A journey with MacMahon and Ramanujan series

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EM seminar @ Rutgers University
25 September, 2025

(with many)

The infectious "q-disease"

Notation

$$[x]_q := \frac{1-q^x}{1-q} \qquad \textit{and} \qquad \begin{bmatrix} x \\ k \end{bmatrix}_q := \frac{[x]_q[x-1]_q \cdots [x-k+1]_q}{[1]_q[2]_q \cdots [k]_q}.$$

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Dedekind η-function

$$\eta(q) := q^{\frac{1}{24}}(q)_\infty \qquad \textit{where} \qquad (q)_\infty = \prod_{j=1}^{\infty}(1-q^j).$$

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EXAMPLE

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6,$$

$$[4]_q = 1 + q + q^2 + q^3.$$

Once upon a time . . .

The finite version

Theorem (TA, G. Andrews, R.Tauraso '24)

For each positive integer $\boldsymbol{n}$, *we have*

$$\sum_{1\le k_1\le\cdots\le k_t\le \boldsymbol{n}} \frac{q^{k_1+k_2+\cdots+k_t}}{[k_1]_q^2[k_2]_q^2\cdots[k_t]_q^2},$$

$$=\sum_{1\le k_1\le\cdots\le k_{2t}\le \boldsymbol{n}} \frac{q^{\boldsymbol{n}+k_1+k_3\cdots+k_{2t-1}}+q^{k_2+k_4+\cdots+k_{2t}}}{[\boldsymbol{n}+k_1]_q[k_2]_q\cdots[k_{2t}]_q}$$

$$=\sum_{k=1}^{\boldsymbol{n}} \frac{(-1)^{k-1}(1+q^k)q^{\binom{k}{2}+tk}\begin{bmatrix} n\\ k\end{bmatrix}_q}{[k]_q^{2t}\begin{bmatrix} \boldsymbol{n}+k\\ k\end{bmatrix}_q}.$$

The infinite version

Theorem (TA, G. Andrews, R. Tauraso '24)

For $t \geq 1$, we have the identity

$$\sum_{1\leq k_1\leq\cdots\leq k_t} \prod_{j=1}^{t} \frac{q^{k_j}}{(1-q^{k_j})^2} = \sum_{m\geq 1} \frac{(-1)^{m-1}(1+q^m)q^{\binom{m}{2}+tm}}{(1-q^m)^{2t}}.$$

Andrews recalls . . .

What is it?

Recall yet another

Theorem (G. Andrews, S. Rose '13)

For $t \geq 1$, we have the identity

$$\sum_{0<k_1<\cdots<k_t} \prod_{j=1}^{t} \frac{q^{k_j}}{(1-q^{k_j})^2} = \sum_{m \geq t} \frac{(-1)^{m+t}}{(q)_\infty^3} \frac{2m+1}{2t+1} \binom{m+t}{m-t} q^{\binom{m+1}{2}}.$$

The partition function $p(n)$

Definition

A **partition** of an integer n is a weakly decreasing sequence

$$\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$$

of positive integers which sum to n, denoted $\lambda \vdash n$.

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Notation

The partition function

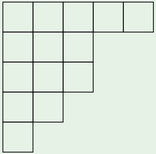
$$p(n) := \#\ \textit{partitions of } n.$$

$$4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1 \implies p(4) = 5.$$

Young diagram

Example

$\lambda = (5,3,3,2,1) \vdash 14$ or in frequency mode
$\lambda = (1^1, 2^1, 3^2, 4^0, 5^1)$:



Underappreciated Fact

Theorem

$\#\{\textit{partitions of } n \textit{ into identical parts}\} = \#\{\textit{divisors of } n\}.$

Major Percy Alexander MacMahon

MacMahon (1920)

Definition

For $t \geq 1$, define the **MacMahon partition function**

$$M_t(n) := \sum_{\substack{n=m_1s_1+\cdots+m_ts_t \\ 0<s_1<\cdots<s_t}} m_1 m_2 \cdots m_t.$$

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Example

If $\sigma_\nu(n) := \sum_{d|n} d^\nu$ then $M_1(n) = \sigma_1(n)$.

