Reflecting (on) the modulo 9 Kanade-Russell (conjectural) identities

joint work with Wadim Zudilin

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**Integer Partitions**

A *partition* of the positive integer $n$ is a *finite list of decreasing natural numbers*.

**Example $n = 5$:**

$(1,1,1,1,1), (2,1,1,1), (2,2,1), (3,1,1), (3,2), (4,1), (5)$.

Size of a partition $\pi$ is the sum of its parts, we denote this by $|\pi|$.
Generating Functions

For a sequence \( \{a_n\}_{n=0}^{\infty} \), the series
\[
\sum_{n \geq 0} a_n q^n
\]
is called a generating function.

Let \( U \) be the set of all partitions.
\[
\sum_{\pi \in U} q^{\vert \pi \vert} = \sum_{n \geq 0} p(n) q^n = 1 + q + 2q^2 + 3q^3 + 5q^4 + \ldots,
\]
where \( p(n) \) is the number of partitions of \( n \).
Generating Functions

\[ (a; q)_L := \prod_{i=0}^{L-1} (1 - aq^i), \]

\[ (a; q)_{\infty} := \lim_{L \to \infty} (a; q)_L, \]

\[ (a_1, a_2, \ldots, a_k; q)_L := \prod_{i=1}^{k} (a_i; q)_L \text{ for } k \in \mathbb{Z}_{\geq 1}. \]
**Generating Functions**

**Example**

- **Ex:** The product
  \[
  \frac{1}{(q; q)_n}
  \]
  is the generating function for the number of partitions with parts \( \leq n \).

- **Ex:** The product
  \[
  (-q; q)_n
  \]
  is the generating function for the number of partitions into distinct parts \( \leq n \).

- **Ex:** For integers \( 1 \leq a \leq b \), products
  \[
  \frac{1}{(q^a; q^b)_n} \text{ and } (-q^a; q^b)_n
  \]
  are the generating functions for the number of partitions into unrestricted and distinct \( a \) modulo \( b \) parts, each \( \leq b(n - 1) + a \), respectively.
Ex: The product
\[ \frac{1}{(q; q)_n} \]
is the generating function for the number of partitions with number of parts $\leq n$.

Ex: The product
\[ (-q; q)_n \]
is the generating function for the number of partitions into distinct parts $\leq n$.

Ex: For integers $1 \leq a \leq b$, products
\[ \frac{1}{(q^a; q^b)_n} \text{ and } (-q^a; q^b)_n \]
are the generating functions for the number of partitions into unrestricted and distinct $a$ modulo $b$ parts, each $\leq b(n - 1) + a$, respectively.
The $q$-binomial coefficient

\[ \binom{m+n}{m}_q := \begin{cases} \frac{(q;q)_{m+n}}{(q;q)_m(q;q)_n} & \text{for } m, n \geq 0, \\ 0 & \text{otherwise.} \end{cases} \]

can be interpreted as the generating function for the number of partitions that fits in an $m \times n$-box.
We call any two sets of partitions being equinumerous for each size $n$, a partition identity.

**Theorem (Euler’s Partition Theorem)**

For any integer $n$, the number of partitions of $n$ into distinct parts

$=$

the number of partitions of $n$ into odd parts.
We call any two sets of partitions being equinumerous for each size \( n \), a \textit{partition identity}.

**Theorem (Euler's Partition Theorem)**

*For any integer \( n \), the number of partitions of \( n \) into distinct parts*

\[ = \]

*the number of partitions of \( n \) into odd parts.*
Partition Identities

We call any two sets of partitions being equinumerous for each size $n$, a partition identity.

**Theorem (Euler’s Partition Theorem)**

For any integer $n$, the number of partitions of $n$ into distinct parts

$= \quad$ the number of partitions of $n$ into odd parts.

**Ex**: $n = 6$

$\begin{align*}
(6) & \quad (5,1) \\
(5,1) & \quad (3,3) \\
(4,2) & \quad (3,1,1,1) \\
(3,2,1) & \quad (1,1,1,1,1,1)
\end{align*}$
Partition Identities

We call any two sets of partitions being equinumerous for each size $n$, a partition identity.

