Origami Flip Graphs of Single Vertex Crease Patterns

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Overview

What is Origami?

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The art of paper folding.

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Crease Pattern

The set of edges (creases) and vertices that one will be folding along.

Mountain/Valley Assignments

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Every crease can folded in one of two ways: like a mountain or like a valley. An MV assignment μ assigns one of $\{$-1,1$\}$ to a crease where -1 represents valleys and 1 represents mountains.

When is a Crease Pattern Flat Foldable?

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Maekawa's Theorem

$M - V = \pm 2$

Big-Little-Big

- If you have an odd number of consecutive smallest angles, $M - V = 0$ for the creases surrounding them.
- If you have an even number of consecutive smallest angles, $M - V = \pm 1$ for the creases surrounding them.

When is a Crease Pattern Flat Foldable?

- **Local flat foldability** is when the creases around every vertex can fold flat.
- **Global flat foldability** is when the crease pattern folds flat entirely.
- For a single vertex crease pattern, these are the same.

When is a Crease Pattern Flat Foldable?

- **Local flat foldability** is when the creases around every vertex can fold flat.
- **Global flat foldability** is when the crease pattern folds flat entirely.
- For a single vertex crease pattern, these are the same.
- We consider an MV assignment on a crease pattern **valid** if it is locally flat foldable.
- $C(v)$ is the number of valid MV assignments for a crease pattern.

What are Face Flips?

- We define a **face flip** as a reversal of parity of every crease bordering the face.

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- We define a **face flip** as a reversal of parity of every crease bordering the face.
- We say that a face flip is **valid** if it results in a new MV assignment that is also valid.
- For example, in a single vertex crease pattern, the face between two creases which are the minority is not flippable, since flipping it would violate Maekawa's rule $M - V = \pm 2$

Flippable Faces

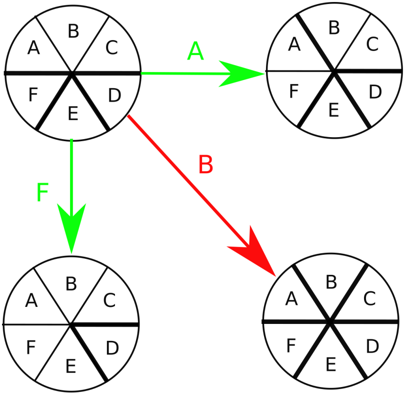
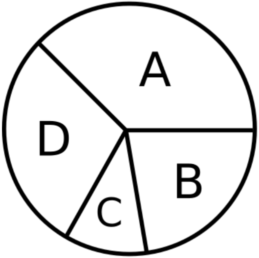


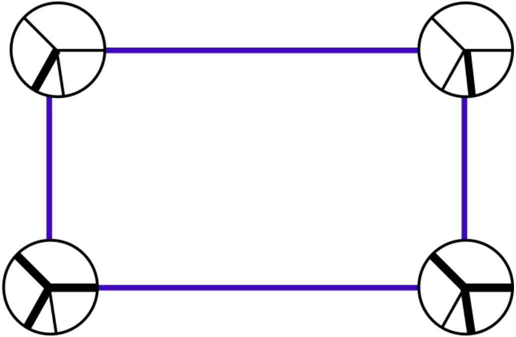
Figure: Flipping the face between two minority creases (Face B) is not a valid face flip.

Constructing the OFG

- The **Origami Flip Graph** or **OFG(*v*)** is the graph where every vertex is a valid MV assignment for *v* and every edge connecting two vertices is a face flip which goes from one MV assingment to the other.
- We studied *single vertex* crease patterns and their OFGs. From now on, we will only be talking about single vertex crease patterns unless we say otherwise.

Some Examples of OFGs for Degree 4 Crease Patterns





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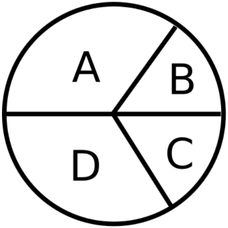

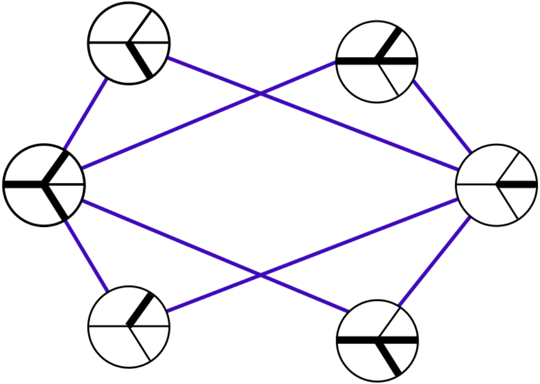

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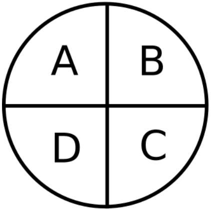
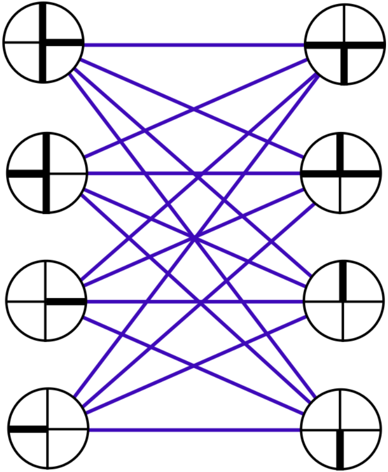