MacMahon's and its allied q-series

Definition (MacMahon)

$$\mathcal{U}_t(q) := \sum_{0<k_1<k_2<\cdots<k_t} \frac{q^{k_1+\cdots+k_t}}{(1-q^{k_1})^2\cdots(1-q^{k_t})^2}.$$

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Definition (TA, G. Andrews, R. Tauraso '24)

$$\mathcal{U}_t^{\star}(q) := \sum_{1\leq k_1\leq k_2\leq\cdots\leq k_t} \frac{q^{k_1+\cdots+k_t}}{(1-q^{k_1})^2\cdots(1-q^{k_t})^2}.$$

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Remark

$$\mathcal{U}_1(q) = \sum_{n\ge 1}\sigma_1(n)q^n = \mathcal{U}_1^\star(q).$$

About Quasimodular forms

Definition (Eisenstein series)

A **quasimodular form** is any q-series in the ring $\mathbb{C}[E_2, E_4, E_6]$, with

$$E_2(q) := 1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n,$$

$$E_4(q) = 1 + 240 \sum_{n \geq 1} \sigma_3(n) q^n,$$

$$E_6(q) = 1 - 504 \sum_{n \geq 1} \sigma_5(n) q^n.$$

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Moreover, the monomial $E_2^{\alpha} E_4^{\beta} E_6^{\gamma}$ has **weight** $2\alpha + 4\beta + 6\gamma$.

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Remark

$$\{\textbf{Quasimodular forms}\} = \color{red}{\mathbb{C}[E_2, E_4, E_6]} = \color{blue}{\mathbb{C}[E_2, E_4, E_6, E_8, E_{10}, \dots]}.$$

Buy one, get one free

Theorem (S. Rose '15; Andrews-Rose '13)

$\mathcal{U}_t(q)$ is a sum of quasimodular forms of weight $\leq 2t$.

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Proof. This is courtesy of combinatorics:

$$\sum_{i=0}^{t}(-1)^i\, e_i(\boldsymbol{x})\, h_{t-i}(\boldsymbol{x}) = 0 \qquad \text{with} \qquad x_i = \frac{q^i}{(1-q^i)^2},$$

a "duet" between elementary and complete homogeneous symmetric functions. ☐

Theorem (Explicit formula - TA, K. Ono, A. Singh '24)

$$\mathcal{U}_t(q) = \sum_{a=0}^{t} w_a(t) \sum_{\substack{\alpha,\beta,\gamma\geq 0\\ \alpha+2\beta+3\gamma=a}} c(\alpha,\beta,\gamma)\, E_2(q)^{\alpha} E_4(q)^{\beta} E_6(q)^{\gamma},$$

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$$w_a(t) := \frac{\binom{2t}{t}}{16^t(2t+1)} \sum_{0\leq \ell_1<\cdots<\ell_a<t} \prod_{j=1}^{a} \frac{1}{(2\ell_j+1)^2},$$

$$\begin{aligned} c(\alpha,\beta,\gamma) :=& -\frac{1}{3}(2\alpha+8\beta+12\gamma+1)\cdot c(\alpha-1,\beta,\gamma) \\ &+\frac{2}{3}(\alpha+1)\cdot c(\alpha+1,\beta-1,\gamma) \\ &+\frac{8}{3}(\beta+1)\cdot c(\alpha,\beta+1,\gamma-1) \\ &+4(\gamma+1)\cdot c(\alpha,\beta-2,\gamma+1). \end{aligned}$$

Explicit formulas for $\mathcal{U}_t^\star(q)$

Theorem (TA, K. Ono, A. Singh '24)

We have that

$$\mathcal{U}_t^\star(q) = \sum_{a=0}^{t} w_a^\star(t) \cdot \mathbb{E}_{2a}^\star(q),$$

EXPLICIT FORMULAS FOR $\mathcal{U}_t^\star(q)$

THEOREM (TA, K. ONO, A. SINGH '24)

