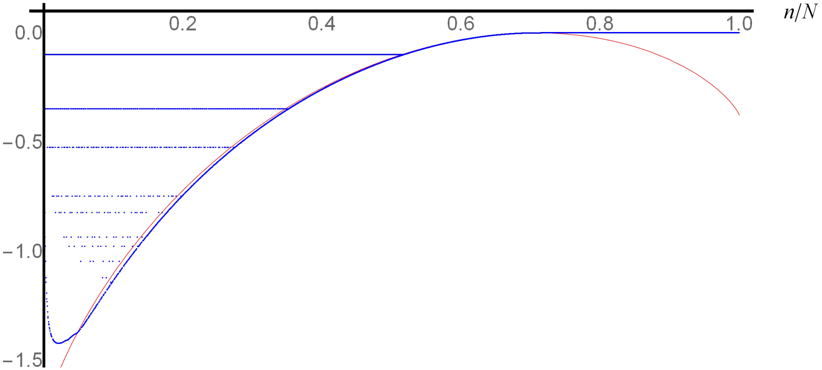


Total plot of $\log_{10}|\delta_{3101,n} - 1|$

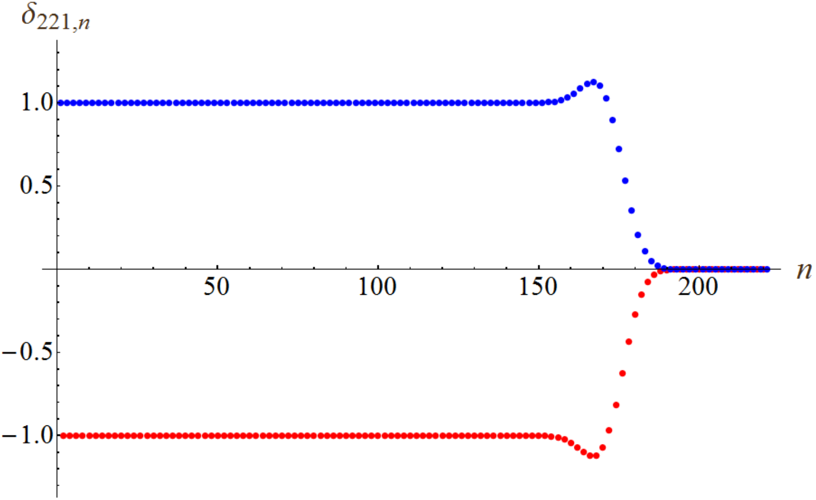


Scaled plot of $(\ln|\delta_{N,n} + (-1)^n|)/N$ for $N = 10001$

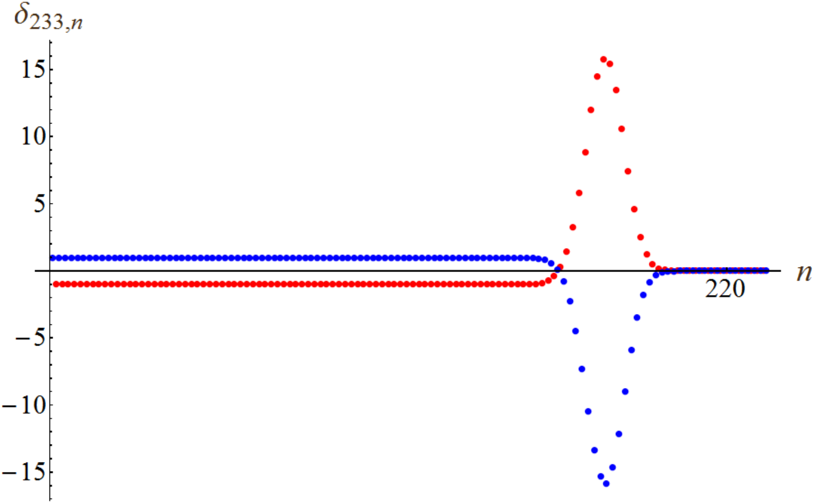


$$\ell(\tfrac{n}{N}) = (\tfrac{n}{N} - 1)\ln(1 - \tfrac{n}{N}) - 2\tfrac{n}{N}\ln(\tfrac{n}{N}) + (\tfrac{n}{N} + 1)\ln(\tfrac{n}{N} + 1) - \ln(2\sqrt{2} + 3)$$

$N = 221$, coefficients $\delta_{221,n}$, red for even n, blue for odd n



$N = 233$, coefficients $\delta_{233,n}$, red for even n, blue for odd n



Where to look for

https://logic.pdmi.ras.ru/~yumat

https://logic.pdmi.ras.ru/~yumat/publications/publications.php?istate=state_show_paper&imykey=94&ilang=eng

https://logic.pdmi.ras.ru/~yumat/publications/publications.php?istate=state_show_paper&imykey=98&ilang=eng

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THANK YOU FOR ATTENTION!

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