Counting Vertices in an OFG

- Number of vertices in an OFG $= C(v)$

Recursive Folding

To fold a crease pattern, find the smallest angle(s) and take the largest consecutive set of them. Fold that and repeat until you get to all equal angles. As the last step, fold them.

- Each step in the rescursive folding contributes a factor to the $C(v)$. If k consecutive angles are folded, there are $\binom{k}{\lfloor k/2 \rfloor}$ ways to assign mountain/valley folds to that set of creases. The last step contributes a factor of $2\binom{2k}{k-1}$.
- For a vertex of degree $2n$,

$$2^n \leq C(v) \leq 2\binom{2n}{n-1}$$

Recursive Folding

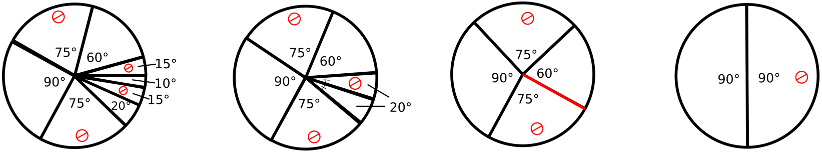


Figure: First, the smallest angle is folded up. Then, two equal angles appear and are folded up. This leaves a small angle, which, when folded, produces two equal angles.

- When folding up an odd number of angles, they collapse into a face with a new angle. If the bigs have angles α and γ and the littles have angle β, then the new face with have angle $\alpha + \gamma - \beta$.
- When folding up an even number of angles, they collapse into a crease.

Single Vertex OFGs are Bipartite

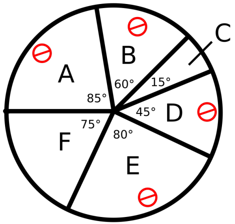
All OFGs for Single-Vertex Crease Patterns are Bipartite

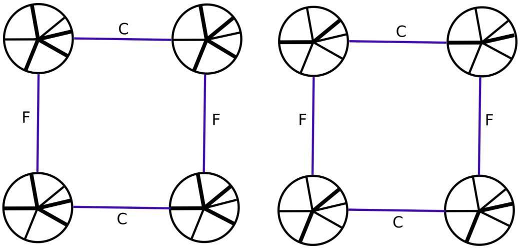
All cycles either contain each face an even number of times or contains every face.

Characterization of an OFG for the Smallest $C(v)$

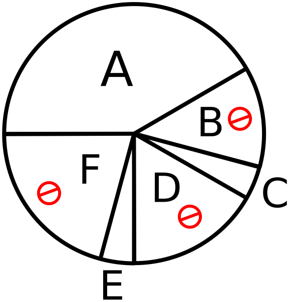
- The smallest possible $C(v) = 2^n$. This occurs when the recursive folding involves $n-1$ smallest angles, one after another, followed by 2 equal angles.
- The only faces that can be flipped are faces between two creases of a small or two creases of the final equal angles. Because these do not influence each other, a given face is either always flippable or never flippable.
- The OFG is 2^{n-k} k-dimensional hypercubes where k is the number of always flippable faces

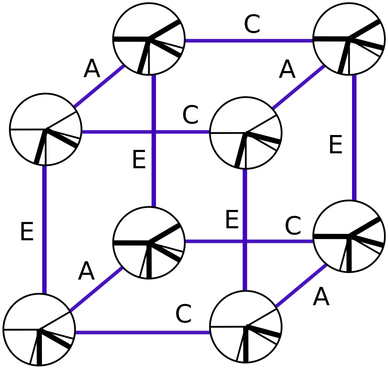
Disconnected Smallest C(v) for degree 6





Connected Smallest C(v) for degree 6





Causes of Disconnectedness for Single Vertex CPs

- Disconnectedness is caused by the interactions of **Never Flippable Faces**
- Never Flippable Faces appear on either side of a big-little-big with an odd number of littles.
- Upon being folded up, the new face is also considered a Never Flippable Face
- If a Never Flippable Face gets folded up into a crease, that crease is called a **Restricted Crease**.
- Restricted Creases that get folded up into creases also cause restricted creases

Never Flippable Faces and Restricted Creases

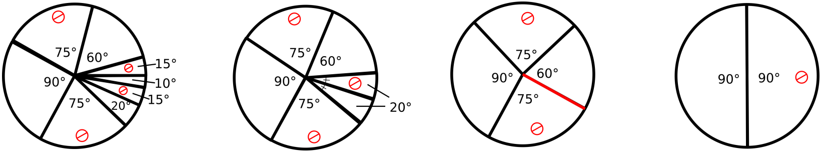


Figure: First, the smallest angle is folded up. Then, two equal angles appear and are folded up. This leaves a small angle, which, when folded, produces two equal angles.