We have that

$$\mathcal{U}_t^\star(q) = \sum_{a=0}^{t} w_a^\star(t) \cdot \mathbb{E}_{2a}^\star(q),$$

where

$$\mathbb{E}_{2t}^\star(q) := \sum_{(1^{m_1},\ldots,t^{m_t})\vdash t} \prod_{j=1}^{t} \frac{1}{m_j!} \left(-\frac{B_{2j}\, E_{2j}(q)}{(2j)\cdot(2j!)}\right)^{m_j}$$

and $w_a^\star(t)$ are explicit combinatorial rational numbers, while B_{2j} are the Bernoulli numbers.

Examples

Example

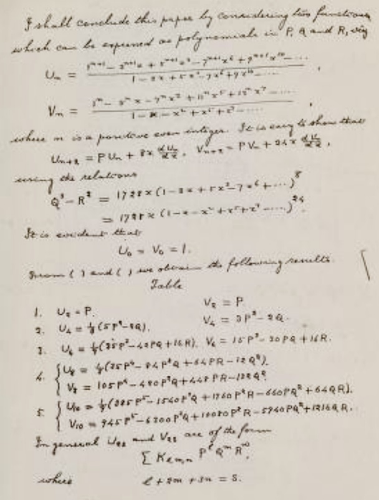
$$\mathcal{U}_3(q) = \frac{5}{7168} - \frac{37E_2}{46080} + \frac{5E_2^2}{27648} - \frac{E_4}{13824} - \frac{E_2^3}{82944} + \frac{E_2E_4}{69120} - \frac{E_6}{181440}$$

$$\mathcal{U}_5^\star(q) = \frac{1295803}{12262440960} + \frac{35}{294912}\mathbb{E}_2^\star(q) - \frac{3229}{967680}\mathbb{E}_4^\star(q) + \frac{47}{1152}\mathbb{E}_6^\star(q) - \frac{7}{24}\mathbb{E}_8^\star(q) + \mathbb{E}_{10}^\star(q).$$

Master Anticipator

Srinivasa Ramanujan

Page from the "Lost Notebook"


I shall conclude this paper by considering two functions which can be expressed as polynomials in P, Q and R, viz

$$U_n = \frac{1^{n+1} - 3^{n+1}x + 5^{n+1}x^3 - 7^{n+1}x^6 + 9^{n+1}x^{10} - \cdots}{1 - 3x + 5x^3 - 7x^6 + 9x^{10} - \cdots},$$

$$V_n = \frac{1^n - 5^n x - 7^n x^2 + 11^n x^5 + 13^n x^7 - \cdots}{1 - x - x^2 + x^5 + x^7 - \cdots},$$

where n is a positive even integer. It is easy to show that

$$U_{n+2} = PU_n + 8x\frac{dU_n}{dx}, \quad V_{n+2} = PV_n + 24x\frac{dV_n}{dx},$$

using the relations

$$Q^3 - R^2 = 1728x(1 - 3x + 5x^3 - 7x^6 + \cdots)^8 = 1728x(1 - x - x^2 + x^5 + x^7 - \cdots)^{24}.$$

It is evident that

$$U_0 = V_0 = 1.$$

From ( ) and ( ) we obtain the following results

Table

1. $U_2 = P.$ $\quad V_2 = P$
2. $U_4 = \frac{1}{3}(5P^2 - 2Q).$ $\quad V_4 = 3P^2 - 2Q.$
3. $U_6 = \frac{1}{9}(35P^3 - 42PQ + 16R).$ $\quad V_6 = 15P^3 - 30PQ + 16R.$
4. $U_8 = \frac{1}{3}(35P^4 - 84P^2Q + 64PR - 12Q^2).$ $\quad V_8 = 105P^4 - 420P^2Q + 448PR - 132Q^2.$
5. $U_{10} = \frac{1}{3}(385P^5 - 1540P^3Q + 1760P^2R - 660PQ^2 + 64QR).$ $\quad V_{10} = 945P^5 - 6300P^3Q + 10080P^2R - 5940PQ^2 + 1216QR.$

