Reflecting (on) the modulo 9 Kanade-Russell (conjectural) identities

joint work with Wadim Zudilin

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Integer Partitions

A *partition* of the positive integer n is a *finite list of decreasing natural numbers*.

Example $n = 5$:

$$(1,1,1,1,1),\ (2,1,1,1),\ (2,2,1),\ (3,1,1),\ (3,2),\ (4,1),\ (5).$$

Size of a partition π is the sum of its parts, we denote this by $|\pi|$.

Generating Functions

For a sequence $\{a_n\}_{n=0}^{\infty}$, the series

$$\sum_{n\geq 0} a_n q^n$$

is called a *generating function*.

Let $\mathcal{U}$ be the set of all partitions.

$$\sum_{\pi\in\mathcal{U}} q^{|\pi|} = \sum_{n\geq 0} p(n)q^n = 1 + q + 2q^2 + 3q^3 + 5q^4 + \ldots,$$

where $p(n)$ is the number of partitions of n.

$\emptyset$	$(1,1,1),\ (2,1),\ (3)$
(1)	$(1,1,1,1),\ (2,1,1),\ (2,2),\ (3,1),\ (4)$
$(1,1),\ (2)$	

Generating Functions

$$(a;q)_L := \prod_{i=0}^{L-1}(1-aq^i),$$

$$(a;q)_\infty := \lim_{L\to\infty}(a;q)_L,$$

$$(a_1,a_2,\ldots,a_k;q)_L := \prod_{i=1}^{k}(a_i;q)_L \text{ for } k\in\mathbb{Z}_{\geq 1}.$$

Generating Functions
Example

Ex: The product

$$\frac{1}{(q;q)_n}$$

is the generating function for the number of partitions with parts $\leq n$.

Ex: The product

$$(-q;q)_n$$

is the generating function for the number of partitions into distinct parts $\leq n$.

Ex: For integers $1 \leq a \leq b$, products

$$\frac{1}{(q^a;q^b)_n} \text{ and } (-q^a;q^b)_n$$

are the generating functions for the number of partitions into unrestricted and distinct a modulo b parts, each $\leq b(n-1)+a$, respectively.

Generating Functions

Example

Ex: The product

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is the generating function for the number of partitions with number of parts $\leq n$.

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are the generating functions for the number of partitions into unrestricted and distinct a modulo b parts, each $\leq b(n-1)+a$, respectively.

Generating Functions

q-binomial Coefficients

$$\begin{bmatrix} m+n \\ m \end{bmatrix}_q := \begin{cases} \frac{(q;q)_{m+n}}{(q;q)_m (q;q)_n} & \text{for } m, n \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

The q-binomial coefficient

$$\begin{bmatrix} m+n \\ m \end{bmatrix}_q$$

can be interpreted as the generating function for the number of partitions that fits in an $m \times n$-box.

Partition Identities

We call any two sets of partitions being equinumerous for each size n, a *partition identity*.

Theorem (Euler's Partition Theorem)

For any integer n, the number of partitions of n into distinct parts
$=$
the number of partitions of n into odd parts.

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Ex: $n = 6$

(6)	$(5,1)$
$(5,1)$	$(3,3)$
$(4,2)$	$(3,1,1,1)$
$(3,2,1)$	$(1,1,1,1,1,1)$

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Theorem (Generating Function version of Euler's Partition Theorem)

$$(-q;q)_\infty = \frac{1}{(q;q^2)_\infty}.$$

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Theorem (Generating Function version of Euler's Partition Theorem)

$$\sum_{j\geq 0} \frac{q^{j(j+1)/2}}{(q;q)_j} = \frac{1}{(q;q^2)_\infty}.$$

Partition Identities

Rogers–Ramanujan Identity

Theorem (Rogers–Ramanujan Identities)

For any integer n and $i = 1$ or 2,
the number of partitions of n into non-consecutive parts $\geq i$
$=$
the number of partitions of into $\pm i$ modulo 5 parts.

