A Map of the Holonomic Forest: Searching for Irrationality With the Conservative Matrix Field

Shachar Weinbaum - Ramanujan Machine Group

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TECHNION
Israel Institute of Technology

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Claims

Connecting formulas for constants

$$\pi \ \zeta(3) \ \gamma \ \ldots$$

Recreating and expanding irrationality results

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n^{1+\delta}}$$

Generalize mathematical tools

WZ Pairs, Gröbner bases, Ore algebra

Hints at deeper math

New conjectures

Pop Quiz!

$$1, 2, 6, 24, 120, \ldots$$

$$1, 2, 5, 14, 42, 132, \ldots$$

$$1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}, \ldots$$

$$1, 5, 73, 1445, 33001, 819005, \ldots$$

Pop Quiz!

$$1, 2, 6, 24, 120, \ldots, n!, \ldots$$

$$1, 2, 5, 14, 42, 132, \ldots, C_n, \ldots$$

$$1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}, \ldots, H_n, \ldots$$

$$1, 5, 73, 1445, 33001, 819005, \ldots, u_n, \ldots$$

Pop Quiz!

$$1, 2, 6, 24, 120, \ldots, n!, \;\; \mathbf{n! = n \cdot (n-1)!}$$

$$1, 2, 5, 14, 42, 132, \ldots, C_n, \;\; \mathbf{C_n} = \frac{2(2n-1)}{n+1}\mathbf{C_{n-1}}$$

$$1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}, \ldots, H_n, \;\; \mathbf{H_{n+2}} = \frac{5+2n}{3+n}\mathbf{H_{n+1}} - \frac{2+n}{3+n}\mathbf{H_n}$$

$$1, 5, 73, 1445, 33001, 819005, \ldots, u_n$$

$$-n^3\mathbf{u_n} + (2n-1)(17n^2 - 17n + 5)\mathbf{u_{n-1}} - (n-1)^3\mathbf{u_{n-2}} = 0$$

A Curious Pair of Matrices

$$M_X(x,y) = \begin{pmatrix} 0 & -(2x+1)x \\ 1 & 3x+y+2 \end{pmatrix}, M_Y(x,y) = \begin{pmatrix} y-x & -(2x+1)x \\ 1 & 2x+2y+1 \end{pmatrix}$$

These Matrices Commute!

A Curious Pair of Matrices

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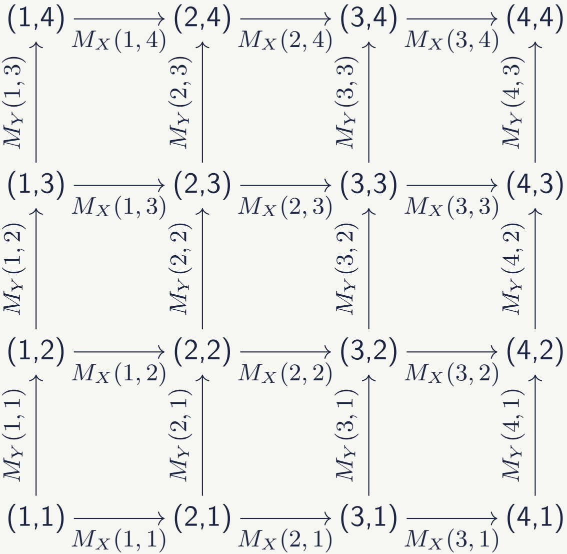
These Matrices Commute! Sort of...

$$M_X(x,y)M_Y(x+1,y) = M_Y(x,y)M_X(x,y+1)$$

A Curious Pair of Matrices



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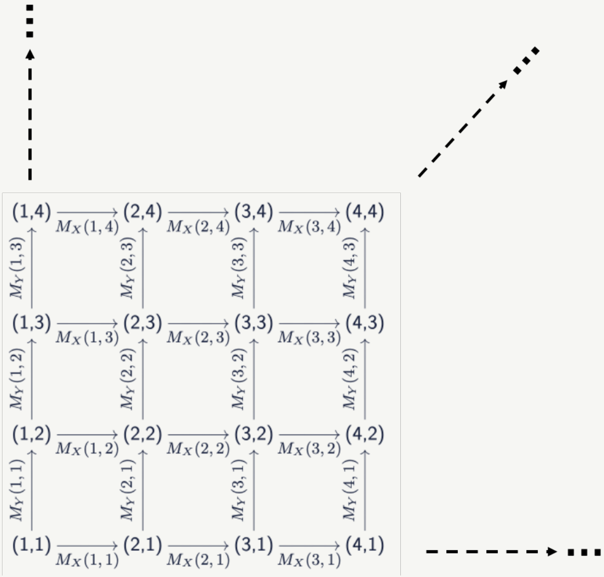

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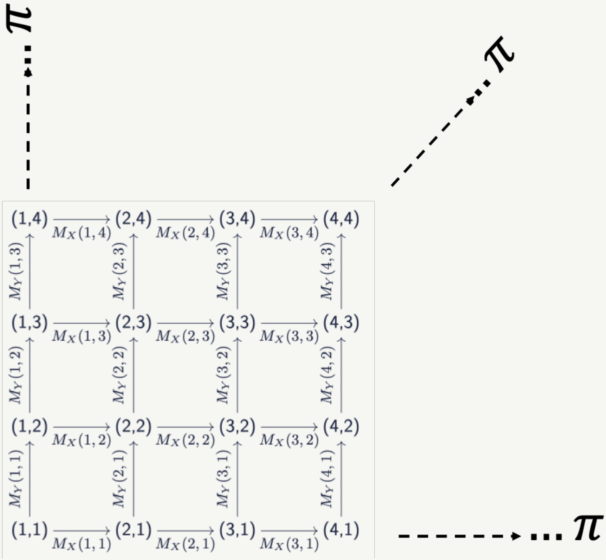


A Curious Pair of Matrices

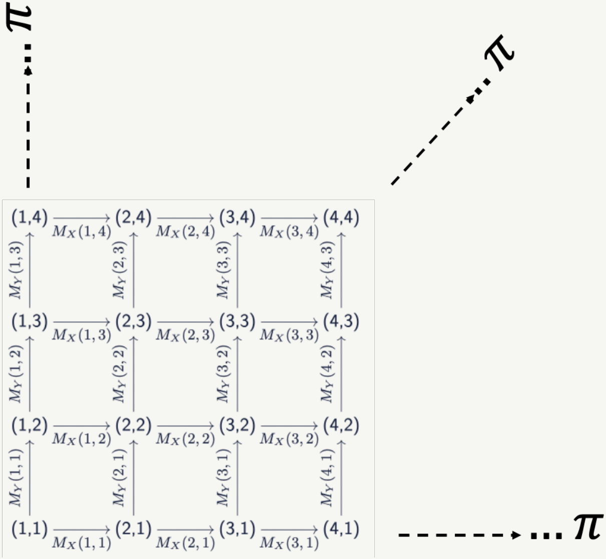
$$(1,1) \xrightarrow[M_X(1,1)]{} (2,1) \xrightarrow[M_X(2,1)]{} (3,1) \xrightarrow[M_X(3,1)]{} (4,1) \dashrightarrow \ldots \pi$$

$$M_X(x,y) = \begin{pmatrix} 0 & -(2x+1)x \\ 1 & 3x+y+2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -3 \\ 1 & 6 \end{pmatrix}, \begin{pmatrix} -3 & -27 \\ 6 & 44 \end{pmatrix}, \begin{pmatrix} -27 & -261 \\ 44 & 402 \end{pmatrix}, \ldots$$

$$0, \frac{-3}{6}, \frac{-27}{44}, \frac{-261}{402} \cdots \to \frac{10-3\pi}{\pi-4}$$