**Theorem (Euler’s Partition Theorem)**

*For any integer $n$, the number of partitions of $n$ into distinct parts = the number of partitions of $n$ into odd parts.*

**Theorem (Generating Function version of Euler’s Partition Theorem)**

$$(-q; q)_\infty = \frac{1}{(q; q^2)_\infty}.$$
Partition Identities

We call any two sets of partitions being equinumerous for each size $n$, a partition identity.

**Theorem (Euler’s Partition Theorem)**

For any integer $n$, the number of partitions of $n$ into distinct parts

\[ \equiv \]

the number of partitions of $n$ into odd parts.

**Theorem (Generating Function version of Euler’s Partition Theorem)**

\[
\sum_{j \geq 0} \frac{q^{j(j+1)/2}}{(q; q)_j} = \frac{1}{(q; q^2)_\infty}.
\]
Theorem (Rogers–Ramanujan Identities)

For any integer $n$ and $i = 1$ or $2$, the number of partitions of $n$ into non-consecutive parts $\geq i$ is equal to the number of partitions of $n \pm i$ modulo 5 parts.

Theorem (Generating Function version of RR-identities)

For $i = 1$ or $2$, 

$$
\sum_{j \geq 0} \frac{q^{j^2+(i-1)j}}{(q; q)_j} = \frac{1}{(q^i, q^{5-i}; q^5)_{\infty}}.
$$
Theorem (Capparelli’s First Partition Theorem)

For any integer $n$, the number of partitions of $n$ into parts, not equal to 1, where the minimal difference between consecutive parts is 4, except if the consecutive parts are

- $3k$ and $3k + 3$ (yielding a difference of 3), or
- $3k - 1$ and $3k + 1$ (yielding a difference of 2) for some $k \in \mathbb{N}$,

the number of partitions of $n$ into distinct parts where no part is congruent to $\pm 1$ modulo 6.

Capparelli’s Second Partition Theorem

Theorem (Capparelli’s Second Partition Theorem)

For any integer \( n \), the number of partitions of \( n \) into parts, not equal to 2, where the minimal difference between consecutive parts is 4, except if the consecutive parts are

- \( 3k \) and \( 3k + 3 \) (yielding a difference of 3), or
- \( 3k - 1 \) and \( 3k + 1 \) (yielding a difference of 2) for some \( k \in \mathbb{N} \),

is equal to the number of partitions of \( n \) into distinct parts where no part is congruent to \( \pm 2 \) modulo 6.

Kanade–Russell Conjectures

In 2014, Shashank Kanade and Matthew Russell experimentally found 6 Difference-Congruence type conjectural partition theory identities.

Similar to the Capparelli’s theorem case, the Generating Function counterparts for these conjectures were not known. Four of the six identities were related to modulo 9 products and the other two were related to modulo 12.

Russell presented one more modulo 9 identity in his PhD thesis in 2016, and in their follow up paper Kanade-Russell extended the list of modulo 12 identities to 11 identities total.


Conjecture (KR - Conjecture I1)

The number of partitions with difference at least 3 at distance 2 such that if two consecutive parts differ by at most 1, then their sum is divisible by 3

\[=\]

The number of partitions of a non-negative integer into parts congruent to 1, 3, 6, or 8 mod 9.

In 2018, Kurşungöz and Kanade-Russell independently (through combinatorial construction and through experimentation, resp.) found the sum generating functions for the original Kanade–Russell conjectures.