Theorem (Generating Function version of RR-identities)

For $i = 1$ or 2,

$$\sum_{j\geq 0} \frac{q^{j^2+(i-1)j}}{(q;q)_j} = \frac{1}{(q^i, q^{5-i}; q^5)_\infty}.$$

Capparelli's First Partition Theorem

Theorem (Capparelli's First Partition Theorem)

For any integer n, the number of partitions of n into parts, not equal to 1, where the minimal difference between consecutive parts is 4, except if the consecutive parts are

- $3k$ *and* $3k+3$ *(yielding a difference of 3), or*
- $3k-1$ *and* $3k+1$ *(yielding a difference of 2) for some* $k \in \mathbb{N}$,

$=$

the number of partitions of n into distinct parts where no part is congruent to ± 1 *modulo* 6.

G. E. Andrews, *Schur's theorem. Capparelli's conjecture and q-trinomial coefficients*, Contemporary Mathematics **166** (1994), 141-154.

Capparelli's Second Partition Theorem

Theorem (Capparelli's Second Partition Theorem)

For any integer n, the number of partitions of n into parts, not equal to 2, where the minimal difference between consecutive parts is 4, except if the consecutive parts are

- $3k$ *and* $3k+3$ *(yielding a difference of 3), or*
- $3k-1$ *and* $3k+1$ *(yielding a difference of 2) for some* $k \in \mathbb{N}$,

$=$

the number of partitions of n into distinct parts where no part is congruent to ± 2 *modulo* 6.

S. Capparelli, *A combinatorial proof of a partition identity related to the level 3 representation of twisted affine Lie algebra*, Communications in Algebra **23** (1995), no. 8, 2959-2969..

Kanade–Russell Conjectures

In 2014, Shashank Kanade and Matthew Russell experimentally found 6 Difference-Congruence type conjectural partition theory identities.

Similar to the Capparelli's theorem case, the Generating Function counterparts for these conjectures were not known. Four of the six identities were related to modulo 9 products and the other two were related to modulo 12.

Russell presented one more modulo 9 identity in his PhD thesis in 2016, and in their follow up paper Kanade-Russell extended the list of modulo 12 identities to 11 identities total.

S. Kanade and M. C. Russell, `IdentityFinder` *and some new identities of Rogers–Ramanujan type*, Exp. Math. **24** (2015), no. 4, 419-423.

S. Kanade and M. C. Russell, Staircases to Generating Function sum-sides for many new integer partition identities of Rogers–Ramanujan type, Electron. J. Combin. **26** (2019), Art. 1.6.

Kanade–Russell Conjectures

One sample Mod 9 identity

Conjecture (KR - Conjecture I1)

The number of partitions with difference at least 3 at distance 2 such that if two consecutive parts differ by at most 1, then their sum is divisible by 3

$\stackrel{?}{=}$

The number of partitions of a non-negative integer into parts congruent to 1, 3, 6, or 8 mod 9.

In 2018, Kurşungöz and Kanade-Russell independently (through combinatorial construction and through experimentation, resp.) found the sum generating functions for the original Kanade–Russell conjectures.

S. Kanade and M. C. Russell, Staircases to Generating Function sum-sides for many new integer partition identities of Rogers–Ramanujan type, Electron. J. Combin. **26** (2019), Art. 1.6.

K. Kurşungöz, *Andrews–Gordon type series for Kanade–Russell conjectures*, *Ann. Comb.* **23** (2019), no. 3-4, 835–888.

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Conjecture (KR - I1 - Generating Function Counterpart)

$$\sum_{m,n\geq 0} \frac{q^{m^2+3mn+3n^2}}{(q;q)_m(q^3;q^3)_n} \stackrel{?}{=} \frac{1}{(q,q^3,q^6,q^8;q^9)_\infty}$$

Kanade–Russell Conjectures

Generating Function Versions of the Modulo 9 Identities

$$\sum_{m,n\geq 0} \frac{q^{m^2+3mn+3n^2}}{(q;q)_m(q^3;q^3)_n} \stackrel{?}{=} \frac{1}{(q,q^3,q^6,q^8;q^9)_\infty},$$

$$\sum_{m,n\geq 0} \frac{q^{m^2+3mn+3n^2+m+3n}}{(q;q)_m(q^3;q^3)_n} \stackrel{?}{=} \frac{1}{(q^2,q^3,q^6,q^7;q^9)_\infty},$$

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$$\sum_{m,n\geq 0} \frac{q^{m^2+3mn+3n^2+2m+4n}}{(q;q)_m(q^3;q^3)_n}(1+q+q^{m+3n+2}) \stackrel{?}{=} \frac{1}{(q,q^4,q^6,q^7;q^9)_\infty}.$$

Kanade–Russell Conjectures

Capparelli's identities

Kurşungöz and Kanade–Russell also found Generating Function counterparts for Capparelli's identities.

