Continuous parking sequences

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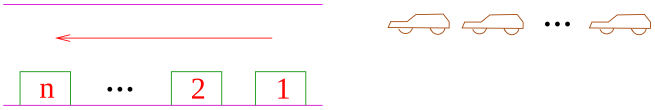
Parking functions

Parking functions were introduced by Konheim and Weiss (1966) under the name of "parking disciplines" in their study of the hash storage structure, and have since found many applications in combinatorics, probability, algebra, and computer science.

Consider a parking lot with n parking spots placed sequentially along a one-way street. A line of n cars enters the lot, one by one. The ith car drives to its preferred spot π_i and parks there if possible; if the spot is already occupied then the car parks in the first available spot after that. The list of preferences $\vec{\pi} = (\pi_1, \dots, \pi_n)$ is called a (classical) parking function of length n if all n cars successfully park.



$n = 1$: 1

$n = 2$: 11, 12, 21

$n = 3$: 111, 112, 121, 211, 113, 131, 311, 122,
212, 221, 123, 132, 213, 231, 312, 321

Pigeonhole principle

It is well-known and easy to see that $\vec{\pi}$ is a parking function if and only if

$$|\{k : \pi_k \leq i\}| \geq i, \text{ for } i = 1, \dots, n.$$

Two immediate observations:

Parking functions are invariant under the action of the symmetric group $\mathfrak{S}_n$ permuting the n cars, that is, permuting the list of preferences $\pi_1, \dots, \pi_n$.

Decreasing parking preferences of a valid parking function always produces a valid parking function.

Continuous parking sequences

Despite the abundance of research into parking functions and their generalizations, one critical aspect about the real life parking scenario has nevertheless not been explored. That is, the length of the car as well as its preferred spot are not necessarily integers; they could very well just be a real number. Our current work aims to fill this gap by considering continuous parking sequences.

Let $\vec{a} = (a_0, a_1, \ldots, a_n)$ be $n+1$ positive real numbers. Assume there is a street of length t, that is, the half open interval $[0, t)$. A trailer of length a_0 occupies the beginning of the street, that is, the interval $[0, a_0)$. Let there be n cars $C_1, \ldots, C_n$ of lengths a_1 through a_n. Each car C_i has a preferred spot $x_i \in [0, t)$.

Definition

The sequence $\vec{x} = (x_1, x_2, \dots, x_n)$ is a continuous parking sequence if the cars in the order C_1 through C_n can park according to the following rule. Initially let the set S_0 be the interval $[0, a_0)$, the spot occupied by the trailer.

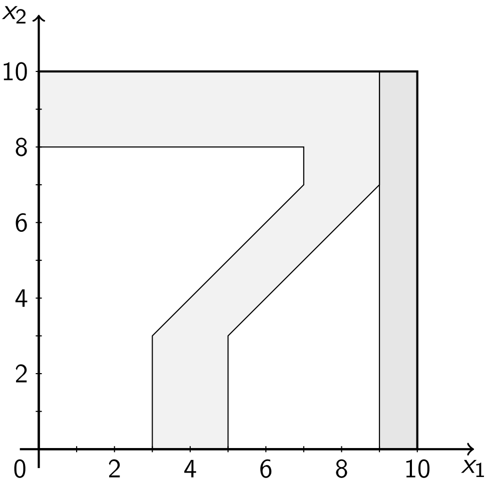
> *Starting at position x_i, car C_i looks for the first empty spot. That is, the position $y_i = \min\{x \geq x_i : x \notin S_{i-1}\}$. If $y_i + a_i \geq t$ then the car is off the street and the result is not a parking sequence. If the intersection $[y_i, y_i + a_i) \cap S_{i-1}$ is non-empty then there will be a collision, and the result is not a parking sequence. Otherwise let the ith car C_i park at $[y_i, y_i + a_i)$ and set $S_i = S_{i-1} \cup [y_i, y_i + a_i)$.*

Here a car seeks to park only once by attempting to park when it locates the first available spot y_i at or after its preference x_i, and "gives up" if it does not fit into $[y_i, y_i + a_i)$.