A Curious Pair of Matrices



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The Holonomic Forest

Apéry's Garden - $\zeta(3)$[1]

$\longrightarrow$

$$-n^3 u_n + (2n-1)(17n^2 - 17n + 5)u_{n-1} - (n-1)^3 u_{n-2} = 0$$

Irrationality - by Dirichlet[2]

$$\exists_{\delta>0} \left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n^{1+\delta}}$$

[2]Dirichlet - *Schubfachprinzip* (1834)

Irrationality - of $\zeta(3)$

$$-n^3 u_n + (2n-1)(17n^2 - 17n + 5)u_{n-1} - (n-1)^3 u_{n-2} = 0$$

$$p_n = 0, 6, \frac{351}{4}, \frac{62531}{36}, \ldots$$

$$q_n = 1, 5, 73, 1445, 33001 \ldots$$

$$\frac{p_n}{q_n} \to \zeta(3)$$

$$\left|\zeta(3) - \frac{p_n}{q_n}\right| < \frac{1}{q_n^{1.08}}$$

Apéry's Garden - $\zeta(3)$

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

$$\downarrow$$

$$-n^3 u_n + (2n-1)(17n^2 - 17n + 5)u_{n-1} - (n-1)^3 u_{n-2} = 0$$

Beauker's Integral - $\zeta(2)$ [3]

$$\int_0^1 \int_0^1 \frac{(x(1-x)y(1-y))^n}{(1-xy)^{n+1}}\, dx\, dy$$

$$\downarrow$$

$$-n^2 u_n + (11n^2 - 11n + 3)u_{n-1} + (n-1)^2 u_{n-2} = 0$$

[3]A Note on the Irrationality of $\zeta(2)$ and $\zeta(3)$ - F. Beukers

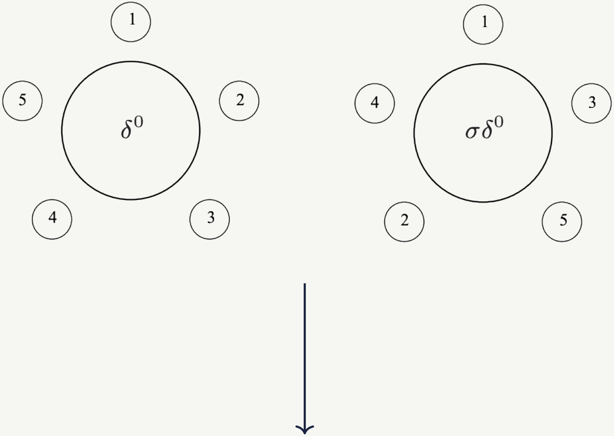
Zudilin's Integral - G[4]

$$\int_0^1 \int_0^1 \frac{x^{a_2-1}(1-x)^{b_2-a_2-1}y^{a_3-1}(1-y)^{b_3-a_3-1}}{(1-xy)^{a_0}}\, dx\, dy$$

$$\downarrow$$

$$\begin{aligned}(2n)^2(2n+1)^2&(20n^2-20n+3)\tilde{u}_{n+1} \\ &-(3520n^6-2672n^4+196n^2-9)\tilde{u}_n \\ &-(2n)^2(2n+1)(2n-3)(20n^2+20n+3)\tilde{u}_{n-1}=0\end{aligned}$$

[4]W.Zudilin - A few remarks on linear forms involving Catalan's constant (2002)



$$p_0(n)u_n + \ldots + p_k(n)u_{n+k} = 0$$

[5]Irrationality proofs for zeta values, moduli spaces and dinner parties -Francis Brown

Aptekarev's Determinants - γ [6]

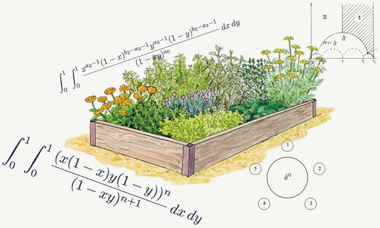
$$\Delta_n^{(ab)} := \det \begin{pmatrix} a_{n+1} & a_n \\ b_{n+1} & b_n \end{pmatrix}$$

$$\downarrow$$

$$(16n+1)(16n-15)u_{n+1} = (16n-15)(256n^3+528n^2+352n+73)u_n$$
$$-(16n+17)(128n^3+40n^2-82n-45)u_{n-1}+n^2(16n+17)(16n+1)u_{n-2}$$

[6] On linear forms containing the Euler constant - A.I.Aptekarev

A Conspiracy?



$$\longrightarrow \quad p_0(n)u_n + \ldots + p_k(n)u_{n+k} = 0$$

A Map of the Holonomic Forest

The Ramanujan Machine

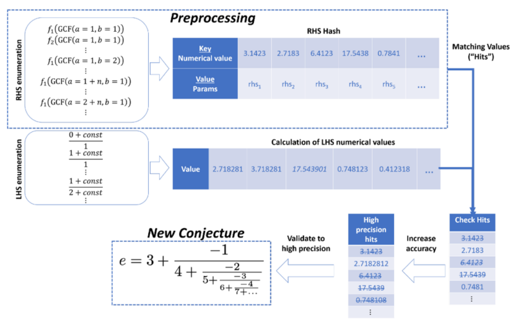
The Ramanujan Machine

Our GitHub

Publications - ramanujanmachine.com

The Ramanujan Library — Automated Discovery on the Hypergraph of Integer Relations (arXiv 2024)
Unsupervised Discovery of Formulas for Mathematical Constants (NeurIPS 2024)
Algorithm-assisted Discovery of an Intrinsic Order Among Mathematical Constants (PNAS 2024)
The conservative matrix field (arXiv 2023)
Automated Search for Conjectures on Mathematical Constants using Analysis of Integer Sequences (PMLR 2023)
On the Connection Between Irrationality Measures and Polynomial Continued Fractions (Arnold Mathematical Journal 2024)
Generating conjectures on fundamental constants with the Ramanujan Machine (Nature 2021)

The Ramanujan Machine[7]