Theorem (Generating Function Version of the First Capparelli's identity)

$$\sum_{m,n\geq 0} \frac{q^{2(m^2+3mn+3n^2)}}{(q;q)_m(q^3;q^3)_n} = (-q^3;q^3)_\infty(-q^2,-q^4;q^6)_\infty.$$

Conjecture (KR - I1 - Generating Function Counterpart)

$$\sum_{m,n\geq 0} \frac{q^{m^2+3mn+3n^2}}{(q;q)_m(q^3;q^3)_n} \stackrel{?}{=} \frac{1}{(q,q^3,q^6,q^8;q^9)_\infty}.$$

K. Kurşungöz, *Andrews–Gordon type series for Capparelli's and Göllnitz–Gordon identities*, J. Combin. Theory Ser. A **165** (2019), 117–138.

q-Difference Equations

Capparelli Case

$$C_1(z) := \sum_{m,n\geq 0} \frac{q^{2(m^2+3mn+3n^2)}}{(q;q)_m(q^3;q^3)_n} z^{m+2n} = \sum_{k\geq 0} c_k(q) z^k.$$

$$C_1(z) - (1+q^3z)C_1(q^3z) - q^2z(1+q^2+q^4z)C_1(q^6z) - q^9z^2C_1(q^9z) = 0.$$

$$C_1(z) = (z;q^3)_\infty G(Z), \text{ where } G(z) := \sum_{n\geq 0} g_n(q) z^n.$$

Then,

$$q^3(1+q^{3+3n})(1-q^{3+6n})g_n - (1+q^3+q^{8+6n}+q^{10+6n})g_{n+1} + (1-q^{6+3n})g_{n+2} = 0.$$

...

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$$q^{15+9k}c_k - q^{12+6k}(1+q^{6+3k})c_{k+1} - q^{9+3k}(1+q^{5+3k}+q^{7+3k})c_{k+2} + (1-q^{9+3k})c_{k+3} = 0$$

$$C_1(z) = (z;q^3)_\infty G(Z), \text{ where } G(z) := \sum_{n\geq 0} g_n(q) z^n.$$

Then,

$$q^3(1+q^{3+3n})(1-q^{3+6n})g_n - (1+q^3+q^{8+6n}+q^{10+6n})g_{n+1} + (1-q^{6+3n})g_{n+2} = 0.$$

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$$C_1(z) - (1+q^3 z)C_1(q^3 z) - q^2 z(1+q^2+q^4 z)C_1(q^6 z) - q^9 z^2 (1-q^6 z)C_1(q^9 z) = 0.$$

$$C_1(z) = (z;q^3)_\infty G(Z), \text{ where } G(z) := \sum_{n\geq 0} g_n(q) z^n.$$

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...

q-Difference Equations

KR I1 Case

$$\mathrm{KR}_1(z) := \sum_{m,n\geq 0} \frac{q^{m^2+3mn+3n^2}}{(q;q)_m(q^3;q^3)_n} z^{m+2n}.$$

$$\begin{aligned}
&\left(q^6z+q^4z+1\right)KR_1(z)\\
&\quad+q^{11}z^3\left(q^{10}z^2+q^8z^2+q^7z+q^6z+q^5z+q^4z+q^3z+q^2+1\right)KR_1\left(q^6z\right)\\
&\quad-\left(q^{12}z^3+q^{10}z^3+q^9z^3+q^9z^2+q^8z^2+q^7z^3+q^7z^2+2q^6z^2+q^6z+q^5z^2\right.\\
&\qquad\left.+q^4z^2+q^4z+q^3z^2+q^3z+q^2z+qz+1\right)KR_1\left(q^3z\right)\\
&\quad+q^{27}z^5\left(q^3z+qz+1\right)KR_1\left(q^9z\right)=0
\end{aligned}$$

q-Partial Difference Equations
KR I1 Case

$$\mathrm{KR}_1(x,y) := \sum_{m,n\geq 0} \frac{q^{m^2+3mn+3n^2}}{(q;q)_m(q^3;q^3)_n} x^{m+3n} y^m.$$

$$-\mathrm{KR}_1(x,y) + \mathrm{KR}_1(x,qy) + qxy\,\mathrm{KR}_1(qx,qy) = 0.$$

For example,

$$f(x,y) = \sum_{m\geq 0} \frac{q^{m^2}(q^{m+1}x;q)_\infty}{(q;q)_m}(xy)^m$$

satisfies the q-(partial) difference equation above but we do not have enough information to continue with this technique.