In general U_{2s} and V_{2s} are of the form

$$\sum K_{\ell,m,n} P^\ell Q^m R^n,$$

where

$$\ell + 2m + 3n = s.$$

Ramanujan's Claim

For non-negative integers t, consider

$$U_{2t}(q) = \frac{1^{2t+1} - 3^{2t+1}q + 5^{2t+1}q^3 - 7^{2t+1}q^6 + 9^{2t+1}q^{10} - \cdots}{1 - 3q + 5q^3 - 7q^6 + 9q^{10} - \cdots}$$

$$V_{2t}(q) = \frac{1^{2t} - 5^{2t}q - 7^{2t}q^2 + 11^{2t}q^5 + 13^{2t}q^7 - \cdots}{1 - q - q^2 + q^5 + q^7 - \cdots}.$$

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We have identities such as

$$U_0 = 1,\ U_2 = E_2,\ U_4 = \frac{1}{3}(5E_2^2 - 2E_4),\ U_6 = \frac{1}{9}(35E_2^3 - 42E_2E_4 + 16E_6), \ldots$$

$$V_0 = 1,\ V_2 = E_2,\ V_4 = 3E_2^2 - 2E_4,\qquad V_6 = 15E_2^2 - 30E_2E_4 + 16E_6, \ldots.$$

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"In general they are of the form $\sum K_{\ell,m,n} E_2^\ell E_4^m E_6^n$."

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Theorem (Berndt, Chan, Liu, Yee and Yesilyurt (2003 & 2004))

Ramanujan's claim is true using Ramanujan's identities

$$D(E_2) = \frac{E_2^2 - E_4}{12}, \; D(E_4) = \frac{E_2 E_4 - E_6}{3} \quad \text{and} \quad D(E_6) = \frac{E_2 E_6 - E_4^2}{2}.$$

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Remarks (G. Andrews & B. Berndt '09)

(1) The proofs **do not** *give explicit formulas for $U_{2t}(q)$ and $V_{2t}(q)$.*

(2) **(Problem) Give explicit formulas** *as quasimodular forms.*

Partition Eisenstein series

Definition (TA, K. Ono, A. Singh '24)

Let λ be a **partition of** n given in frequency notation

$$\lambda = (1^{m_1}, 2^{m_2}, \ldots, n^{m_n}) \vdash n.$$

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The weight $2n$ **partition Eisenstein series** is the quasimodular form

$$\lambda = (1^{m_1}, 2^{m_2}, \ldots, n^{m_n}) \vdash n \quad \longmapsto \quad E_\lambda(q) := E_2(q)^{m_1} E_4(q)^{m_2} \cdots E_{2n}(q)^{m_n}.$$

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Definition (TA, K. Ono, A. Singh '24)

For a function $\phi : \mathcal{P} \mapsto \mathbb{C}$, its **weight** $2n$ **Eisenstein trace** is

$$\mathrm{Tr}_n(\phi; q) := \sum_{\lambda \vdash n} \phi(\lambda) E_\lambda(q).$$

Partitions of 3: $(3^1), (1^1, 2^1), (1^3)$

Examples

(1) If $\phi(\lambda) := 1$, then

$$\mathrm{Tr}_3(\phi; q) = E_6 + E_2E_4 + E_2^3.$$

Partitions of 3: $(3^1), (1^1, 2^1), (1^3)$

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(1) If $\phi(\lambda) := 1$, then

$$\mathrm{Tr}_3(\phi; q) = E_6 + E_2E_4 + E_2^3.$$

(2) If $\phi(\lambda) := \#\text{parts}$, then

$$\mathrm{Tr}_3(\phi; q) = E_6 + 2E_2E_4 + 3E_2^3.$$

Answering Andrews and Berndt

Theorem 1 (TA, K. Ono, A. Singh '24)

For all $t \geq 1$, we have that

$$U_{2t}(q) = \mathrm{Tr}_t(\phi_u; q) \quad \text{and} \quad V_{2t}(q) = \mathrm{Tr}_t(\phi_v; q),$$