Meet in the Middle Algorithm

$\frac{1}{2G} = 1 - \frac{2}{7 - \frac{32}{19 - \frac{162}{37 - \frac{512}{\ddots}}}}$	$a_n = 3n^2 + 3n + 1,\ b_n = -2n^4$
$\frac{2}{2G-1} = 1 - \frac{4}{7 - \frac{48}{19 - \frac{216}{37 - \frac{640}{\ddots}}}}$	$a_n = 3n^2 + 3n + 1,\ b_n = -2n^3(n+1)$
$\frac{24}{18G-11} = 1 - \frac{6}{7 - \frac{64}{19 - \frac{270}{37 - \frac{768}{\ddots}}}}$	$a_n = 3n^2 + 3n + 1,\ b_n = -2n^3(n+2)$
$\frac{720}{450G-299} = 1 - \frac{8}{7 - \frac{80}{19 - \frac{324}{37 - \frac{896}{\ddots}}}}$	$a_n = 3n^2 + 3n + 1,\ b_n = -2n^3(n+3)$
$\frac{2}{2G-1} = 3 - \frac{6}{13 - \frac{64}{29 - \frac{270}{51 - \frac{768}{\ddots}}}}$	$a_n = 3n^2 + 7n + 3,\ b_n = -2n^3(n+2)$
$\frac{4}{2G+1} = 3 - \frac{12}{13 - \frac{96}{29 - \frac{360}{51 - \frac{960}{\ddots}}}}$	$a_n = 3n^2 + 7n + 3,\ b_n = -2n^2(n+1)(n+2)$

Families of Continued Fractions of Catalan's Constant

[7]G. Raayoni et al - *Generating conjectures on fundamental constants with the Ramanujan Machine*. **Nature** (2021)

$$u_{n+1} = a_n u_n + b_n u_{n-1}$$

$$(u_n, u_{n+1}) = (u_{n-1}, u_n) \begin{pmatrix} 0 & b_n \\ 1 & a_n \end{pmatrix}$$

Hence:

$$\begin{pmatrix} u_n & u_{n+1} \\ v_n & v_{n+1} \end{pmatrix} = \begin{pmatrix} u_0 & u_1 \\ v_0 & v_1 \end{pmatrix} \prod_{i=1}^{n} \begin{pmatrix} 0 & b_i \\ 1 & a_i \end{pmatrix}$$

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Connecting Infinite Families[8] - Ex. $\zeta(3)$

$$\xrightarrow[\begin{pmatrix}0 & -1\\ 1 & 9+6m(m-1)\end{pmatrix}]{} \xrightarrow[\begin{pmatrix}0 & -64\\ 1 & 35+10m(m-1)\end{pmatrix}]{} \dashrightarrow \xrightarrow[\begin{pmatrix}0 & -n^6\\ 1 & n^3(n+1)^3+2m(m-1)(2n+1)\end{pmatrix}]{} \cdots \Rightarrow \sim\zeta(3)$$

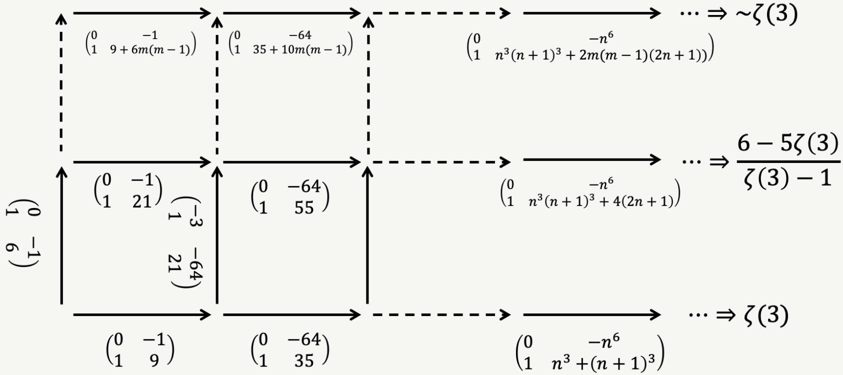
$$\vdots$$

$$\xrightarrow[\begin{pmatrix}0 & -1\\ 1 & 21\end{pmatrix}]{} \xrightarrow[\begin{pmatrix}0 & -64\\ 1 & 55\end{pmatrix}]{} \dashrightarrow \xrightarrow[\begin{pmatrix}0 & -n^6\\ 1 & n^3(n+1)^3+4(2n+1)\end{pmatrix}]{} \cdots \Rightarrow \frac{6-5\zeta(3)}{\zeta(3)-1}$$

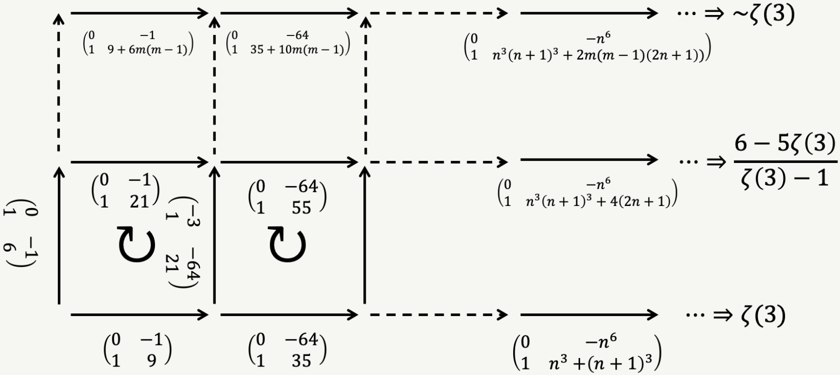
$$\xrightarrow[\begin{pmatrix}0 & -1\\ 1 & 9\end{pmatrix}]{} \xrightarrow[\begin{pmatrix}0 & -64\\ 1 & 35\end{pmatrix}]{} \dashrightarrow \xrightarrow[\begin{pmatrix}0 & -n^6\\ 1 & n^3+(n+1)^3\end{pmatrix}]{} \cdots \Rightarrow \zeta(3)$$

[8] Elimelech et al *Algorithm-assisted discovery of an intrinsic order among mathematical constants.* **PNAS** (2024)

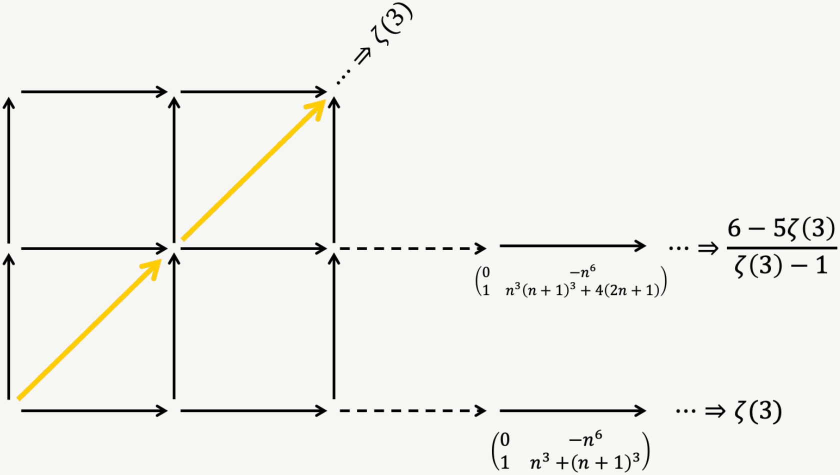
Connecting Infinite Families[8] - Ex. $\zeta(3)$



