On Round-Robin Tournaments with a Unique Maximum Score

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Outline

- Background and Problem Statement
- Main Result
Part 1: Introduction and Background
Richard Arnold Epstein Problem

Richard Arnold Epstein (1927-2016) published the first edition of "The Theory of Gambling and Statistical Logic" in 1967. He introduced some material on round-robin tournaments (complete oriented graphs) with \( n \) labeled vertices in Chapter 9; in particular, he stated, without proof, that the probability that there is a unique vertex with the maximum score tends to one as \( n \) tends to infinity.

In a classical round-robin tournament, each of $n$ players wins or loses a game against each of the other $n - 1$.

Let $X_{ij}$ equal 1 or 0 according as player $i$ wins or loses the game played against player $j$, for $1 \leq i, j \leq n$, $i \neq j$, where

$$X_{ij} + X_{ji} = 1.$$ 

We assume that all $\binom{n}{2}$ pairs $(X_{ij}, X_{ji})$ are independently distributed with

$$P(X_{ij} = 1) = P(X_{ji} = 0) = 1/2.$$ 

Let

$$s_i = \sum_{j=1, j\neq i}^{n} X_{ij}$$

denote the score of player $i$, $1 \leq i \leq n$, after playing against all the other $n - 1$ players.
$(s_1, s_2, \ldots, s_n)$ is the score sequence of the tournament.

$$G(n) = \prod_{1 \leq i < j \leq n} (a_i + a_j).$$

For example,

$$G(3) = a_1^2a_2 + a_1a_2^2 + a_2^2a_3 + a_1a_3^2 + a_2^2a_3 + a_2a_3^2 + 2a_1a_2a_3,$$

(ordered) score sequence $(1, 1, 1)$ occurs twice

$(0, 1, 2)$ occurs six times in the $2^3 = 8$ possible outcomes.

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Maximum Score

- Let $r_n$ denote the probability that an ordinary tournament with $n$ labeled vertices has a unique vertex with maximum score, assuming all the $2^{(n/2)}$ such tournaments are equally likely.

- Epstein (1963), p. 353:

$$r_4 = 0.5, r_5 = 0.586, r_6 = 0.627, r_7 = 0.581, r_8 = 0.634$$

with no explanation of how these numbers were calculated.

- However, the paper of David (1959)\(^a\) is included among the references Epstein gave at the end of the chapter containing these values. From David’s Table 1 follows these values except that the value for $r_8 = 160,241,152/2^{28} = 0.596$.

It follows from MacMahon’s data that

\[ r_9 = \frac{42,129,744,768}{2^{36}} = 0.613. \]

Doron Zeilberger had extended MacMahon’s work and had generated the score vectors and their frequencies for tournaments with up to 15 vertices using the Maple program.