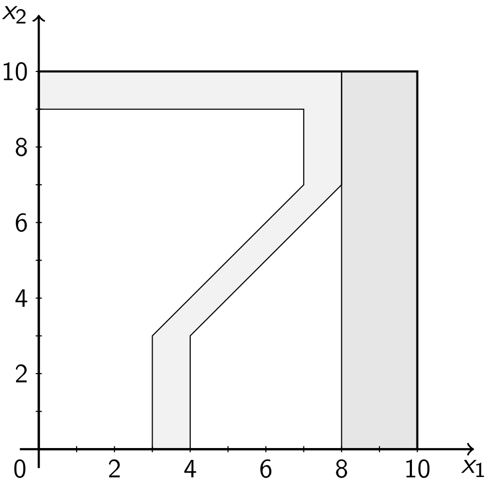
Continuous parking set

We let $\mathrm{PS}(t;\vec{a}) \subseteq [0,t)^n$ be the set of all parking sequences $\vec{x} = (x_1, \ldots, x_n)$ where the list $\vec{a} = (a_0, a_1, \ldots, a_n)$ records the lengths of the trailer and the cars. The set $\mathrm{PS}(t;\vec{a})$ is referred to as the continuous parking set, or simply parking set. Furthermore, let $f(t;\vec{a})$ be the volume of the parking set $\mathrm{PS}(t;\vec{a})$, that is,

$$f(t;\vec{a}) = \mathrm{Vol}(\mathrm{PS}(t;\vec{a})).$$



The parking set with $(a_0, a_1, a_2) = (3, 1, 2)$ and $t = 10$. The dark gray region is where the first car fails to park. The light gray region is where the second car fails to park. The white region is where both cars can park.



The parking set with $(a_0, a_1, a_2) = (3, 2, 1)$ and $t = 10$. The dark gray region is where the first car fails to park. The light gray region is where the second car fails to park. The white region is where both cars can park.

Parking functions vs. Parking sequences

While all parking functions are permutation invariant, this is not the case for parking sequences in general. For example, $(4,3) \notin \mathrm{PS}(10;(3,1,2))$ while $(3,4) \in \mathrm{PS}(10;(3,2,1))$.

Unlike parking functions, decreasing parking preferences of a valid parking sequence may not necessarily produce a valid parking sequence. For example, $(4,5) \in \mathrm{PS}(10;(3,1,2))$ while $(4,3) \notin \mathrm{PS}(10;(3,1,2))$.

Even the volume of the parking set $f(t;\vec{a})$ depends on the order in which the cars enter. For example, $f(10;(3,1,2)) = 56$ while $f(10;(3,2,1)) = 63$.

A central idea in our derivation is the parking function (sequence) shuffle construction, which breaks up a parking function (sequence) into disjoint non-interacting segments separated by empty spots. Though the segments themselves are disjoint and non-interacting, exactly which car belongs to which segment could be changed, and swapping cars between segments is called a shuffle.

Volume of parking set

Theorem

The volume of the parking set $\mathrm{PS}(t;\vec{a})$ *is given by*

$$\begin{aligned} f(t;\vec{a}) &= u_{a_0+a_1+\cdots+a_n}(t)\cdot(t-a_1-\cdots-a_n) \\ &\qquad \cdot(t-a_2-\cdots-a_n)\cdots(t-a_n) \\ &:= u_{a_0+a_1+\cdots+a_n}(t)\cdot g(t-a_0-a_1-\cdots-a_n;\vec{a}). \end{aligned}$$

Outline of proof

We show that $f(t;\vec{a})$ and $u_{a_0+a_1+\cdots+a_n}(t)\cdot g(t-a_0-a_1-\cdots-a_n;\vec{a})$ satisfy the same base case and recursion.

Base case:

$$f(t;a_0)=g(t;a_0).$$

Recursion:

$$f(t;\vec{a})=a_0\cdot f(t;(a_0+a_1,a_2,\ldots,a_n))+\sum_{\substack{I\subseteq[0,n]\\0\in I,1\notin I}} f(t;\vec{a}_I)*f(t;\vec{a}_{\overline{I}}).$$

$$g(t;\vec{a})=a_0\cdot g(t;(a_0+a_1,a_2,\ldots,a_n))+\sum_{\substack{I\subseteq[0,n]\\0\in I,1\notin I}} g(t;\vec{a}_I)*g(t;\vec{a}_{\overline{I}}).$$

Consider how we park the first car C_1. Its desired position is $x_1 = x$.

If $x < a_0$ it will park at $y = a_0$. Then we may view the trailer and the first car C_1 as a new trailer of length $a_0 + a_1$. The associated volume in this case is

$$a_0 \cdot f(t; (a_0 + a_1, a_2, \ldots, a_n)).$$

If $x \geq a_0$ it will park at its desired spot $y = x$. Let J be a subset of $[2, n]$ corresponding to the indices of cars parking before car C_1 and let $I = \{0\} \cup J$. Then to park the cars before C_1, the associated volume is $f(x; \vec{a}_I)$. Similarly, to park the cars after C_1, the associated volume is $f(t - x; \vec{a}_{\bar{I}})$ since we view the first car as the trailer in this case. Integrating x between a_0 and t (and thus 0 and t), and summing over all index sets J (and thus I) yield

$$\sum_{\substack{I \subseteq [0,n] \\ 0 \in I, 1 \notin I}} \int_0^t f(x; \vec{a}_I) \cdot f(t - x; \vec{a}_{\bar{I}})\, dx.$$

Distribution of the first car

Theorem

The probability that the first car does not park in its desired spot is $a_0/(t - a_1 - \cdots - a_n)$.