Polynomial Analogues

Capparelli Case

Theorem (Kurşungöz , 2018)

$$\sum_{m,n\geq 0} \frac{q^{2m^2+6mn+6n^2}}{(q;q)_m(q^3;q^3)_n} = (-q^2,-q^4;q^6)_\infty(-q^3;q^3)_\infty,$$

$$\sum_{m,n\geq 0} \frac{q^{2m^2+6mn+6n^2+m+3n}}{(q;q)_m(q^3;q^3)_n} + \sum_{m,n\geq 0} \frac{q^{2m^2+6mn+6n^2+3m+6n+1}}{(q;q)_m(q^3;q^3)_n} = (-q,-q^5;q^6)_\infty(-q^3;q^3)_\infty.$$

K. Kurşungöz, *Andrews–Gordon type series for Capparelli's and Göllnitz–Gordon identities*, J. Combin. Theory Ser. A **165** (2019), 117–138.

Polynomial Analogues
Capparelli Case

Theorem (Berkovich-U, 2018)

$$\sum_{m,n\geq 0} q^{2m^2+6mn+6n^2} \begin{bmatrix} 3(N-2n-m) \\ m \end{bmatrix}_q \begin{bmatrix} 2(N-2n-m)+n \\ n \end{bmatrix}_{q^3} = \sum_{l=0}^{N} q^{3\binom{N-2l}{2}} \begin{bmatrix} N \\ 2l \end{bmatrix}_{q^3} (-q^2,-q^4;q^6)_l,$$

$$\sum_{m,n\geq 0} q^{2m^2+6mn+6n^2+m+3n} \begin{bmatrix} 3(N-2n-m)+2 \\ m \end{bmatrix}_q \begin{bmatrix} 2(N-2n-m)+n+1 \\ n \end{bmatrix}_{q^3}$$

$$+ \sum_{m,n\geq 0} q^{2m^2+6mn+6n^2+3m+6n+1} \begin{bmatrix} 3(N-2n-m) \\ m \end{bmatrix}_q \begin{bmatrix} 2(N-2n-m)+n \\ n \end{bmatrix}_{q^3}$$

$$= \sum_{l=0}^{N} q^{3\binom{N-2l}{2}} \begin{bmatrix} N+1 \\ 2l+1 \end{bmatrix}_{q^3} (-q;q^6)_{l+1} (-q^5;q^6)_l.$$

A. Berkovich and A. K. Uncu, *Polynomial identities implying Capparelli's partition theorems*, J. Number Theory **201** (2019), 77-107.

Polynomial Analogues

KR Mod 9 conjectures Cases

$$\sum_{m,n\geq 0}\frac{q^{m^2+3mn+3n^2}}{(q;q)_m(q^3;q^3)_n}\stackrel{?}{=}\frac{1}{(q,q^3,q^6,q^8;q^9)_\infty},$$

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Polynomial Analogues

KR Mod 9 conjectures Cases

$$\mathrm{KR}_1(q) := \sum_{m,n\geq 0} \frac{q^{m^2+3mn+3n^2}}{(q;q)_m(q^3;q^3)_n},$$

$$\mathrm{KR}_2(q) := \sum_{m,n\geq 0} \frac{q^{m^2+3mn+3n^2+m+3n}}{(q;q)_m(q^3;q^3)_n},$$

$$\mathrm{KR}_3(q) := \sum_{m,n\geq 0} \frac{q^{m^2+3mn+3n^2+2m+3n}}{(q;q)_m(q^3;q^3)_n},$$

$$\mathrm{KR}_4(q) := \sum_{m,n\geq 0} \frac{q^{m^2+3mn+3n^2+m+2n}}{(q;q)_m(q^3;q^3)_n},$$

$$\mathrm{KR}_5(q) := \sum_{m,n\geq 0} \frac{q^{m^2+3mn+3n^2+2m+4n}}{(q;q)_m(q^3;q^3)_n}(1+q+q^{m+3n+2}).$$