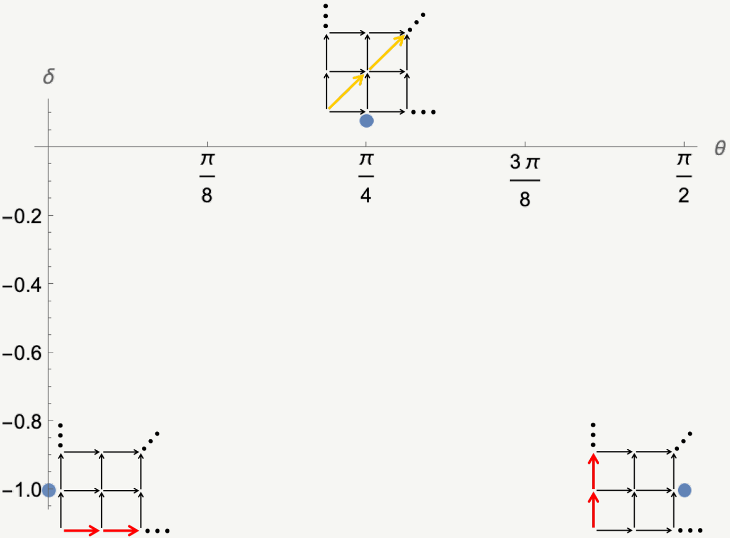
CONNECTING INFINITE FAMILIES[8] - EX. $\zeta(3)$



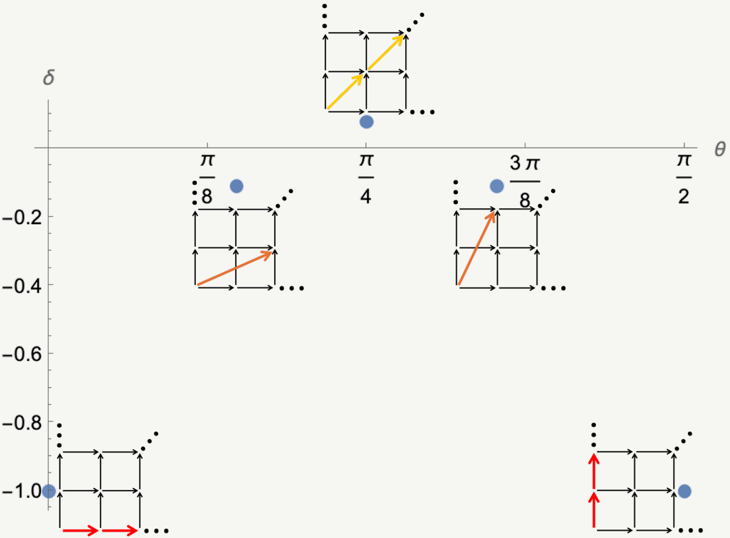
Connecting Infinite Families[8] - Ex. $\zeta(3)$



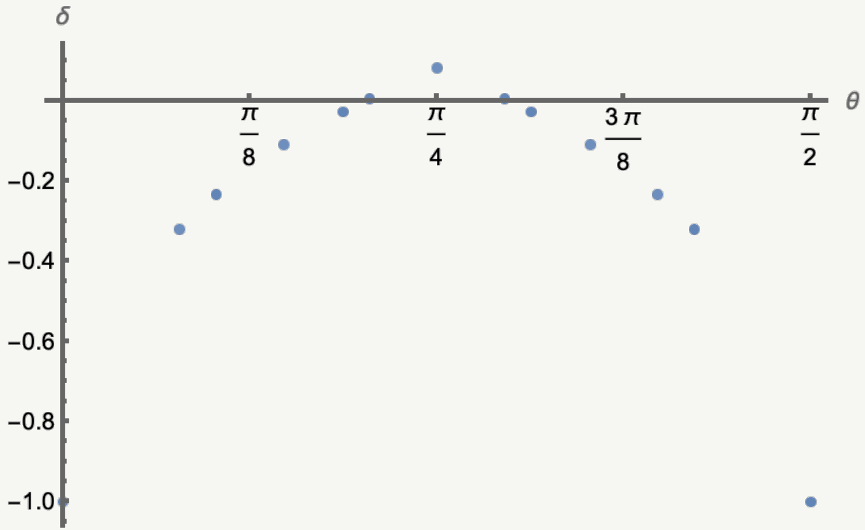
δ VS ANGLE



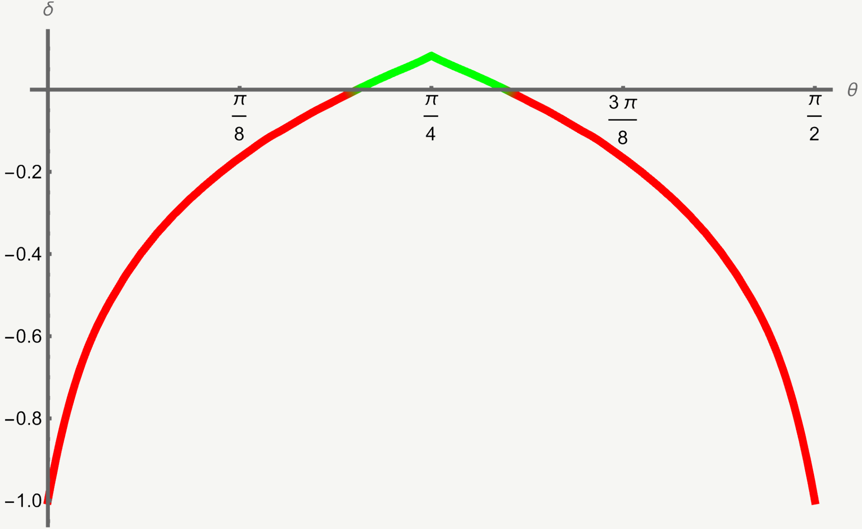
δ VS ANGLE



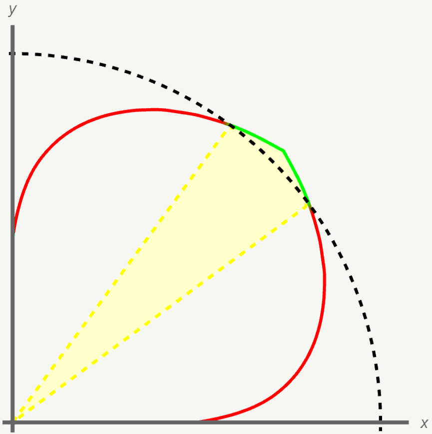
δ VS Angle



Miracle - Continuous δ



δ as function of angle, $\zeta(3)$ CMF



δ as function of angle, $\zeta(3)$ CMF - polar graph

The Conservative Matrix Field

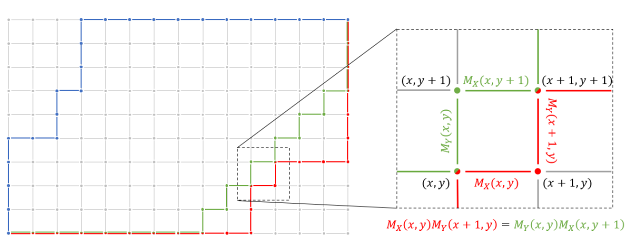
$$\phi : \mathbb{Z}^d \curvearrowright \mathrm{GL}(\mathbb{Q}(x_1, \ldots, x_d)), (\phi_v M)(\mathbf{X}) := M(\mathbf{X} + v) \quad (v \in \mathbb{Z}^d)$$