Characterization of an OFG for the Largest $C(v)$

- The largest possible $C(v) = 2\binom{2n}{n-1}$. This occurs when all the angles are equal.
- Let A_{2n} be the crease pattern with $2n$ equal angles.
- The only faces that cannot be flipped from a given MV assignment are faces surrounded by minority creases.
- To count the number of edges in this graph, we counted the number of possible edges (if every face was flippable) and subtracted the number of faces that cannot be flipped from a given MV assignment for every MV assignment.

Flippable Faces for All Equal Angles

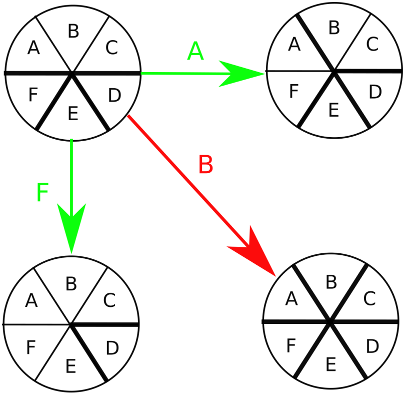
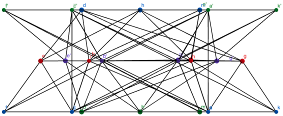
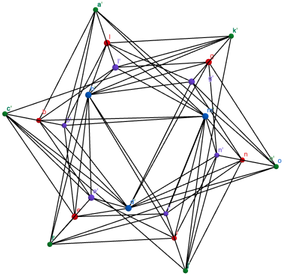


Figure: Flipping the face between two minority creases (Face B) is not a valid face flip.

Images of $OFG(A_6)$

Click here for the interactive file.





Vertices and Edges in $OFG(A_{2n})$

n	# Vertices	# Edges
1	2	2
2	8	16
3	30	84
4	112	400
5	420	1820
6	1584	8064
7	6006	35112
8	22880	151008
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(A162551)

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(A162551)

$$E_{2n} = \text{Number of edges in } OFG(A_{2n})$$
$$= \frac{(n+1)(3n-2)}{2n-1}\binom{2n}{n-1}$$

Edges in OFG for All Equal Angles

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$$\begin{aligned} f_{2n}(k) &= \text{Number of vertices of degree k in } OFG(A_{2n}) \\ &= \frac{4n}{n+1}\binom{n+1}{k-n-1}\binom{n-2}{k-n-2} \end{aligned}$$

Connectivity for A_{2n}

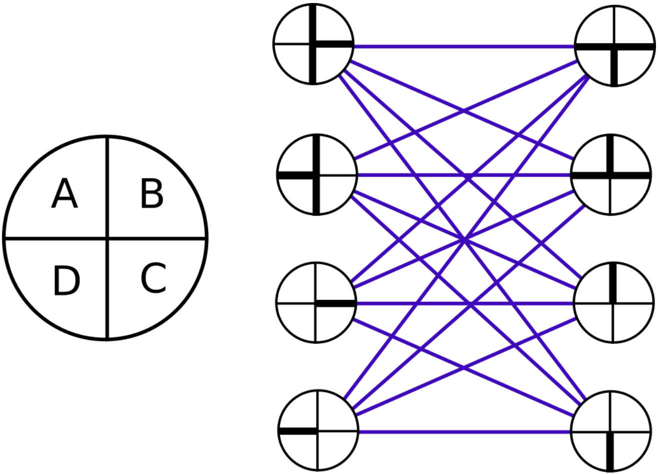
- $OFG(A_{2n})$ is connected
- $OFG(A_{2n})$ has diameter n

Looking at Subgraphs

Theorem

Let C be a flat-foldable, single-vertex crease pattern of degree $2n$ that is not A_{2n}. Then (C) is isomorphic to at least $2n$ distinct subgraphs of (A_{2n}).

Looking at Subgraphs: An Example



Thank You!

Thank you to Sarah Nash and Manny Morales, who I did this work with, Dr. Thomas C. Hull, the director of the MathILy-EST REU, Robert Dougherty-Bliss for conjecturing the edges formula, Jonah Ostroff for the combinatorial proof, dr. sarah-marie belcastro, the PI, Mathematical Staircase, Inc. and everyone who contributed thoughts and ideas to this research.

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