Polynomial Analogues

KR Mod 9 conjectures Cases

$$\mathrm{KR}_1(q,N) = \sum_{m,n\geq 0} q^{m^2+3mn+3n^2} \begin{bmatrix} N-m-3n+1 \\ m \end{bmatrix}_q \begin{bmatrix} \lfloor \frac{2}{3}N \rfloor - m - n + 1 \\ n \end{bmatrix}_{q^3},$$

$$\mathrm{KR}_2(q,N) = \sum_{m,n\geq 0} q^{m^2+3mn+3n^2+m+3n} \begin{bmatrix} N-m-3n \\ m \end{bmatrix}_q \begin{bmatrix} \lfloor \frac{2}{3}N \rfloor - m - n \\ n \end{bmatrix}_{q^3},$$

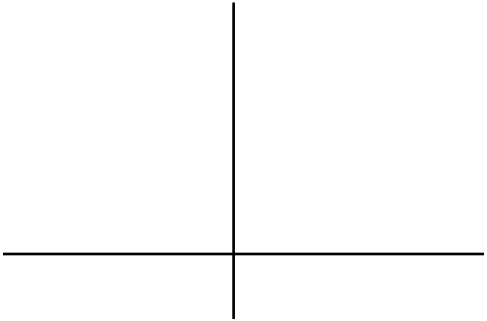
$$\mathrm{KR}_3(q,N) = \sum_{m,n\geq 0} q^{m^2+3mn+3n^2+2m+3n} \begin{bmatrix} N-m-3n-1 \\ m \end{bmatrix}_q^* \begin{bmatrix} \lfloor \frac{2}{3}N \rfloor - m - n \\ n \end{bmatrix}_{q^3},$$

$$\mathrm{KR}_4(q,N) = \sum_{m,n\geq 0} q^{m^2+3mn+3n^2+m+2n} \begin{bmatrix} N-m-3n \\ m \end{bmatrix}_q \begin{bmatrix} \lfloor \frac{2}{3}(N-1) \rfloor - m - n + 1 \\ n \end{bmatrix}_{q^3},$$

$\mathrm{KR}_5(q,N) =$ *Looks like a sum of two lines above.*

Reflecting the Polynomials

Let $\mathrm{KR}_1(q, N) := \sum_{n \geq 0} a_n q^n$, and think about graphing them on the Cartesian coordinates as (n, a_n):



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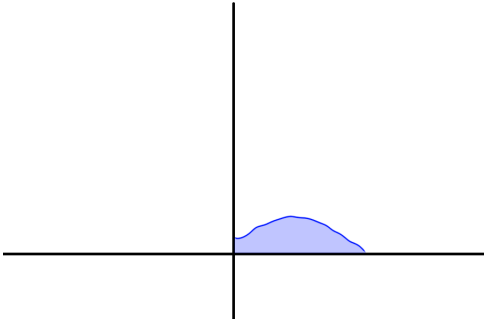

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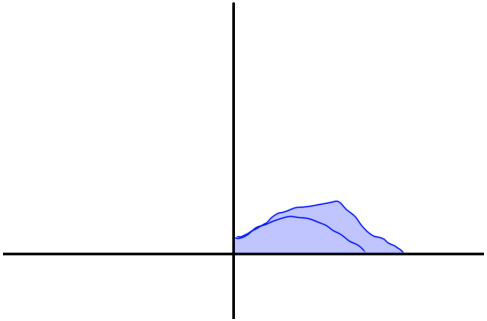

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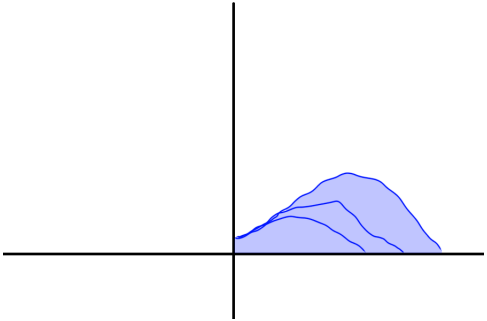