Kanade–Russell Conjectures

One sample Mod 9 identity

**Conjecture (KR - I1)**

>The number of partitions with difference at least 3 at distance 2 such that if two consecutive parts differ by at most 1, then their sum is divisible by 3

\[
\sum_{m,n \geq 0} \frac{q^{m^2+3mn+3n^2}}{(q; q)_m (q^3; q^3)_n} = \frac{1}{(q, q^3, q^6, q^8; q^9)_\infty}
\]
Kanade–Russell Conjectures
Generating Function Versions of the Modulo 9 Identities

\[
\sum_{m,n \geq 0} \frac{q^{m^2 + 3mn + 3n^2}}{(q; q)_m(q^3; q^3)_n} \equiv \frac{1}{(q, q^3, q^6, q^8; q^9)_\infty},
\]

\[
\sum_{m,n \geq 0} \frac{q^{m^2 + 3mn + 3n^2 + m + 3n}}{(q; q)_m(q^3; q^3)_n} \equiv \frac{1}{(q^2, q^3, q^6, q^7; q^9)_\infty},
\]

\[
\sum_{m,n \geq 0} \frac{q^{m^2 + 3mn + 3n^2 + 2m + 3n}}{(q; q)_m(q^3; q^3)_n} \equiv \frac{1}{(q^3, q^4, q^5, q^6; q^9)_\infty},
\]

\[
\sum_{m,n \geq 0} \frac{q^{m^2 + 3mn + 3n^2 + m + 2n}}{(q; q)_m(q^3; q^3)_n} \equiv \frac{1}{(q^2, q^3, q^5, q^8; q^9)_\infty},
\]

\[
\sum_{m,n \geq 0} \frac{q^{m^2 + 3mn + 3n^2 + 2m + 4n}}{(q; q)_m(q^3; q^3)_n}(1 + q + q^{m+3n+2}) \equiv \frac{1}{(q, q^4, q^6, q^7; q^9)_\infty}.
\]
Kurşungöz and Kanade–Russell also found Generating Function counterparts for Capparelli’s identities.

**Theorem (Generating Function Version of the First Capparelli’s identity)**

\[
\sum_{m,n \geq 0} q^{2(m^2 + 3mn + 3n^2)} \frac{1}{(q; q)_m (q^3; q^3)_n} = (-q^3; q^3)_\infty (-q^2, -q^4; q^6)_\infty.
\]

**Conjecture (KR - I1 - Generating Function Counterpart)**

\[
\sum_{m,n \geq 0} q^{m^2 + 3mn + 3n^2} \frac{1}{(q; q)_m (q^3; q^3)_n} \overset{?}{=} \frac{1}{(q, q^3, q^6, q^8; q^9)_\infty}.
\]

\[ C_1(z) := \sum_{m,n\geq 0} \frac{q^2(m^2+3mn+3n^2)}{(q; q)_m(q^3; q^3)_n} z^{m+2n} = \sum_{k\geq 0} c_k(q)z^k. \]

\[ C_1(z) - (1 + q^3 z)C_1(q^3 z) - q^2 z(1 + q^2 + q^4 z)C_1(q^6 z) - q^9 z^2 C_1(q^9 z) = 0. \]

\[ C_1(z) = (z; q^3)_\infty G(Z), \text{ where } G(z) := \sum_{n\geq 0} g_n(q)z^n. \]

Then,

\[ q^3(1 + q^3+3n)(1 - q^{3+6n})g_n - (1 + q^3 + q^{8+6n} + q^{10+6n})g_{n+1} + (1 - q^{6+3n})g_{n+2} = 0. \]
\( q \)-Difference Equations

Capparelli Case

\[
C_1(z) := \sum_{m,n \geq 0} \frac{q^{2(m^2 + 3mn + 3n^2)}}{(q; q)_m(q^3; q^3)_n} z^{m+2n} = \sum_{k \geq 0} c_k(q)z^k.
\]

\[
C_1(z) - (1 + q^3 z) C_1(q^3 z) - q^2 z (1 + q^2 + q^4 z) C_1(q^6 z) - q^9 z^2 (1 - q^6 z) C_1(q^9 z) = 0.
\]

\[
C_1(z) = (z; q^3) \infty G(Z), \text{ where } G(z) := \sum_{n \geq 0} g_n(q)z^n.
\]