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where for $\lambda = (1^{m_1}, \ldots, n^{m_n}) \vdash n$, in terms of Bernoulli numbers we let

$$\phi_u(\lambda) := 4^n (2n+1)! \cdot \prod_{k=1}^{n} \frac{1}{m_k!} \left(\frac{B_{2k}}{(2k)\,(2k)!} \right)^{m_k},$$

$$\phi_v(\lambda) := 4^n (2n)! \cdot \prod_{k=1}^{n} \frac{1}{m_k!} \left(\frac{(4^k - 1) B_{2k}}{(2k)\,(2k)!} \right)^{m_k}.$$

Example

For $t = 3$ we find that

$$\phi_u((3^1)) = \frac{16}{9},\ \phi_u((1^1, 2^1)) = -\frac{42}{9}, \quad \text{and} \quad \phi_u((1^3)) = \frac{35}{9}.$$

This reproduces Ramanujan's identity

$$U_6 = \mathrm{Tr}_3(\phi_u; q) = \frac{1}{9}(16E_6 - 42E_2E_4 + 35E_2^3).$$

Example

For $t = 4$ we compute

$$\phi_v((4^1)) = -272, \qquad \phi_v((1^1, 3^1)) = 448,$$
$$\phi_v((2^2)) = 140, \qquad \phi_v((1^2, 2^1)) = -420,$$
$$\phi_v((1^4)) = 105.$$

Therefore, we have that

$$V_8 = \mathrm{Tr}_4(\phi_v; q) = -272E_8 + 448E_2E_6 + 140E_4^2 - 420E_2^2E_4 + 105E_2^4.$$

Pólya's Theory of Cycle Index

Definition

For $\lambda = (\lambda_1, \ldots, \lambda_{\ell(\lambda)}) \vdash t$, the number of perms in $\mathfrak{S}_t$ of cycle type λ is

$$\frac{t!}{z_\lambda}, \qquad \text{where} \ \ z_\lambda := 1^{m_1} \cdots t^{m_t} m_1! \cdots m_t!.$$

The **cycle index polynomial** is

$$Z(\mathfrak{S}_t) = \sum_{\lambda \vdash t} \frac{1}{z_\lambda} \prod_{i=1}^{\ell(\lambda)} x_{\lambda_i} = \sum_{\lambda \vdash t} \prod_{k=1}^{t} \frac{1}{m_k!} \left(\frac{x_k}{k}\right)^{m_k}.$$

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Lemma (Pólya)

As a power series in y, their generating function is

$$\sum_{t \geq 0} Z(\mathfrak{S}_t)\, y^t = \exp\left(\sum_{k \geq 1} x_k \frac{y^k}{k}\right).$$

MacMahon series as an avatar of Ramanujan series

Theorem (TA, R. Barman, A. Singh '25)

We have that

$$\mathcal{U}_t(q) = \sum_{a=0}^{t} w_a(t) \cdot U_{2a}(q).$$

Theorem (Explicit formula - TA, K. Ono, A. Singh '24)

$$U_{2t}(q) = \sum_{\substack{\alpha,\beta,\gamma \geq 0 \\ \alpha+2\beta+3\gamma=t}} c_u(\alpha,\beta,\gamma)\, E_2(q)^{\alpha} E_4(q)^{\beta} E_6(q)^{\gamma},$$

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where

$$\begin{aligned} c_u(\alpha,\beta,\gamma) := &-\frac{1}{3}(2\alpha+8\beta+12\gamma+1)\cdot c_u(\alpha-1,\beta,\gamma)\\ &+\frac{2}{3}(\alpha+1)\cdot c_u(\alpha+1,\beta-1,\gamma)\\ &+\frac{8}{3}(\beta+1)\cdot c_u(\alpha,\beta+1,\gamma-1)\\ &+4(\gamma+1)\cdot c_u(\alpha,\beta-2,\gamma+1). \end{aligned}$$