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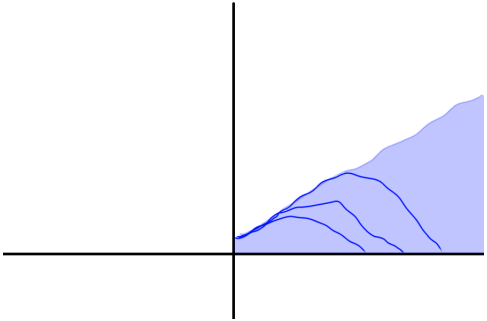
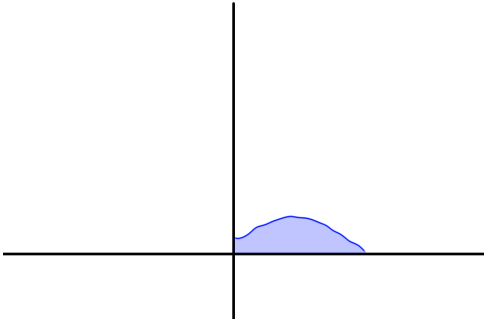


Reflecting the Polynomials

Let $\mathrm{KR}_1(q, N) := \sum_{n \geq 0} a_n q^n$, and think about graphing them on the Cartesian coordinates as (n, a_n):



Reflecting the Polynomials

$q \mapsto 1/q$

Let $\mathrm{KR}_1(q, N) := \sum_{n \geq 0} a_n q^n$, and think about graphing them on the Cartesian coordinates as (n, a_n):

Reflecting the Polynomials

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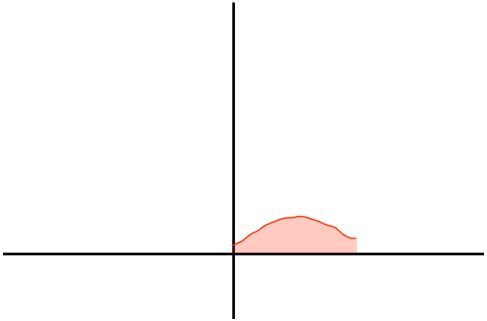

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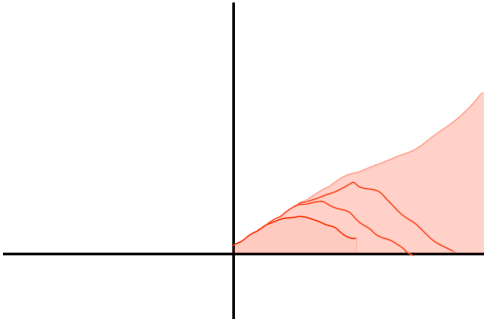

Reflecting the KR polynomials

Reflecting the I4 conjecture

We start with the $\mathrm{KR}_4(1/q, N)$. The degrees of these polynomials differ for 3 residue classes, so we handle these cases separately:

$$\begin{aligned}
\text{ϰ}\text{ϒ}_4(q, 3M) &= q^{M(3M+2)}\,\mathrm{KR}_4(1/q, 3M),\\
\text{ϰ}\text{ϒ}_4(q, 3M+1) &= q^{M(3M+5)}\,\mathrm{KR}_4(1/q, 3M+1),\\
\text{ϰ}\text{ϒ}_4(q, 3M+2) &= q^{(M+1)(3M+2)}\,\mathrm{KR}_4(1/q, 3M+2).
\end{aligned}$$

$$\begin{aligned}
\lim_{M\to\infty} \text{ϰ}\text{ϒ}_4(q, 3M) &\stackrel{?}{=} \frac{1}{(q^2;q^3)_\infty (q^3, q^9, q^{12}, q^{21}, q^{30}, q^{36}, q^{39}; q^{45})_\infty},\\
\lim_{M\to\infty} \text{ϰ}\text{ϒ}_4(q, 3M+1) &\stackrel{?}{=} \frac{1}{(q^2;q^3)_\infty (q^3, q^{12}, q^{18}, q^{21}, q^{27}, q^{30}, q^{39}; q^{45})_\infty},\\
\lim_{M\to\infty} \text{ϰ}\text{ϒ}_4(q, 3M+2) &\stackrel{?}{=} \text{ϰ}\text{ϒ}_4(q, 3\infty) + q^2\,\text{ϰ}\text{ϒ}_4(q, 3\infty+1).
\end{aligned}$$

