On Round-Robin Tournaments with a Unique Maximum Score

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Rutgers Experimental Mathematics Seminar

November 17, 2022

Outline

- Background and Problem Statement
- Main Result

Part 1: Introduction and Background

Richard Arnold Epstein Problem[a]

Richard Arnold Epstein (1927-2016) published the first edition of "The Theory of Gambling and Statistical Logic" in 1967. He introduced some material on round-robin tournaments (complete oriented graphs) with n labeled vertices in Chapter 9; in particular, he stated, without proof, that the probability that there is a unique vertex with the maximum score tends to one as n tends to infinity.

[a] Epstein, R. A. (1967). The theory of gambling and statistical logic. First edition. Academic Press, New York.

Classical Round-Robin Tournament

- In a classical round-robin tournament, each of n players wins or loses a game against each of the other $n-1$.
- Let X_{ij} equal 1 or 0 according as player i wins or loses the game played against player j, for $1 \leqslant i, j \leqslant n$, $i \neq j$, where
$$X_{ij} + X_{ji} = 1.$$
- We assume that all $\binom{n}{2}$ pairs (X_{ij}, X_{ji}) are independently distributed with
$$P(X_{ij} = 1) = P(X_{ji} = 0) = 1/2.$$
- Let
$$s_i = \sum_{j=1, j\neq i}^{n} X_{ij}$$
denote the score of player $i, 1 \leqslant i \leqslant n$, after playing against all the other $n-1$ players.

P.A. MacMahon: Counting Score Sequences[a]

- $(s_1, s_2, \ldots, s_n)$ is the score sequence of the tournament.
- $$G(n) = \prod_{1 \leqslant i < j \leqslant n} (a_i + a_j).$$
- For example,

$$G(3) = a_1^2 a_2 + a_1 a_2^2 + a_1^2 a_3 + a_1 a_3^2 + a_2^2 a_3 + a_2 a_3^2 + 2a_1 a_2 a_3,$$

(ordered) score sequence $(1, 1, 1)$ occurs twice

$(0, 1, 2)$ occurs six time in the $2^{\binom{3}{2}} = 8$ possible outcomes.

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[a] MacMahon, P. A. (1923). An American tournament treated by the calculus of symmetric functions. Quart. J. Pure Appl. Math. **v**ol. XLIX, No. 193, 1–36.

Maximum Score

- Let r_n denote the probability that an ordinary tournament with n labeled vertices has a unique vertex with maximum score, assuming all the $2^{\binom{n}{2}}$ such tournaments are equally likely.
- Epstein (1963), p. 353:

$$r_4 = 0.5, r_5 = 0.586, r_6 = 0.627, r_7 = 0.581, r_8 = 0.634$$

  with no explanation of how these numbers were calculated.
- However, the paper of David(1959) [a] is included among the references Epstein gave at the end of the chapter containing these values. From David's Table 1 follows these values except that the value for $r_8 = 160,241,152/2^{28} = 0.596$.

[a] David, H. A. (1959). Tournaments and Paired Comparisons. Biometrika **4**6, 139–149.

Percy MacMahon and Doron Zeilberger [a]

It follows from MacMahon's data that

$$r_9 = 42,129,744,768/2^{36} = 0.613.$$

Doron Zeilberger had extended MacMahon's work and had generated the score vectors and their frequencies for tournaments with up to 15 vertices using the Maple program.

[a] Zeilberger, D. (2016). On the Most Commonly-Occuring Score Vectors of American Tournaments of n-players, and their Corresponding Records. https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/Percy.html