Then,

\[
q^3(1 + q^3 + 3n)(1 - q^3 + 6n)g_n - (1 + q^3 + q^8 + 6n + q^{10 + 6n})g_{n+1} + (1 - q^6 + 3n)g_{n+2} = 0.
\]
\[ C_1(z) := \sum_{m,n \geq 0} \frac{q^{2(m^2+3mn+3n^2)}}{(q; q)_m(q^3; q^3)_n} z^{m+2n} = \sum_{k \geq 0} c_k(q)z^k. \]

\[ C_1(z) - (1 + q^3 z)C_1(q^3 z) - q^2 z(1 + q^2 + q^4 z)C_1(q^6 z) - q^9 z^2(1 - q^6 z)C_1(q^9 z) = 0. \]

\[ q^{15+9k}c_k - q^{12+6k}(1+q^6+3k)c_{k+1} - q^{9+3k}(1+q^5+3k+q^7+3k)c_{k+2} + (1-q^9+3k)c_{k+3} = 0 \]

\[ C_1(z) = (z; q^3)_\infty G(Z), \text{ where } G(z) := \sum_{n \geq 0} g_n(q)z^n. \]

Then,

\[ q^3(1 + q^{3+3n})(1 - q^{3+6n})g_n - (1 + q^3 + q^{8+6n} + q^{10+6n})g_{n+1} + (1 - q^{6+3n})g_{n+2} = 0. \]
\[ C_1(z) := \sum_{m,n \geq 0} \frac{q^{2(m^2+3mn+3n^2)}}{(q;q)_m(q^3;q^3)_n} z^{m+2n} = \sum_{k \geq 0} c_k(q) z^k. \]

\[ C_1(z) - (1 + q^3 z) C_1(q^3 z) - q^2 z (1 + q^2 + q^4 z) C_1(q^6 z) - q^9 z^2 (1 - q^6 z) C_1(q^9 z) = 0. \]

\[ C_1(z) = (z; q^3)_\infty G(Z), \quad \text{where} \quad G(z) := \sum_{n \geq 0} g_n(q) z^n. \]

Then,

\[ q^3 (1 + q^3 + 3n) (1 - q^3 + 6n) g_n - (1 + q^3 + q^8 + 6n + q^{10 + 6n}) g_{n+1} + (1 - q^6 + 3n) g_{n+2} = 0. \]
\[ KR_1(z) := \sum_{m,n \geq 0} \frac{q^{m^2+3mn+3n^2}}{(q;q)_m(q^3;q^3)_n} z^{m+2n}. \]

\[
(q^6 z + q^4 z + 1) KR_1(z) \\
\quad + q^{11} z^3 (q^{10} z^2 + q^8 z^2 + q^7 z + q^6 z + q^5 z + q^4 z + q^3 z + q^2 + 1) KR_1(q^6 z) \\
\quad - (q^{12} z^3 + q^{10} z^3 + q^9 z^3 + q^9 z^2 + q^8 z^2 + q^7 z^3 + q^7 z^2 + 2q^6 z^2 + q^6 z + q^5 z^2 \\
\quad \quad \quad + q^4 z^2 + q^4 z + q^3 z^2 + q^3 z + q^2 z + qz + 1) KR_1(q^3 z) \\
\quad + q^{27} z^5 (q^3 z + qz + 1) KR_1(q^9 z) = 0
\]
**q-Partial Difference Equations**

**KR I1 Case**

\[
KR_1(x, y) := \sum_{m,n \geq 0} \frac{q^{m^2+3mn+3n^2}}{(q; q)_m(q^3; q^3)_n} x^{m+3n} y^m.
\]

\[-KR_1(x, y) + KR_1(x, qy) + qxy KR_1(qx, qy) = 0.\]