A **Conservative Matrix Field** (CMF) of dimension d and rank r with respect to the shift action ϕ is a map:

$$\mathcal{M} : \mathbb{Z}^d \to \mathrm{GL}_r(\mathbb{Q}(x_1, \ldots, x_d)), \quad v \mapsto \mathcal{M}_v$$

with the following **Cocycle** property:

$$\mathcal{M}_{v+w} = \mathcal{M}_v \cdot \phi_v \mathcal{M}_w$$



The Conservative Matrix Field[9][10]

$$e : \mathcal{M}_{(1,0)} = \begin{pmatrix} 0 & x_1 + 1 \\ 1 & -(x_1 + x_2 + 1) \end{pmatrix}, \mathcal{M}_{(0,1)} = \begin{pmatrix} -1 & x_1 + 1 \\ 1 & -(x_1 + x_2 + 2) \end{pmatrix}$$

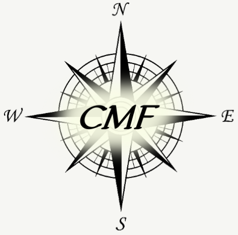
$$\pi : \mathcal{M}_{(1,0)} = \begin{pmatrix} 0 & -(2x_1 + 1)x_1 \\ 1 & 3x_1 + x_2 + 2 \end{pmatrix}, \mathcal{M}_{(0,1)} = \begin{pmatrix} x_2 - x_1 & -(2x_1 + 1)x_1 \\ 1 & 2x_1 + 2x_2 + 1 \end{pmatrix}$$

$$\zeta(2), \zeta(3), G, \ldots$$

[9] O.David: The conservative matrix field (arXiv 2023)

[10] W.Gosper: Strip Mining in the Abandoned Orefields of Nineteenth Century Mathematics (1990)

A Map of the Holonomic Forest



The Hypergeometric Conservative Matrix Field

Reminder - Hypergeometric Functions

Generalized Hypergeometric Function

$${}_pF_q : \mathbb{C}^{p+q} \to \mathbb{C}[[z]]$$

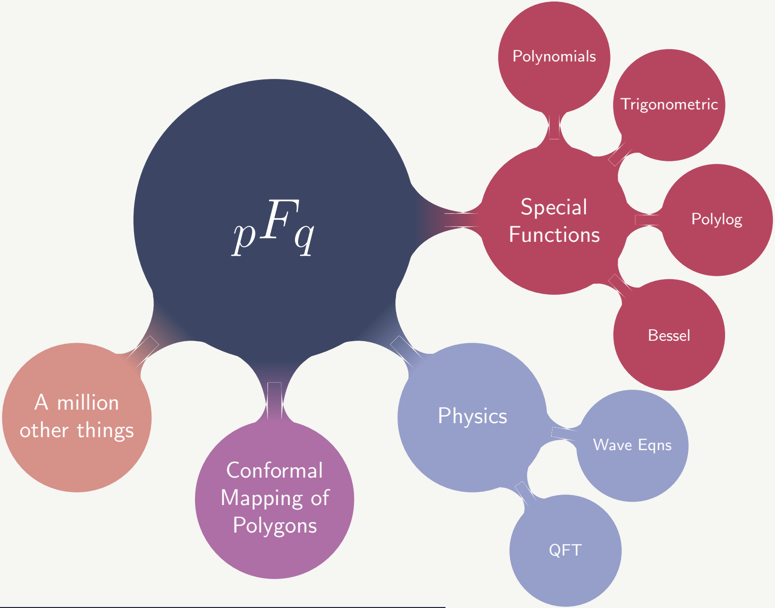
$${}_pF_q\left[\begin{matrix} x_1, \ldots, x_p \\ y_1, \ldots, y_q \end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{\prod_i (x_i)_n z^n}{\prod_i (y_i)_n n!}$$

with $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$

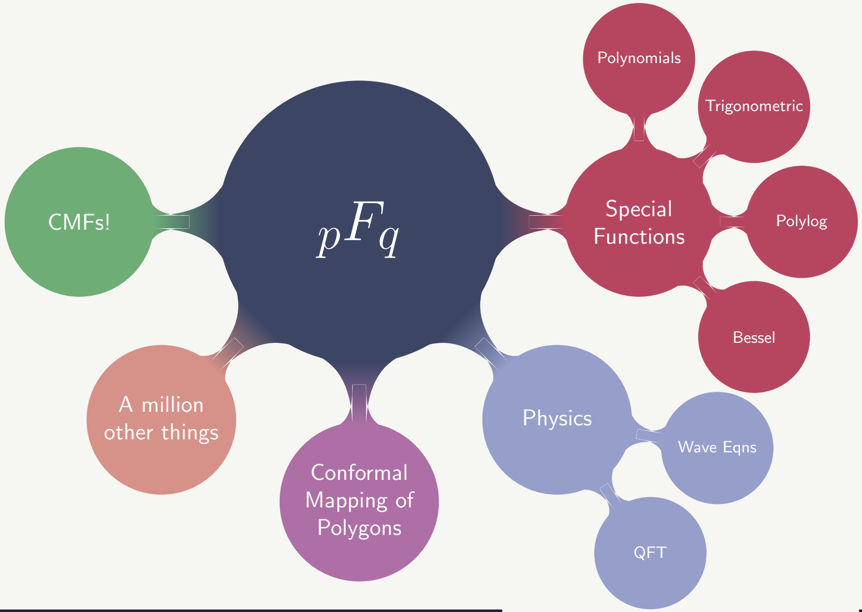
Example

$${}_2F_1\left[\begin{matrix} 1, 1 \\ 2 \end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{n!^2 z^n}{(n+1)!n!} = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)} = -\frac{\log(1-z)}{z}$$

Hypergeometric Functions



Hypergeometric Functions



Algorithm suggestion

Constant of interest

↓

Find ${}_pF_q$ form

↓

CMF

↓

Irrationality

Theorem

For every $_pF_q\left[\begin{smallmatrix} x_1,\dots,x_p \\ y_1,\dots,y_q \end{smallmatrix}; z\right]$ *there exists a Conservative Matrix Field* $\mathcal{M}_v$, *called the* **Hypergeometric Conservative Matrix Field**, *which defines an action* like* ϕ_v *on the parameter space of the Hypergeomteric:*

$$_pF_q\left[\begin{matrix} x_1,\dots,x_p \\ y_1,\dots,y_q \end{matrix}; z\right] \overset{\mathcal{M}_{(1,0\dots,0)}}{\longmapsto} {}_pF_q\left[\begin{matrix} x_1+1,\dots,x_p \\ y_1,\dots,y_q \end{matrix}; z\right]$$