## Monte-Carlo Simulations

<table>
<thead>
<tr>
<th>$n$</th>
<th>$M$</th>
<th>$r_n$</th>
<th>$\hat{r}_n(M)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>$10^6$</td>
<td>0.5</td>
<td>0.5003 \ldots</td>
</tr>
<tr>
<td>5</td>
<td>$10^6$</td>
<td>$600/2^{10} = 0.5859 \ldots$</td>
<td>0.5862 \ldots</td>
</tr>
<tr>
<td>6</td>
<td>$10^6$</td>
<td>$20,544/2^{15} = 0.6269 \ldots$</td>
<td>0.6262 \ldots</td>
</tr>
<tr>
<td>7</td>
<td>$10^6$</td>
<td>$1,218,224/2^{21} = 0.5808 \ldots$</td>
<td>0.5806 \ldots</td>
</tr>
<tr>
<td>8</td>
<td>$10^6$</td>
<td>$160,241,152/2^{28} = 0.5969 \ldots$</td>
<td>0.5966 \ldots</td>
</tr>
<tr>
<td>9</td>
<td>$10^6$</td>
<td>$42,129,744,768/2^{36} = 0.6130 \ldots$</td>
<td>0.6129 \ldots</td>
</tr>
<tr>
<td>10</td>
<td>$10^6$</td>
<td>$21,293,228,876,800/2^{45} = 0.6051 \ldots$</td>
<td>0.6054 \ldots</td>
</tr>
<tr>
<td>11</td>
<td>$10^6$</td>
<td>$22,220,602,090,444,032/2^{55} = 0.6167 \ldots$</td>
<td>0.6169 \ldots</td>
</tr>
<tr>
<td>12</td>
<td>$10^6$</td>
<td>$45,959,959,305,969,143,808/2^{66} = 0.6228 \ldots$</td>
<td>0.6231 \ldots</td>
</tr>
<tr>
<td>13</td>
<td>$10^6$</td>
<td></td>
<td>0.6240 \ldots</td>
</tr>
<tr>
<td>14</td>
<td>$10^6$</td>
<td></td>
<td>0.6323 \ldots</td>
</tr>
<tr>
<td>15</td>
<td>$10^6$</td>
<td></td>
<td>0.6355 \ldots</td>
</tr>
<tr>
<td>30</td>
<td>$10^6$</td>
<td></td>
<td>0.6881 \ldots</td>
</tr>
<tr>
<td>50</td>
<td>$10^6$</td>
<td></td>
<td>0.7290 \ldots</td>
</tr>
<tr>
<td>100</td>
<td>$10^6$</td>
<td></td>
<td>0.7808 \ldots</td>
</tr>
<tr>
<td>500</td>
<td>$10^4$</td>
<td></td>
<td>0.8673 \ldots</td>
</tr>
<tr>
<td>1,000</td>
<td>$10^4$</td>
<td></td>
<td>0.8996 \ldots</td>
</tr>
<tr>
<td>10,000</td>
<td>300</td>
<td></td>
<td>0.9533 \ldots</td>
</tr>
</tbody>
</table>
Noga Alon referred us to a paper Paul Erdős and Robin Wilson that contained a Lemma stating that almost all labeled graphs in which pairs of vertices are joined by an edge with probability $1/2$ have a unique vertex of maximum degree.

\cite{ErdosWilson1977}

Main Result

\[ t_{n-1} = \frac{n-1}{2} + x_{n-1} \sqrt{\frac{n-1}{4}}, \quad x_{n-1} = \sqrt{(2 \log(n-1) - (1 + \epsilon) \log(\log(n-1)))}. \]

\[ s^* = \max \{s_1, \ldots, s_n\} \]

in a random n-vertex tournament \( T_n \).

Result 1

(i) \( P(s^* > t_{n-1}) \to 1 \) as \( n \to \infty \); 

(ii) If \( W_n = W_n(T_n) \) denotes the number of ordered pairs of distinct vertices \( u \) and \( v \) in \( T_n \) such that \( s_u = s_v = h \) for some integer \( h \) such that \( t_{n-1} \leq h \leq n - 1 \), then

\[ P(W_n > 0) \to 0 \quad \text{as} \quad n \to \infty. \]

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Proof Sketch of (i)

- Let $Y_{t_{n-1}} = \sum_{j=1}^{n} I(s_j > t_{n-1})$. 

\[ E(Y_{t_{n-1}}) \sim n \frac{(\log(n - 1))^{\epsilon/2}}{\sqrt{4\pi(n - 1)}}. \]

\[ \text{Var}(Y_{t_{n-1}}) \leq E(Y_{t_{n-1}}). \]

- Therefore,

\[ P(Y_{t_{n-1}} = 0) \leq P\left(|Y_{t_{n-1}} - E(Y_{t_{n-1}})| \geq E(Y_{t_{n-1}})\right) \leq \frac{\text{Var}(Y_{t_{n-1}})}{(E(Y_{t_{n-1}}))^2} \]

\[ \leq \frac{1}{E(Y_{t_{n-1}})} \to 0, \]

as $n \to \infty$.

Since $P(s^* > t_{n-1}) = P(Y_{t_{n-1}} > 0)$, we obtain (i).
Proof Sketch of (ii)

- Recall that

\[ W_n = \sum_{1 \leq v < u \leq n} I(t_{n-1} < s_u = s_v). \]

- Appealing to \( W_n = W_nI(W_n > 0) \geq I(W_n > 0) \), we find that

\[ P(W_n > 0) \leq E(W_n) \to 0, \]

as required.
Thank You !