For example,

\[
f(x, y) = \sum_{m \geq 0} \frac{q^{m^2} (q^{m+1} x; q)_\infty}{(q; q)_m} (xy)^m
\]

satisfies the \(q\)-(partial) difference equation above but we do not have enough information to continue with this technique.
Theorem (Kurşungöz, 2018)

\[
\sum_{m,n \geq 0} \frac{q^{2m^2 + 6mn + 6n^2}}{(q; q)_m(q^3; q^3)_n} = (q, -q^5; q^6) \infty (-q^3; q^3) \infty,
\]

\[
\sum_{m,n \geq 0} \frac{q^{2m^2 + 6mn + 6n^2 + m + 3n}}{(q; q)_m(q^3; q^3)_n} + \sum_{m,n \geq 0} \frac{q^{2m^2 + 6mn + 6n^2 + 3m + 6n + 1}}{(q; q)_m(q^3; q^3)_n} = (q, -q^5; q^6) \infty (-q^3; q^3) \infty.
\]

Polynomial Analogues
Capparelli Case

Theorem (Berkovich-U, 2018)

\[
\sum_{m,n \geq 0} q^{2m^2+6mn+6n^2} \left[ \frac{3(N - 2n - m)}{m} \right]_q \left[ \frac{2(N - 2n - m) + n}{n} \right]_q^3 = \sum_{l=0}^{N} q^{\frac{(N-2l)}{2}} \left[ \frac{N}{2l} \right]_q^3 \left( -q^2, -q^4; q^6 \right)_l.
\]

\[
\sum_{m,n \geq 0} q^{2m^2+6mn+6n^2+6m+3n} \left[ \frac{3(N - 2n - m) + 2}{m} \right]_q \left[ \frac{2(N - 2n - m) + n + 1}{n} \right]_q^3 + \sum_{m,n \geq 0} q^{2m^2+6mn+6n^2+3m+6n+1} \left[ \frac{3(N - 2n - m)}{m} \right]_q \left[ \frac{2(N - 2n - m) + n}{n} \right]_q^3
\]

\[
= \sum_{l=0}^{N} q^{\frac{(N-2l)}{2}} \left[ \frac{N+1}{2l+1} \right]_q^3 \left( -q^6 \right)_{l+1} \left( -q^5; q^6 \right)_l.
\]

Polynomial Analogue
KR Mod 9 conjectures Cases

\[
\sum_{m,n \geq 0} \frac{q^{m^2 + 3mn + 3n^2}}{(q; q)_m(q^3; q^3)_n} \equiv \frac{1}{(q, q^3, q^6, q^8; q^9)_\infty},
\]

\[
\sum_{m,n \geq 0} \frac{q^{m^2 + 3mn + 3n^2 + m + 3n}}{(q; q)_m(q^3; q^3)_n} \equiv \frac{1}{(q^2, q^3, q^6, q^7; q^9)_\infty},
\]

\[
\sum_{m,n \geq 0} \frac{q^{m^2 + 3mn + 3n^2 + 2m + 3n}}{(q; q)_m(q^3; q^3)_n} \equiv \frac{1}{(q^2, q^3, q^5, q^6; q^9)_\infty},
\]

\[
\sum_{m,n \geq 0} \frac{q^{m^2 + 3mn + 3n^2 + m + 2n}}{(q; q)_m(q^3; q^3)_n} \equiv \frac{1}{(q^2, q^3, q^6, q^7; q^9)_\infty},
\]

\[
\sum_{m,n \geq 0} \frac{q^{m^2 + 3mn + 3n^2 + 2m + 4n}}{(q; q)_m(q^3; q^3)_n} \equiv \frac{1}{(q, q^4, q^6, q^7; q^9)_\infty}. 
\]
Polynomial Analogues
KR Mod 9 conjectures Cases

\[ \text{KR}_1(q) := \sum_{m,n \geq 0} q^{m^2 + 3mn + 3n^2} \frac{1}{(q; q)_m(q^3; q^3)_n}, \]

\[ \text{KR}_2(q) := \sum_{m,n \geq 0} q^{m^2 + 3mn + 3n^2 + m + n} \frac{1}{(q; q)_m(q^3; q^3)_n}, \]

