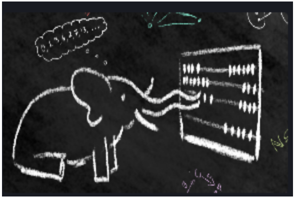
Tiling a Square with Similar Rectangles

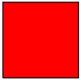
Presented by: George Spahn

Background

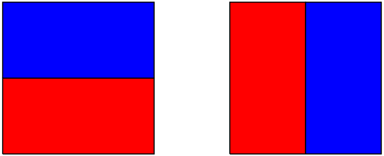
- How many ways can we partition a square into n similar rectangles?
- Hot question on Mathstodon - "math twitter"
- Group effort organized by John Carlos Baez
- Got the attention of the New York Times



Case: $n = 1$

- Divide a square into 1 similar rectangle.
- 1 way to do it!
- 

Case: $n = 2$

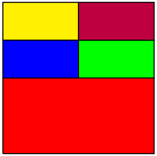
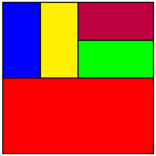


Both cases: The rectangles have the same size

Both cases: The rectangles have a 2:1 aspect ratio

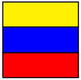
Dealing with Symmetry

- We could try to exclude rotations/reflections.

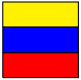
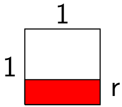



- Let's say all partitions with the same aspect ratio (of the rectangles) are equivalent.

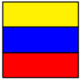
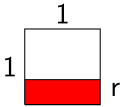
Case: $n = 3$

- Here's one solution with $r = 1/3$ 
- How to be more rigorous?

Case: $n = 3$

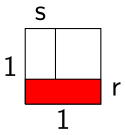
- Here's one solution with $r = 1/3$ 
- How to be more rigorous?
- First make a cut 

Case: $n = 3$

- Here's one solution with $r = 1/3$ 
- How to be more rigorous?
- First make a cut 

- Without loss of generality we can make our next cut in the white region.
- Making another horizontal cut forces the solution we found already.

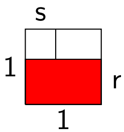
Case: $n = 3$



By observation, we see $r < 1$. The red rectangle must be oriented horizontally (long size is horizontal). There are now 3 possibilities:

1. Both white rectangles are oriented horizontally
2. Both are oriented vertically
3. One is oriented horizontally and the other is oriented vertically

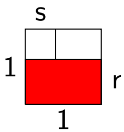
Case: $n = 3$, both horizontal



- The white rectangles must be congruent, therefore $s = 1/2$
- In order for the white rectangles to be similar to the red rectangle

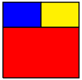
$$\frac{r}{1} = \frac{1-r}{s}$$

Case: $n = 3$, both horizontal

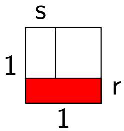


- The white rectangles must be congruent, therefore $s = 1/2$
- In order for the white rectangles to be similar to the red rectangle

$$\frac{r}{1} = \frac{1-r}{s}$$

- Linear equation in r with solution $r = 2/3$ 

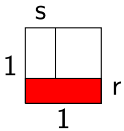
Case: $n = 3$, both vertical



- The white rectangles must be congruent, therefore $s = 1/2$
- In order for the white rectangles to be similar to the red rectangle

$$\frac{r}{1} = \frac{s}{1-r}$$

Case: $n = 3$, both vertical



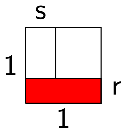
- The white rectangles must be congruent, therefore $s = 1/2$
- In order for the white rectangles to be similar to the red rectangle

$$\frac{r}{1} = \frac{s}{1-r}$$

$$r * (1 - r) = 1/2$$

- No real solution!