Theorem (Explicit formula - TA, R. Barman, A. Singh '25)

$$V_{2t}(q) = \sum_{\substack{\alpha,\beta,\gamma\geq 0\\ \alpha+2\beta+3\gamma=t}} c_v(\alpha,\beta,\gamma)\, E_2(q)^\alpha E_4(q)^\beta E_6(q)^\gamma,$$

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Combinatorial interpretation (in progress)

Given $\lambda = (1^{m_1}, 2^{m_2}, \cdots, n^{m_n})$, focus on

$$|\phi(\lambda)| = (2n)! \cdot \prod_{k=1}^{n} \frac{1}{m_k!} \left(\frac{4^k(4^k-1)|B_{2k}|}{(2k)\,(2k)!} \right)^{m_k}.$$

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$\phi(\lambda) \in \mathbb{Z}$ *and* $\sum_{\lambda \vdash n} |\phi(\lambda)| = E_{2n}$.

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Theorem (TA, J. Shareshian, R. Stanley)

$$|\phi(\lambda)| = \#\{w \in \mathfrak{U}_{2n} : rp(\hat{w}) = \lambda\}.$$

Freeman Dyson's Conjecture

Definition

The **crank** $c(\lambda)$ of a partition $\lambda = (1^{m_1}, 2^{m_2}, \ldots, n^{m_n})$ is

$$c(\lambda) := \begin{cases} \text{largest part} & \text{if } m_1 = 0, \\ \#\{\text{parts larger than } m_1\} - m_1 & \text{if } m_1 > 0. \end{cases}$$

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Theorem (Andrews-Garvan (1988))

For every n and a, we have

$$\#\{\lambda \vdash 5n+4 \ : \ c(\lambda) \equiv a \pmod 5\} = \frac{p(5n+4)}{5},$$

$$\#\{\lambda \vdash 7n+5 \ : \ c(\lambda) \equiv a \pmod 7\} = \frac{p(7n+5)}{7},$$

$$\#\{\lambda \vdash 11n+6 \ : \ c(\lambda) \equiv a \pmod{11}\} = \frac{p(11n+6)}{11}.$$

In particular, Ramanujan's congruences are true.

Crank Moments

Definition

If $M(m;n) := \#\{\lambda \vdash n \ : \ c(\lambda) = m\}$, then the d**th moment GF** is

$$C_d(q) := \sum_{n \geq 0} \sum_{m \in \mathbb{Z}} m^d M(m;n)\, q^n.$$

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Theorem 3 (TA, M. Griffin, K. Ono, A. Singh '25)

For $k \geq 1$, we have

$$C_{2k}(q) = \frac{1}{(q)_\infty} \cdot \sum_{n=0}^{k} \frac{(2k)_{2k-2n}}{4^n \cdot (2n+1)} \cdot \mathrm{Tr}_{k-n}(\phi_c;\tau),$$

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where $(x)_m := x(x-1)\cdots(x-m+1)$ *and*

$$\phi_c((1^{m_1}, 2^{m_2}, \ldots, k^{m_k})) := \frac{1}{\prod_{j=1}^{k} m_j!\,((2j)!)^{m_j}}.$$

Atkin-Garvan

They expressed C_2, C_4, C_6, C_8 in terms of the **Lambert series**

$$\Phi_{2j-1}(q) := \sum_{i\geq 1} \frac{i^{2j-1}q^i}{1-q^i}.$$

Used induction to proclaim C_{2k} is a polynomial in $\{\Phi_{2j-1}(q)\}_j$.