\[ \text{KR}_3(q) := \sum_{m,n \geq 0} q^{m^2 + 3mn + 3n^2 + 2m + 3n} \frac{1}{(q; q)_m(q^3; q^3)_n}, \]

\[ \text{KR}_4(q) := \sum_{m,n \geq 0} q^{m^2 + 3mn + 3n^2 + m + 2n} \frac{1}{(q; q)_m(q^3; q^3)_n}, \]

\[ \text{KR}_5(q) := \sum_{m,n \geq 0} q^{m^2 + 3mn + 3n^2 + 2m + 4n} \frac{1}{(q; q)_m(q^3; q^3)_n} (1 + q + q^{m+3n+2}). \]
Polynomial Analogues

KR Mod 9 conjectures Cases

\[ \text{KR}_1(q, N) = \sum_{m,n \geq 0} q^{m^2+3mn+3n^2} \left[ N - m - 3n + 1 \right] \left[ \frac{2}{3} N - m - n + 1 \right] q^m \left[ \frac{2}{3} \left\lfloor \frac{2}{3} N \right\rfloor - m - n \right] q^n, \]

\[ \text{KR}_2(q, N) = \sum_{m,n \geq 0} q^{m^2+3mn+3n^2 + m + 3n} \left[ N - m - 3n \right] \left[ \frac{2}{3} N - m - n \right] q^m \left[ \frac{2}{3} \left\lfloor \frac{2}{3} N \right\rfloor - m - n \right] q^n, \]

\[ \text{KR}_3(q, N) = \sum_{m,n \geq 0} q^{m^2+3mn+3n^2 + 2m + 3n} \left[ N - m - 3n - 1 \right] \left[ \frac{2}{3} N - m - n \right] q^m \left[ \frac{2}{3} \left\lfloor \frac{2}{3} (N - 1) \right\rfloor - m - n \right] q^n, \]

\[ \text{KR}_4(q, N) = \sum_{m,n \geq 0} q^{m^2+3mn+3n^2 + m + 2n} \left[ N - m - 3n \right] \left[ \frac{2}{3} (N - 1) - m - n + 1 \right] q^m \left[ \frac{2}{3} \left\lfloor \frac{2}{3} (N - 1) \right\rfloor - m - n \right] q^n, \]

\[ \text{KR}_5(q, N) = \text{Looks like a sum of two lines above.} \]
Let $KR_1(q, N) := \sum_{n \geq 0} a_n q^n$, and think about graphing them on the Cartesian coordinates as $(n, a_n)$:
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Reflecting the Polynomials

Let \( KR_1(q, N) := \sum_{n \geq 0} a_n q^n \), and think about graphing them on the Cartesian coordinates as \((n, a_n)\):
Reflecting the Polynomials

Let $KR_1(q, N) := \sum_{n \geq 0} a_n q^n$, and think about graphing them on the Cartesian coordinates as $(n, a_n)$:
Reflecting the Polynomials

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Let $KR_1(q, N) := \sum_{n \geq 0} a_n q^n$, and think about graphing them on the Cartesian coordinates as $(n, a_n)$:
We start with the KR$_4(1/q, N)$. The degrees of these polynomials differ for 3 residue classes, so we handle these cases separately:

\[KR_4(q, 3M) = q^{M(3M+2)} KR_4(1/q, 3M),\]
\[KR_4(q, 3M + 1) = q^{M(3M+5)} KR_4(1/q, 3M + 1),\]
\[KR_4(q, 3M + 2) = q^{(M+1)(3M+2)} KR_4(1/q, 3M + 2).\]

\[\lim_{M \to \infty} KR_4(q, 3M) \equiv \frac{1}{(q^2; q^3)_{\infty} (q^3, q^9, q^{12}, q^{21}, q^{30}, q^{36}, q^{39}; q^{45})_{\infty}},\]
\[\lim_{M \to \infty} KR_4(q, 3M + 1) \equiv \frac{1}{(q^2; q^3)_{\infty} (q^3, q^{12}, q^{18}, q^{21}, q^{27}, q^{30}, q^{39}; q^{45})_{\infty}},\]
\[\lim_{M \to \infty} KR_4(q, 3M + 2) \equiv KR_4(q, 3\infty) + q^2 KR_4(q, 3\infty + 1).\]
Reflecting the KR polynomials