* - The action it defines will be explained shortly

${}_pF_q$ Contiguous Relation and ODE

$$F = {}_pF_q\left[\begin{matrix} x_1, \ldots, x_p \\ y_1, \ldots, y_q \end{matrix}; z\right] \ , \ \phi_{e_i} F = {}_pF_q\left[\begin{matrix} x_1, \ldots, x_i + 1, \ldots, x_p \\ y_1, \ldots, y_q \end{matrix}; z\right]$$

$$\theta = z\frac{d}{dz} \ , \ X = (F, \theta F, \ldots, \theta^{r-1} F)$$

Hypergeometric ODE:

$$\theta X = X M_\theta$$

with $M_\theta \in \mathrm{GL}_r(\mathbb{Q}(z))$

Hypergeometric Contiguous Relations:

$$\phi_{e_i} F = F + \frac{1}{x_i}\theta F$$

a CMF Construction

$$X(I + \frac{1}{x_i}M_\theta) = X + \frac{1}{x_i}XM_\theta = X + \frac{1}{x_i}\theta X$$

$$X(I + \frac{1}{x_i}M_\theta) = (F + \frac{1}{x_i}\theta F, \theta F + \frac{1}{x_i}\theta^2 F, \ldots, \theta^{r-1}F + \frac{1}{x_i}\theta^r F)$$

$$X(I + \frac{1}{x_i}M_\theta) = (\phi_{e_i}F, \phi_{e_i}\theta F, \ldots, \phi_{e_i}\theta^{r-1}F) = \phi_{e_i}X$$

$$\boxed{X(I + \frac{1}{x_i}M_\theta) = \phi_{e_i}X}$$

Example - ${}_2F_1$

$$F = {}_2F_1\left[\begin{matrix} x_1, x_2 \\ y_1 \end{matrix}; z\right]$$

$$(F, \theta F)\mathcal{M}_{(1,0,0)} = (F, \theta F)\begin{pmatrix} 1 & \frac{x_2 z}{1-z} \\ \frac{1}{x_1} & \frac{1+x_1+x_2 z+1-y_1}{x_1(1-z)} \end{pmatrix} = \phi_{(1,0,0)}(F, \theta F)$$

$$(F, \theta F)\mathcal{M}_{(0,1,0)} = (F, \theta F)\begin{pmatrix} 1 & \frac{x_1 z}{1-z} \\ \frac{1}{x_2} & \frac{1+x_2+x_1 z+1-y_1}{x_1(1-z)} \end{pmatrix} = \phi_{(0,1,0)}(F, \theta F)$$

$$(F, \theta F)\mathcal{M}_{(0,0,1)} = (F, \theta F)\frac{y_1}{x_1 x_2 - x_1 y_1 - x_2 y_1 + y_1^2}\begin{pmatrix} -x_1 - x_2 + y_1 & x_1 x_2 \\ \frac{1-z}{z} & \frac{y_1(z-1)}{z} \end{pmatrix} = \phi_{(0,0,1)}(F, \theta F)$$

Example - ${}_2F_1$

$$\begin{array}{ccc}
\left({}_2F_1\left[\begin{matrix}1,2\\y_1\end{matrix};z\right],\theta\,{}_2F_1\left[\begin{matrix}1,2\\y_1\end{matrix};z\right]\right) & \xrightarrow[\mathcal{M}_{(1,0,0)}(1,2)]{} & \left({}_2F_1\left[\begin{matrix}2,2\\y_1\end{matrix};z\right],\theta\,{}_2F_1\left[\begin{matrix}2,2\\y_1\end{matrix};z\right]\right) \\
\Big\uparrow \mathcal{M}_{(0,1,0)}(1,1) & & \Big\uparrow \mathcal{M}_{(0,1,0)}(2,1) \\
\left({}_2F_1\left[\begin{matrix}1,1\\y_1\end{matrix};z\right],\theta\,{}_2F_1\left[\begin{matrix}1,1\\y_1\end{matrix};z\right]\right) & \xrightarrow[\mathcal{M}_{(1,0,0)}(1,1)]{} & \left({}_2F_1\left[\begin{matrix}2,1\\y_1\end{matrix};z\right],\theta\,{}_2F_1\left[\begin{matrix}2,1\\y_1\end{matrix};z\right]\right)
\end{array}$$

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Constant of interest

↓

Find ${}_pF_q$ form

↓

CMF

↓

Irrationality

with $F = {}_2F_1\left[\begin{smallmatrix}1,1\\2\end{smallmatrix};z\right] = -\frac{\log(1-z)}{z}$, we have for $z=-1$:

$$X = (\log(2), \frac{1}{2} - \log(2))$$

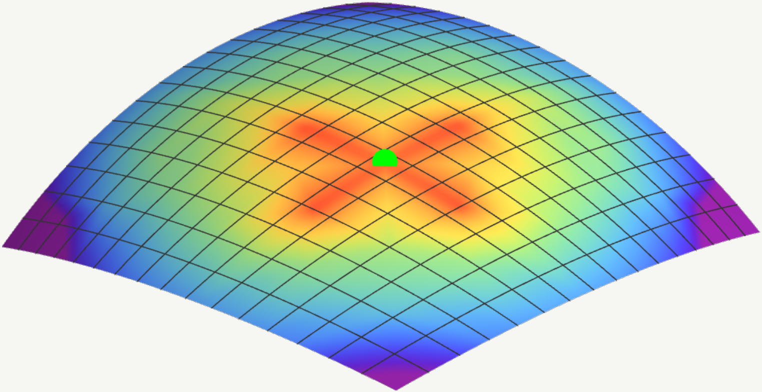
$$X_n = \phi_{(n,n,2n)} X = X\mathcal{M}_{(n,n,2n)}$$

$$X_n = \left({}_2F_1\left[\begin{matrix} n+1, n+1 \\ 2n+2 \end{matrix}; -1\right], \theta_2 F_1\left[\begin{matrix} n+1, n+1 \\ 2n+2 \end{matrix}; -1\right] \right)$$

$$X_n \longrightarrow C(0,1)$$

$$\mathcal{M}_{(n,n,2n)} \left(\frac{\log(2)}{\frac{1}{2} - \log(2)} \right) \longrightarrow 0 \ \cdots \implies \left| \log(2) - \frac{p_n}{q_n} \right| < \frac{1}{q_n^{1.3}}$$

Continuous δ



Delta as function of angle: ${}_2F_1\left[\begin{matrix}1+xn, 1+yn\\ 2+zn\end{matrix}; -1\right]$

Recreation/Extention of Known Results

Recreated using a ${}_pF_q$ CMF:

- $\zeta(2)$: R.Apéry (1979) F.Beaukers (1979)
- $\zeta(3)$: R.Apéry (1979) F.Beaukers (1979)
- G: Zudilin (2001) - **Found better δ!**
- $L(2, \chi_{-3})$: D.Zagier (1984)

Extended using Gröbner CMF:

- $\zeta(5)$: F.Brown, W.Zudilin (2022) - Using Kampé de Fériet functions
- π: D.Zeilberger, W.Zudilin (2019)

Improvements (In the spirit of[11])