Monte-Carlo Simulations

n	M	r_n	$\hat{r}_n(M)$
4	10^6	0.5	$0.5003\cdots$
5	10^6	$600/2^{10} = 0.5859\cdots$	$0.5862\cdots$
6	10^6	$20,544/2^{15} = 0.6269\cdots$	$0.6262\cdots$
7	10^6	$1,218,224/2^{21} = 0.5808\cdots$	$0.5806\cdots$
8	10^6	$160,241,152/2^{28} = 0.5969\cdots$	$0.5966\cdots$
9	10^6	$42,129,744,768/2^{36} = 0.6130\cdots$	$0.6129\cdots$
10	10^6	$21,293,228,876,800/2^{45} = 0.6051\cdots$	$0.6054\cdots$
11	10^6	$22,220,602,090,444,032/2^{55} = 0.6167\cdots$	$0.6169\cdots$
12	10^6	$45,959,959,305,969,143,808/2^{66} = 0.6228\cdots$	$0.6231\cdots$
13	10^6		$0.6240\cdots$
14	10^6		$0.6323\cdots$
15	10^6		$0.6355\cdots$
30	10^6		$0.6881\cdots$
50	10^6		$0.7290\cdots$
100	10^6		$0.7808\cdots$
500	10^4		$0.8673\cdots$
1,000	10^4		$0.8996\cdots$
10,000	300		$0.9533\cdots$

Erdős and Wilson (1977) [a]

Noga Alon referred us to a paper Paul Erdős and Robin Wilson that contained a Lemma stating that almost all labeled graphs in which pairs of vertices are joined by an edge with probability $1/2$ have a unique vertex of maximum degree.

[a] Erdős, P., Wilson, R. J. (1977). On the chromatic index of almost all graphs. J. Combinatorial Theory Ser. B **2**3, 255–257.

Main Result

$$t_{n-1} = \frac{n-1}{2} + x_{n-1}\sqrt{\frac{n-1}{4}},\ x_{n-1} = \sqrt{(2\log(n-1) - (1+\epsilon)\log(\log(n-1)))}.$$

$$s^\star = \max\{s_1, \ldots, s_n\}$$

in a random n-vertex tournament T_n.

Result 1

(i) $P(s^\star > t_{n-1}) \to 1$ *as* $n \to \infty$ [a];

(ii) *If* $W_n = W_n(T_n)$ *denotes the number of ordered pairs of distinct vertices* u *and* v *in* T_n *such that* $s_u = s_v = h$ *for some integer* h *such that* $t_{n-1} \leqslant h \leqslant n-1$, *then*

$$P(W_n > 0) \to 0 \quad \text{as} \quad n \to \infty.$$

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[a] Huber, P. J. (1963). A remark on a paper of Trawinski and David entitled: Selection of the best treatment in a paired comparison experiment. Ann.Math.Statist. **3**4, 92–94.

Proof Sketch of (i)

- Let $Y_{t_{n-1}} = \sum_{j=1}^{n} I\left(s_j > t_{n-1}\right)$.

$$E\left(Y_{t_{n-1}}\right) \sim n\frac{(\log(n-1))^{\epsilon/2}}{\sqrt{4\pi}(n-1)}.$$

- $$Var(Y_{t_{n-1}}) \leqslant E(Y_{t_{n-1}}).$$

- Therefore,

$$P\left(Y_{t_{n-1}} = 0\right) \leqslant P\left(|Y_{t_{n-1}} - E(Y_{t_{n-1}})| \geqslant E(Y_{t_{n-1}})\right) \leqslant \frac{Var(Y_{t_{n-1}})}{(E(Y_{t_{n-1}}))^2}$$

$$\leqslant \frac{1}{E(Y_{t_{n-1}})} \to 0,$$

  as $n \to \infty$.

Since $P(s^\star > t_{n-1}) = P(Y_{t_{n-1}} > 0)$, we obtain (i).

Proof Sketch of (ii)

- Recall that

$$W_n = \sum_{1 \leqslant v < u \leqslant n} I(t_{n-1} < s_u = s_v).$$

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$$E(W_n) \sim \frac{(\log(n-1))^{1/2+\epsilon}}{\pi\sqrt{2(n-1)}} \to 0,$$

  as $n \to \infty$.

- Appealing to $W_n = W_n I(W_n > 0) \geqslant I(W_n > 0)$, we find that

$$P(W_n > 0) \leqslant E(W_n) \to 0,$$

  as required.

Thank You !