Conjecture (Reflected I4 Identities - U.–Zudilin, 2021)

\[
\sum_{a,b \geq 0} q^{a^2 - 3ab + 3b^2 + b} \left( \frac{3b - a}{(q^3; q^3)_b} a \right)_q = \frac{1}{(q^2; q^3)_\infty (q^3, q^9, q^{12}, q^{21}, q^{30}, q^{36}, q^{39}; q^{45})_\infty},
\]

\[
\sum_{a,b \geq 0} q^{a^2 - 3ab + 3b^2 + b - 2} \left( \frac{3b - a - 1}{(q^3; q^3)_b} a \right)_q = \frac{1}{(q^2; q^3)_\infty (q^3, q^{12}, q^{18}, q^{21}, q^{27}, q^{30}, q^{39}, q^{45})_\infty}.
\]

Theorem (One reflection of the I4 Conjecture - U.–Zudilin, 2021)

\[
\sum_{a,b \geq 0} q^{a^2 - 3ab + 3b^2 + b} \left( \frac{3b - a + 1}{(q^3; q^3)_b} a \right)_q = KR_4(q, 3\infty) + q^2 KR_4(q, 3\infty + 1).
\]
Reflecting the KR polynomials
Reflecting the 15* conjecture

We have

\[ KR_5(q, 3M) = q^{M(3M+1)} KR_5(1/q, 3M), \]
\[ KR_5(q, 3M + 1) = q^{(M+1)(3M+1)} KR_5(1/q, 3M + 1), \]
\[ KR_5(q, 3M + 2) = q^{(M+2)(3M+1)} KR_5(1/q, 3M + 2). \]

\[
\lim_{M \to \infty} KR_5(q, 3M + 1) \overset{?}{=} \frac{1}{(q; q^3)_\infty (q^6, q^9, q^{15}, q^{24}, q^{33}, q^{36}, q^{42}; q^{45})_\infty},
\]
\[
\lim_{M \to \infty} KR_5(q, 3M + 2) \overset{?}{=} \frac{1}{(q; q^3)_\infty (q^6, q^{15}, q^{18}, q^{24}, q^{27}, q^{33}, q^{42}; q^{45})_\infty},
\]
\[
\lim_{M \to \infty} KR_5(q, 3M) \overset{?}{=} \lim_{M \to \infty} KR_5(q, 3M + 1) + q^2 \lim_{M \to \infty} KR_5(q, 3M + 2).
\]
Reflecting the KR polynomials
Reflecting the I5* conjecture

Conjecture (Reflected I5* Identities - U.–Zudilin, 2021)

\[
\sum_{a,b\geq 0} \frac{q^{a^2-3ab+3b^2+2b(1+q)}}{(q^3; q^3)_b} \left[ \frac{3b-a}{a} \right]_q + \sum_{a,b\geq 0} \frac{q^{a^2-3ab+3b^2-a+5b+2}}{(q^3; q^3)_b} \left[ \frac{3b-a+1}{a} \right]_q
\]

\[
? = \frac{1}{(q; q^3)_{\infty}(q^6, q^9, q^{15}, q^{24}, q^{33}, q^{36}, q^{42}; q^{45})_{\infty}},
\]

\[
\sum_{a,b\geq 0} \frac{q^{a^2-3ab+3b^2+2b-2(1+q)}}{(q^3; q^3)_b} \left[ \frac{3b-a-1}{a-1} \right]_q + \sum_{a,b\geq 0} \frac{q^{a^2-3ab+3b^2-a+5b}}{(q^3; q^3)_b} \left[ \frac{3b-a}{a} \right]_q
\]

\[
? = \frac{1}{(q; q^3)_{\infty}(q^6, q^{15}, q^{18}, q^{24}, q^{27}, q^{33}, q^{42}; q^{45})_{\infty}}.
\]
Reflecting the KR polynomials