- $$\frac{{}_2F_1\left[\begin{matrix}\frac{4}{3},\frac{4}{3}\\2\end{matrix};-\frac{1}{4}\right]}{{}_2F_1\left[\begin{matrix}\frac{1}{3},\frac{1}{3}\\1\end{matrix};-\frac{1}{4}\right]} \in \mathbb{R}\setminus\mathbb{Q}$$
- $$-\frac{4\sqrt{2}\pi\Gamma\left(\frac{-1}{4}\right)}{\Gamma\left(\frac{1}{4}\right)^3} = \frac{{}_2F_1\left[\begin{matrix}\frac{3}{2},\frac{3}{2}\\2\end{matrix};\frac{1}{2}\right]}{{}_2F_1\left[\begin{matrix}\frac{1}{2},\frac{1}{2}\\1\end{matrix};\frac{1}{2}\right]} \in \mathbb{R}\setminus\mathbb{Q}$$
- ...

what about:

$$\frac{5}{4}{}_4F_3\left[\begin{matrix}1,1,1,2\\\frac{4}{3},\frac{3}{2},\frac{5}{3}\end{matrix};-1/4\right] - \frac{1}{3}{}_4F_3\left[\begin{matrix}1,1,1,1\\\frac{4}{3},\frac{3}{2},\frac{5}{3}\end{matrix};-1/4\right]$$

[11]Bliss,Koutschan,Zeilberger - Tweaking the Beukers Integrals In Search of More Miraculous Irrationality Proofs A La Apéry (2021)

Improvements

- $$\frac{{}_2F_1\left[\begin{matrix}\frac{4}{3},\frac{4}{3}\\ 2\end{matrix};-\frac{1}{4}\right]}{{}_2F_1\left[\begin{matrix}\frac{1}{3},\frac{1}{3}\\ 1\end{matrix};-\frac{1}{4}\right]} \in \mathbb{R}\setminus\mathbb{Q}$$
- $$-\frac{4\sqrt{2}\pi\Gamma\left(-\frac{1}{4}\right)}{\Gamma\left(\frac{1}{4}\right)^3} = \frac{{}_2F_1\left[\begin{matrix}\frac{3}{2},\frac{3}{2}\\ 2\end{matrix};\frac{1}{2}\right]}{{}_2F_1\left[\begin{matrix}\frac{1}{2},\frac{1}{2}\\ 1\end{matrix};\frac{1}{2}\right]} \in \mathbb{R}\setminus\mathbb{Q}$$
- ...

what about:[12]

$$L(2,\chi_{-3}) = \frac{5}{4}\,{}_4F_3\left[\begin{matrix}1,1,1,2\\ \frac{4}{3},\frac{3}{2},\frac{5}{3}\end{matrix};-1/4\right] - \frac{1}{3}\,{}_4F_3\left[\begin{matrix}1,1,1,1\\ \frac{4}{3},\frac{3}{2},\frac{5}{3}\end{matrix};-1/4\right]$$

[12]F. Calegari, V. Dimitrov, Y. Tang - The linear independence of 1, $\zeta(2)$, and $L(2,\chi_{-3})$

Improvements - γ [13]

${}_2F_2$ Matrix Field $\rightarrow$ infinitely many Holonomic approximations of γ

$$\prod_{n=1}^{\infty} M_n \iff \text{Recursive Sequence}$$

$$M_n = \begin{pmatrix} n^2(n+2) & 2n^2(n+1) & n^3(n+1) \\ n^2+6n+2 & n^3+10n^2+10n+2 & 6n^3+5n^2-6n-4 \\ -2n-1 & -n^2+n+1 & 9n^2+18n+8 \end{pmatrix}$$

$$M_n = \begin{pmatrix} n^2(2n(2n+1)-1) & n^2\left(3n^2+6n+1\right) & 2n^2\left(n^3+8n^2+12n+4\right) \\ 2n\left(2n^2+n-1\right) & 2n\left(2n^3+10n^2+17n+5\right) & 5n\left(3n^3+18n^2+24n+8\right) \\ -3n^2-6n-1 & -2n^3-13n^2-14n-3 & -8n^3-13n^2-12n-4 \end{pmatrix}$$

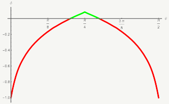
$$\vdots$$

[13] J. Lagarias: Euler's constant: Euler's work and modern developments. (2013)

Connecting formulas for constants

$\pi \ \zeta(3) \ \gamma \ \ldots$

Recreating and expanding irrationality results



Generalize mathematical tools

WZ Pairs, Gröbner bases, Ore algebra

Hints at deeper math

New conjectures

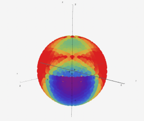
Hints at Deeper Mathematics

The Path to Continuous δ

Poincaré/Perron Asymptotics

$$u_n \approx \sum_{i=1}^{r} c_i f_i(n)$$

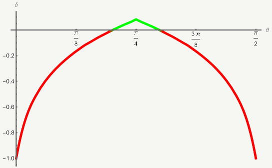
Conjecture: Continuous Asymptotics



Conjecture: P-adic Asymptotics

$$u_n \underset{\text{p-adic}}{\sim} \sum_{i=1}^{r} c_i p^{g_i(n)}$$

Conjecture: Cont. P-adic Asymptotics



Poincaré/Perron Asymptotics[14]

$$\delta = -1 - \lim_{n\to\infty} \frac{\log\left|c - \frac{p_n}{q_n}\right|}{\log|\tilde{q_n}|}$$ Where: $\tilde{q_n}$ = effective denominator

For Apéry:

$$p_n \sim c_1(17+12\sqrt{2})^n n^{-\frac{3}{2}} + c_2(17-12\sqrt{2})^n n^{-\frac{3}{2}}$$

$$q_n \sim \tilde{c_1}(17+12\sqrt{2})^n n^{-\frac{3}{2}} + \tilde{c_2}(17-12\sqrt{2})^n n^{-\frac{3}{2}}$$

$$\frac{p_n}{q_n} \sim \frac{c_1}{\tilde{c_1}} + d\cdot\left(\frac{17-12\sqrt{2}}{17+12\sqrt{2}}\right)^n \implies \log\left|\frac{c_1}{\tilde{c_1}} - \frac{p_n}{q_n}\right| \sim n\log\left(\frac{17-12\sqrt{2}}{17+12\sqrt{2}}\right)$$

[14] O. Perron: Über einen Satz des Herrn Poincaré. (1909)

Arithmetic Asymptotics

$$\delta = -1 - \lim_{n\to\infty} \frac{\log\left|c - \frac{p_n}{q_n}\right|}{\log|\tilde{q_n}|}$$ Where: $\tilde{q_n}$ = effective denominator

For Apéry, q_n are integers, but the denominators of p_n grow like LCM$[1, \ldots, n]^3$. thus

$$|q_n| \sim (17 + 12\sqrt{2})^n \implies |\tilde{q_n}| \sim (17 + 12\sqrt{2})^n e^{3n}$$

$$\log|\tilde{q_n}| \sim n(\log\left(17 + 12\sqrt{2}\right) + 3)$$

$$\delta = -1 - \frac{n\log\left(\frac{17-12\sqrt{2}}{17+12\sqrt{2}}\right)}{n(\log\left(17 + 12\sqrt{2}\right) + 3)} \approx 0.0805294$$

Conjecture:

$$u_n \underset{\text{p-adic}}{\sim} \sum_{i=1}^{r} c_i p^{g_i(n)}$$

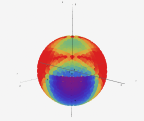
For Example, in Apéry:

$$\forall_p : \; g_1(n) = -3\log_p(n) \; , \; g_2(n) = 0$$

Poincaré/Perron Asymptotics

$$u_n \approx \sum_{i=1}^{r} c_i f_i(n)$$

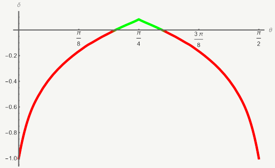
Conjecture: Continuous Asymptotics



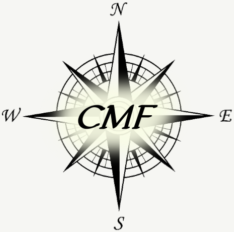
Conjecture: P-adic Asymptotics

$$u_n \underset{\text{p-adic}}{\sim} \sum_{i=1}^{r} c_i p^{g_i(n)}$$

Conjecture: Cont. P-adic Asymptotics



A Map of the Holonomic Forest



Thank You!