Reflecting the KR polynomials

Reflecting the I4 conjecture

Conjecture (Reflected I4 Identities - U.–Zudilin, 2021)

$$\sum_{a,b\geq 0} \frac{q^{a^2-3ab+3b^2+b}}{(q^3;q^3)_b} \begin{bmatrix} 3b-a \\ a \end{bmatrix}_q \stackrel{?}{=} \frac{1}{(q^2;q^3)_\infty (q^3,q^9,q^{12},q^{21},q^{30},q^{36},q^{39};q^{45})_\infty},$$

$$\sum_{a,b\geq 0} \frac{q^{a^2-3ab+3b^2+b-2}}{(q^3;q^3)_b} \begin{bmatrix} 3b-a-1 \\ a \end{bmatrix}_q \stackrel{?}{=} \frac{1}{(q^2;q^3)_\infty (q^3,q^{12},q^{18},q^{21},q^{27},q^{30},q^{39};q^{45})_\infty}$$

Theorem (One reflection of the I4 Conjecture - U.–Zudilin, 2021)

$$\sum_{a,b\geq 0} \frac{q^{a^2-3ab+3b^2+b}}{(q^3;q^3)_b} \begin{bmatrix} 3b-a+1 \\ a \end{bmatrix}_q = \text{ϰ}\text{ϱ}_4(q,3\infty) + q^2\,\text{ϰ}\text{ϱ}_4(q,3\infty+1).$$

Reflecting the KR polynomials

Reflecting the I5* conjecture

We have

$$\text{ϰ}\text{ϱ}_5(q,3M) = q^{M(3M+1)}\,\mathrm{KR}_5(1/q,3M),$$
$$\text{ϰ}\text{ϱ}_5(q,3M+1) = q^{(M+1)(3M+1)}\,\mathrm{KR}_5(1/q,3M+1),$$
$$\text{ϰ}\text{ϱ}_5(q,3M+2) = q^{(M+2)(3M+1)}\,\mathrm{KR}_5(1/q,3M+2).$$

$$\lim_{M\to\infty} \text{ϰ}\text{ϱ}_5(q,3M+1) \stackrel{?}{=} \frac{1}{(q;q^3)_\infty(q^6,q^9,q^{15},q^{24},q^{33},q^{36},q^{42};q^{45})_\infty},$$
$$\lim_{M\to\infty} \text{ϰ}\text{ϱ}_5(q,3M+2) \stackrel{?}{=} \frac{1}{(q;q^3)_\infty(q^6,q^{15},q^{18},q^{24},q^{27},q^{33},q^{42};q^{45})_\infty},$$
$$\lim_{M\to\infty} \text{ϰ}\text{ϱ}_5(q,3M) \stackrel{?}{=} \lim_{M\to\infty} \text{ϰ}\text{ϱ}_5(q,3M+1) + q^2 \lim_{M\to\infty} \text{ϰ}\text{ϱ}_5(q,3M+2).$$

Reflecting the KR polynomials

Reflecting the I5* conjecture

Conjecture (Reflected I5* Identities - U.–Zudilin, 2021)

$$\sum_{a,b\geq 0}\frac{q^{a^2-3ab+3b^2+2b}(1+q)}{(q^3;q^3)_b}\begin{bmatrix}3b-a\\ a\end{bmatrix}_q+\sum_{a,b\geq 0}\frac{q^{a^2-3ab+3b^2-a+5b+2}}{(q^3;q^3)_b}\begin{bmatrix}3b-a+1\\ a\end{bmatrix}_q$$
$$\stackrel{?}{=}\frac{1}{(q;q^3)_\infty(q^6,q^9,q^{15},q^{24},q^{33},q^{36},q^{42};q^{45})_\infty},$$

$$\sum_{a,b\geq 0}\frac{q^{a^2-3ab+3b^2+2b-2}(1+q)}{(q^3;q^3)_b}\begin{bmatrix}3b-a-1\\ a\end{bmatrix}_q+\sum_{a,b\geq 0}\frac{q^{a^2-3ab+3b^2-a+5b}}{(q^3;q^3)_b}\begin{bmatrix}3b-a\\ a\end{bmatrix}_q$$
$$\stackrel{?}{=}\frac{1}{(q;q^3)_\infty(q^6,q^{15},q^{18},q^{24},q^{27},q^{33},q^{42};q^{45})_\infty}.$$