Reflecting the I1, I2 and I3 conjectures

Reflections of the conjectures

\[
\sum_{m,n \geq 0} \frac{q^{m^2+3mn+3n^2}}{(q; q)_m(q^3; q^3)_n} = \frac{1}{(q, q^3, q^6, q^8; q^9)_\infty},
\]

\[
\sum_{m,n \geq 0} \frac{q^{m^2+3mn+3n^2+m+3n}}{(q; q)_m(q^3; q^3)_n} = \frac{1}{(q^2, q^3, q^6, q^7; q^9)_\infty},
\]

\[
\sum_{m,n \geq 0} \frac{q^{m^2+3mn+3n^2+2m+3n}}{(q; q)_m(q^3; q^3)_n} = \frac{1}{(q^3, q^4, q^5, q^6; q^9)_\infty},
\]

yield more complicated expressions than the reflection of the asymmetric cases.

Let

\[
\langle c_1, c_2, c_3, c_4 \rangle = \frac{(q^{45}; q^{45})_\infty}{(q^3; q^3)_\infty \prod_{j=1}^{4}(q^{c_j}, q^{45-c_j}; q^{45})_\infty}.
\]
Reflecting the KR polynomials
Reflecting the I1, I2 and I3 conjectures

Conjecture (Reflected I1 Identities - U.–Zudilin, 2021)

\[
\sum_{a,b\geq 0} q^{a^2-3ab+3b^2-1} \frac{1}{(q^3; q^3)_b} \left[ \begin{array}{c} 3b - a - 1 \\ a \end{array} \right]_q \\
\quad \equiv \langle 2, 8, 11, 20 \rangle + q^3 \langle 2, 14, 20, 22 \rangle - q^8 \langle 17, 19, 20, 22 \rangle
\]

\[
\sum_{a,b\geq 0} q^{a^2-3ab+3b^2} \frac{1}{(q^3; q^3)_b} \left[ \begin{array}{c} 3b - a \\ a \end{array} \right]_q \\
\quad \equiv \langle 1, 7, 11, 20 \rangle + q^6 \langle 11, 13, 14, 20 \rangle - q^6 \langle 8, 14, 19, 20 \rangle.
\]
Theorem (U.–Zudilin, 2021)

\[
\sum_{a, b \geq 0} \frac{q^{2(a^2 - 3ab + 3b^2)}}{(q^3; q^3)_{2b}} \binom{3b}{2a}_q = (-q^2, -q^4; q^6)_\infty \sum_{c \geq 0} \frac{q^{3c}}{(q^3; q^3)_{2c}}
\]

\[
= \frac{1}{(q^2, q^{10}; q^{12})_\infty (q^3, q^6, q^9, q^{12}, q^{15}, q^{15}, q^{21}, q^{27}, q^{33}, q^{36}, q^{39}, q^{42}, q^{45}, q^{48})_\infty}
\]

\[
\sum_{a, b \geq 0} \frac{q^{2(a^2 - 3ab + 3b^2) + 2a - 3b - 1}}{(q^3; q^3)_{2b}} \binom{3b}{2a + 1}_q = (-q^2, -q^4; q^6)_\infty \sum_{c \geq 0} \frac{q^{3c}}{(q^3; q^3)_{2c+1}}
\]

\[
= \frac{1}{(q^2, q^{10}; q^{12})_\infty (q^3, q^3, q^6, q^{12}, q^{15}, q^{18}, q^{21}, q^{21}, q^{27}, q^{27}, q^{30}, q^{33}, q^{36}, q^{39}, q^{42}, q^{45}, q^{48})_\infty}
\]
Happy Birthday Doron!

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*Reflecting (on) the modulo 9 Kanade-Russell (conjectural) identities*

https://arxiv.org/abs/2106.02959
Reflecting (on) the modulo 9 Kanade-Russell (conjectural) identities

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Rutgers Experimental Mathematics Seminar

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