Explicit formulas

Theorem (TA, M. Griffin, K. Ono, A. Singh '25)

Letting $(q)_\infty := \prod_{j\geq 1}(1-q^j)$, *we have that*

$$C_{2k}(q) = \frac{(2k)!}{(q)_\infty} \sum_{\lambda\vdash k} \prod_{j=1}^{k} \frac{1}{m_j!}\left(\frac{2}{(2j)!}\right)^{m_j} \cdot \boldsymbol{\Phi}_\lambda(q),$$

where

$$\boldsymbol{\Phi}_\lambda(q) := \prod_{j=1}^{k} \boldsymbol{\Phi}_{2j-1}(q).$$

Hitchin's expository article (2025)

First, consider the $\widehat{A}$-genus of spin manifolds (Borel-Hirzebruch):

$$\widehat{A}_0 = 1, \quad \widehat{A}_1 = -\frac{1}{24}e_1, \quad \widehat{A}_2 = \frac{1}{5760}\left(-4e_2 + 7e_1^2\right),$$
$$\widehat{A}_3 = \frac{1}{967680}\left(-16e_3 + 44e_1e_2 - 31e_1^3\right), \ldots$$

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Notation

$$\widehat{A}(e_1, e_2, \ldots; t) := \sum_{k\geq 0} \widehat{A}_k \, t^k.$$

Second, consider the Hirzebruch L-genus of closed smooth oriented manifolds:

$$L_0 = 1, \quad L_1 = \frac{1}{3}e_1, \quad L_2 = \frac{1}{45}\left(7e_2 - e_1^2\right),$$
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Notation

$$L(e_1, e_2, \ldots; t) := \sum_{k\geq 0} L_k\, t^k.$$

Let $q = e^{2\pi i\tau}$. Weight $2k$ Eisenstein series

$$G_{2k}(\tau) := -\frac{B_{2k}}{2k} + 2\sum_{n} \sigma_{2k-1}(n)\, q^n.$$

If $\phi : \mathcal{P} \to \mathbb{C}$ is on partitions, the **partition Eisenstein trace**

$$\mathrm{Tr}_k^{(G)}(\phi;\tau) := \sum_{\lambda \vdash k} \phi(\lambda) \cdot G_\lambda(\tau)$$

where $G_\lambda(\tau) := G_2^{m_1} G_4^{m_2} \cdots G_{2k}^{m_k}$ and $\lambda = (1^{m_1}, 2^{m_2}, \ldots, k^{m_k})$.

Enter power-sum symmetric functions

Theorem (TA, M. Griffin, K. Ono '25)

We have that

$$\widehat{\mathcal{A}}_k(q) = Tr_k^{(G)}(\phi_{\widehat{A}}; q) \qquad \text{and} \qquad \mathcal{L}_k(q) = Tr_k^{(G)}(\phi_L; q)$$

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Remark

In short, we proclaim that (after minor modification) these genera are essentially **Jacobi forms** *on* $\mathbb{C} \times \mathbb{H}$.

Ramanujan anticipates $\widehat{A}$-genus 100 yrs ago

Theorem (TA, M. Griffin, K. Ono '25)

- *We have that*

$$U_{2k}(q) = 4^k(2k+1)! \cdot \mathrm{Tr}_k^{(E)}(\phi_U;\tau).$$

- *We have that*

$$\widehat{\mathcal{A}}_k(\tau) = (-1)^k \cdot \mathrm{Tr}_k^{(E)}(|\phi_U|;\tau).$$

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Here we have retreated to the "E-traces" with

$$\phi_U(\tau) := \prod_{j=1}^{k} \frac{1}{m_j!} \left(\frac{B_{2k}}{(2j)\,(2j)!} \right)^{m_j}.$$

Compare

Example

Ramanujan's U_6 and the $\widehat{\mathcal{A}}_3$-genus are

$$U_6(q) = \frac{+16E_6 - 42E_2E_4 + 35E_2^3}{9} \qquad \text{and}$$
$$\widehat{\mathcal{A}}_3(\tau) = \frac{-16E_6 - 42E_2E_4 - 35E_2^3}{2903040}.$$

The Witten genus

Remark

Compiling the **modular** *Eisenstein series $G_4, G_6, \ldots,$ we obtain the* **Witten genus**

$$Q_W(z) = \exp\left(\sum_{k\geq 2} \frac{(2\pi i z)^{2k} G_{2k}(\tau)}{(2k)!}\right).$$

Injecting G_2, *the Jacobi theta avatar of the $\widehat{A}$-genus arises.*


That's all Folks!