Reflecting the KR polynomials

Reflecting the I1, I2 and I3 conjectures

Reflections of the conjectures

$$\sum_{m,n\geq 0}\frac{q^{m^2+3mn+3n^2}}{(q;q)_m(q^3;q^3)_n}\stackrel{?}{=}\frac{1}{(q,q^3,q^6,q^8;q^9)_\infty},$$

$$\sum_{m,n\geq 0}\frac{q^{m^2+3mn+3n^2+m+3n}}{(q;q)_m(q^3;q^3)_n}\stackrel{?}{=}\frac{1}{(q^2,q^3,q^6,q^7;q^9)_\infty},$$

$$\sum_{m,n\geq 0}\frac{q^{m^2+3mn+3n^2+2m+3n}}{(q;q)_m(q^3;q^3)_n}\stackrel{?}{=}\frac{1}{(q^3,q^4,q^5,q^6;q^9)_\infty},$$

yield more complicated expressions than the reflection of the asymmetric cases. Let

$$\langle c_1,c_2,c_3,c_4\rangle=\frac{(q^{45};q^{45})_\infty}{(q^3;q^3)_\infty\prod_{j=1}^{4}(q^{c_j},q^{45-c_j};q^{45})_\infty}.$$

Reflecting the KR polynomials

Reflecting the I1, I2 and I3 conjectures

Conjecture (Reflected I1 Identities - U.–Zudilin, 2021)

$$\sum_{a,b\geq 0} \frac{q^{a^2-3ab+3b^2-1}}{(q^3;q^3)_b} \begin{bmatrix} 3b-a-1 \\ a \end{bmatrix}_q \stackrel{?}{=} \langle 2,8,11,20\rangle + q^3\langle 2,14,20,22\rangle - q^8\langle 17,19,20,22\rangle$$

$$\sum_{a,b\geq 0} \frac{q^{a^2-3ab+3b^2}}{(q^3;q^3)_b} \begin{bmatrix} 3b-a \\ a \end{bmatrix}_q \stackrel{?}{=} \langle 1,7,11,20\rangle + q^6\langle 11,13,14,20\rangle - q^6\langle 8,14,19,20\rangle.$$

Reflecting the Capparelli polynomials

Theorem (U.–Zudilin, 2021)

$$\sum_{a,b\geq 0}\frac{q^{2(a^2-3ab+3b^2)}}{(q^3;q^3)_{2b}}\begin{bmatrix}3b\\2a\end{bmatrix}_q=(-q^2,-q^4;q^6)_\infty\sum_{c\geq 0}\frac{q^{3c}}{(q^3;q^3)_{2c}}$$
$$=\frac{1}{(q^2,q^{10};q^{12})_\infty(q^3,q^6,q^9,q^9,q^{12},q^{15},q^{15},q^{21},q^{27},q^{33},q^{33},q^{36},q^{39},q^{39},q^{42},q^{45};q^{48})_\infty},$$

$$\sum_{a,b\geq 0}\frac{q^{2(a^2-3ab+3b^2)+2a-3b-1}}{(q^3;q^3)_{2b}}\begin{bmatrix}3b\\2a+1\end{bmatrix}_q=(-q^2,-q^4;q^6)_\infty\sum_{c\geq 0}\frac{q^{3c}}{(q^3;q^3)_{2c+1}}$$
$$=\frac{1}{(q^2,q^{10};q^{12})_\infty(q^3,q^3,q^9,q^{12},q^{15},q^{18},q^{21},q^{21},q^{27},q^{27},q^{30},q^{33},q^{36},q^{39},q^{45},q^{45};q^{48})_\infty}.$$

Happy Birthday Doron!

A. K. Uncu, and W. Zudilin,
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https://arxiv.org/abs/2106.02959

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joint work with Wadim Zudilin

Ali Kemal Uncu (aku21@bath.ac.uk)

Rutgers Experimental Mathematics Seminar

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