Distribution of the last car

Theorem

Fix the vector of car lengths $(a_0, a_1, \ldots, a_n)$ *and the street length* $t \geq \sum_{i=0}^{n} a_i$, *and take* $(x_1, \ldots, x_n)$ *uniformly from the set of all continuous parking sequences. Then the last car* C_n*'s desired spot* x_n *is equal in distribution to a uniformly distributed random variable on the interval* $[0, t - a_n]$.

Theorem

Fix the vector of car lengths $(a_0, a_1, \ldots, a_n)$ *and the street length* $t \geq \sum_{i=0}^{n} a_i$, *and take* $(x_1, \ldots, x_n)$ *uniformly from the set of all continuous parking sequences. Then the* $(n-1)$*-dimensional volume of the parking set conditioned on the choice* x_n *of car* C_n *identically equals* $f(t; \vec{a})/(t - a_n)$ *on the interval* $[0, t - a_n]$.

Outline of proof

Let $G(t;\vec{a};x)$ be the volume of the parking set $\mathrm{PS}(t;\vec{a})$ restricted to the region where $x_n = x \in [0, t-a_n]$. It suffices to show that $G(t;\vec{a};x)$ is a constant function on the interval $[0, t-a_n]$.

For $n = 1$, the statement is clearly true.

For $0 \le x \le t - a_n$, we have the recursion

$$G(t;\vec{a};x) =$$

$$a_0 \cdot G(t;(a_0+a_1, a_2, \dots, a_n);x) + \sum_{\substack{I\subseteq[0,n]\\ 0,n\in I;1\notin I}} \int_0^t G(\tau;\vec{a}_I;x)\cdot f(t-\tau;\vec{a}_{\bar{I}})\,d\tau$$

$$+ \sum_{\substack{I\subseteq[0,n]\\ 0\in I;1,n\notin I}} \int_0^t f(\tau;\vec{a}_I)\cdot G(t-\tau;\vec{a}_{\bar{I}};x-\tau)\,d\tau.$$

The first term is constant on $[0, t - a_n]$ by the induction hypothesis.

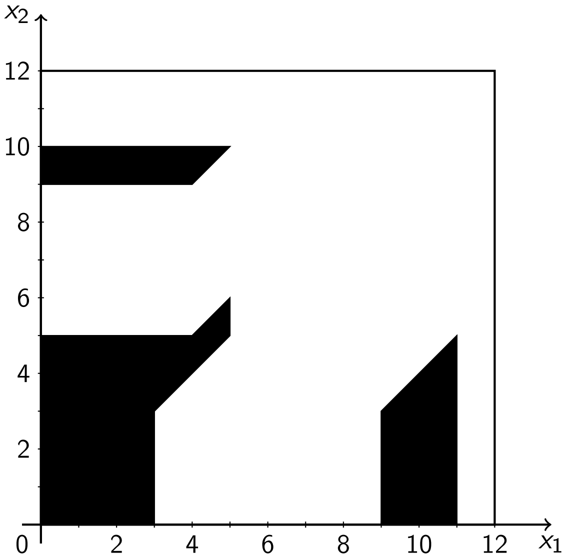
For the remaining part, we combine the terms in the two sums by forming pairs and obtain

$$\sum_{\substack{J \subseteq [0,n-1] \\ 0 \in J; 1 \notin J}} \int_{\underline{u}(J)}^{\overline{u}(J)} f(u; \vec{a}_J) \cdot f(t - u - a_n; \vec{a}_{\overline{J}}) \cdot \big(\chi_u(x) + \chi_{t-u-a_n}(x - u)\big)\, du$$

$$= \sum_{\substack{J \subseteq [0,n-1] \\ 0 \in J; 1 \notin J}} \int_{\underline{u}(J)}^{\overline{u}(J)} f(u; \vec{a}_J) \cdot f(t - u - a_n; \vec{a}_{\overline{J}})\, du,$$

where $\underline{u}(J) = \sum_{j \in J} a_j$, $\overline{u}(J) = t - a_n - \sum_{j \in \overline{J}} a_j$, and $\chi_b(x)$ is the indicator function for $0 \leq x \leq b$.

This shows that $G(t; \vec{a}; x)$ is indeed constant for $x \in [0, t - a_n]$.



The parking set with $(a_0, a_1, a_2, a_3) = (3, 1, 2, 4)$ and $t = 12$ when it is known a priori that the preferred spot of car C_3 is 5. The black region is where all cars can park.

Thank You! Questions?