Questions?

- D-Finite CMFs, and Groebner Bases
- π-Unifier and the coboundary problem
- Regularization of infinite sums, and gauge transforms
- q-Analogs: the Basic Hypergeometric CMF
- Conjectures for simplified case

Our GitHub

Our Publications

Lets use an Ore algebra.

$$\mathbb{A} = \mathbb{C}(a_1, \ldots, a_p; z_1, \ldots, z_q)[S_1, \ldots, S_p, D_1, \ldots D_q]$$

Given a D-Finite Function $F(a_1, \ldots, a_p; z_1, \ldots, z_q) : \mathbb{N}^p \times \mathbb{C}^q \to \mathbb{C}$
Looking at the annihilator $\mathfrak{a} \lhd \mathbb{A}$, of F, we can represent $\mathbb{A}/\mathfrak{a}$ in the base $B \subset \mathbb{C}(a_1, \ldots, a_p; z_1, \ldots, z_q)[D_1, ..., D_q]$. For each shift S_i we encode its equivalence class in $\mathbb{A}/\mathfrak{a}$ using B:

$$M_i = [S_i]_B$$

these will form a CMF! and "act" on F!
Equivalent to Ore Gröbner Basis

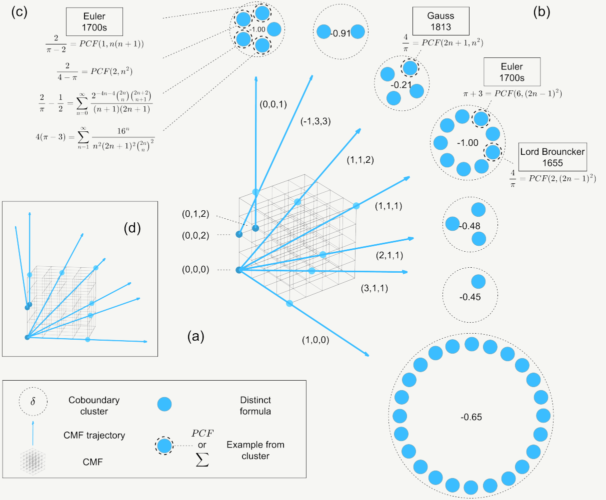
When are two formulas for π related?

$$\text{(Leibnitz)}\ \cfrac{4}{1+\cfrac{1^2}{2+\cfrac{2^2}{2+\ddots}}} = \pi = 3 + \cfrac{1^2}{6+\cfrac{2^2}{6+\cfrac{3^2}{6+\ddots}}}\ \text{(Nīlakaṇṭha)}$$

$$\text{(Leibnitz)}\ A_n = \begin{pmatrix} 0 & (2n-1)^2 \\ 1 & 2 \end{pmatrix},\ \text{(Nīlakaṇṭha)}\ B_n = \begin{pmatrix} 0 & (2n-1)^2 \\ 1 & 6 \end{pmatrix}$$

$$U_n = \begin{pmatrix} 2n-3 & (2n-1)^2 \\ 1 & 2n+1 \end{pmatrix},\ A_n = U_n B_n U_{n+1}^{-1}$$

π - Unifier



T. Raz et al - Formulas of π and their inclusion in a Hypergeometric CMF

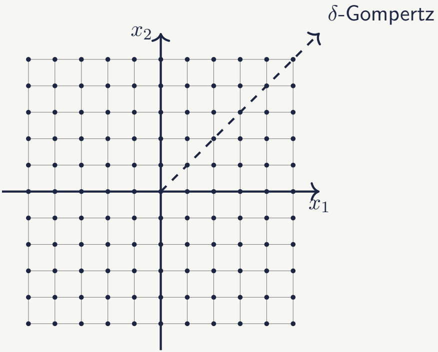
Regularization of ${}_pF_q$s

$$ {}_2F_0\left[\begin{matrix}1,1\\ -\end{matrix};-1\right] = \sum_{n=0}^{\infty}(-1)^n n! \text{ (diverges)} \underset{\text{regularization}}{\longrightarrow} \delta $$

$$ {}_2F_0\left[\begin{matrix}x_1,x_2\\ -\end{matrix};-1\right] \Longrightarrow $$

$$ \mathcal{M}_{(1,0)} = \begin{pmatrix} 1 & -x_2 \\ \frac{1}{x_1} & 1+\frac{-x_2-1}{x_1} \end{pmatrix} $$

$$ \mathcal{M}_{(0,1)} = \begin{pmatrix} 1 & -x_1 \\ \frac{1}{x_2} & 1+\frac{-x_1-1}{x_2} \end{pmatrix} $$



Gauge Transform

for X solution of $\theta X = XM_\theta$, and $Y := XA, A \in \mathrm{GL}_r(\mathbb{C}[[z]])$, then

$$\theta Y = Y\bar{M}_\theta, \bar{M}_\theta := A^{-1}(\theta A + M_\theta A)$$

Claim: if $Y = X\mathcal{M}_v = \phi_v X$ then:

$$\bar{M}_\theta = \mathcal{M}_v^{-1}(\theta\mathcal{M}_v + M_\theta\mathcal{M}_v) = \phi_v M_\theta$$

Meaning: not only is $\mathcal{M}_v$ a CMF, it acts on all solutions to $\theta X = XM_\theta$

Proof: Nitpicky algebra on the ODE structure.

q-Analogs: The Basic Hypergeometric CMF

$$ {}_2\phi_1\left[\begin{matrix} a, b \\ c \end{matrix}; q; z\right] = \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(c;q)_n (q;q)_n} z^n $$

Where:

$$ (x;q)_n = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}) $$

$$ M_{a+} = \begin{pmatrix} 1 & -\frac{zq^{a+1}\left(q^b-1\right)}{-zq^{a+b+1}+zq^{a+b+2}+q^c} \\ \frac{(q-1)q^a}{q^a-1} & \frac{(1-q)q^a\left(-\left((q-1)zq^{a+1}\left(2q^b-1\right)\right)+(q-1)zq^{b+1}-q^c+q\right)}{(q-1)(1-q^a)\left((q-1)zq^{a+b+1}+q^c\right)}+1 \end{pmatrix} $$

Conjectures for constant matrices