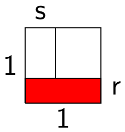
Case: $n = 3$, one vertical + one horizontal



Let's say the left one is vertical.

$$\frac{r}{1} = \frac{s}{1-r} \qquad \frac{r}{1} = \frac{1-r}{1-s}$$

Case: $n = 3$, one vertical + one horizontal



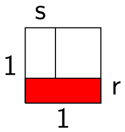
Let's say the left one is vertical.

$$\frac{r}{1} = \frac{s}{1-r} \qquad \frac{r}{1} = \frac{1-r}{1-s}$$

$$s = r - r^2 \qquad r * \left[1 - (r - r^2)\right] = 1 - r$$

$$r^3 - r^2 + 2r - 1 = 0$$

Case: $n = 3$, one vertical + one horizontal



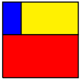
Let's say the left one is vertical.

$$\frac{r}{1} = \frac{s}{1-r} \qquad \frac{r}{1} = \frac{1-r}{1-s}$$

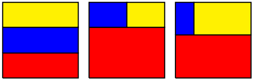
$$s = r - r^2 \qquad r * \left[1 - (r - r^2)\right] = 1 - r$$

$r^3 - r^2 + 2r - 1 = 0$

$r \approx 0.5698$, the reciprocal of the square of the plastic number!



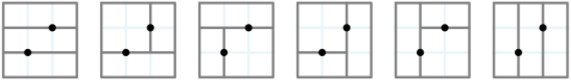
Case: n=3



3 possibilities in total: 2 rational, 1 irrational

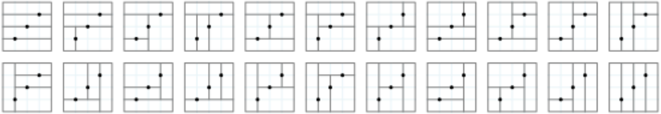
$$r = .3333, .5698, 0.6667$$

Guillotine Cuts

- To solve the $n = 3$ case we started with the square and repeatedly cut a rectangle into two smaller rectangles.
- Such cuts are called guillotine cuts.
- A guillotine partition is a partition of the square into rectangles using guillotine cuts.
- 
- A guillotine partition into n rectangles could potentially give rise to 2^n distinct solutions, since each rectangle can be oriented in 2 ways.

Schroder Numbers

- The number of structurally distinct guillotine partitions into n rectangles is given by the Schroder numbers, S_n. For example, $S_4 = 22$:



- S_n also counts the number of n step lattice paths from $(0,0)$ to $(2n,0)$ using steps of EAST $(2,0)$, NORTHEAST $(1,1)$ and SOUTHEAST $(1,-1)$ that do no fall below the x-axis.

Schroder Numbers

This is A6318 in the OEIS. The generating function is:

$$\frac{1 - x - \sqrt{1 - 6x + x^2}}{2x}$$

Similar to the Catalan numbers.

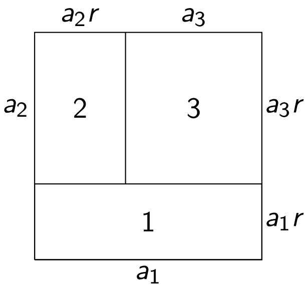
We can compute many terms.

Algorithm using guillotine cuts

- Enumerate all guillotine partitions with n rectangles.
- For each partition, iterate through all 2^n orientations for each rectangle.
- Let $r < 1$ be the aspect ratio, and let a_i be the side length of the longest side of the i^{th} rectangle.
- Generate a system of equations.

Getting a system of Equations

For each cut, we get an equation.



$a_1 = a_2 r + a_3$

$a_2 = a_3 r$

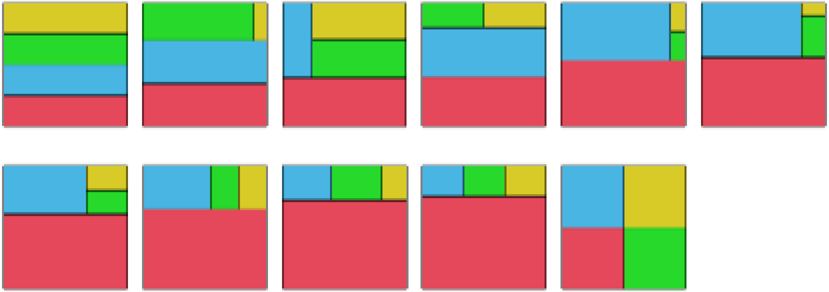
Plus two more for the sides of the square:

$a_1 r + a_3 r = 1$

$a_2 r + a_3 = 1$

Case: $n = 4$

The guillotine cut approach successfully computes $a(4) = 11$:

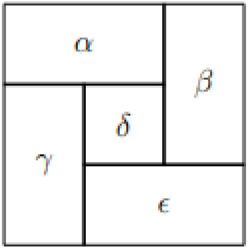


There are 5 rational solutions: $r = 1, 1/4, 3/4, 2/5, 3/5$ – Exactly when all the rectangles are horizontal!

There are 6 irrational solutions, each a root of a 3^{rd} degree polynomial.

Failure of Guillotine Cuts

The following partition of a square into 5 rectangles is not obtainable by guillotine cuts!



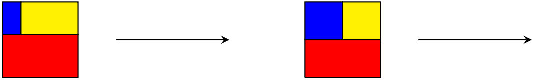
Grid Coloring

We need a better way of generating the “arrangements”.

In a tiling with n rectangles, there are at most $n - 1$ internal horizontal lines (at most one for the top of each rectangle, minus the very top).

We can slide the horizontal lines vertically, without changing the arrangement, until each line is on a gridline of the $n \times n$ grid.

Grid Coloring



$$\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$$

The arrangement is nicely represented by a matrix. There was only 1 internal horizontal line, and 1 internal vertical line, so we used a 2×2 matrix.

Grid Coloring

To generate arrangements with n rectangles, we could start by generating all possible colorings of the $n \times n$ grid into n colors.

Then check whether each color is connected and forms a rectangle.

In practice there are much more efficient ways.

4 Rectangle Arrangements

With a few symmetries removed, here are the possible arrangements for 4 rectangles:

$$\begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 & 4 \\ 3 & 1 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 2 \\ 3 & 2 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 4 & 1 \\ 3 & 3 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 1 \\ 3 & 1 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 1 \\ 3 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$$

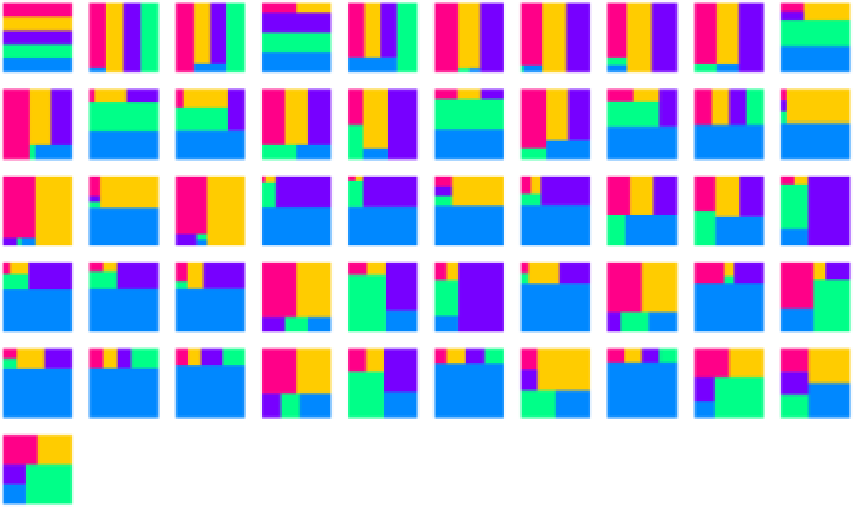
All the ones that used a 4×4 grid had repeated rows or columns, which can be removed.

Grid Coloring Algorithm

1. Generate all arrangements using grid coloring.
2. For each arrangement, check all 2^n orientations.
3. Define the variables and generate the system of equations.
4. Use linear algebra to solve for the polynomial that r must satisfy.
5. Check to see that the polynomial has a real root $\in (0, 1]$ and that none of the rectangles have size 0.

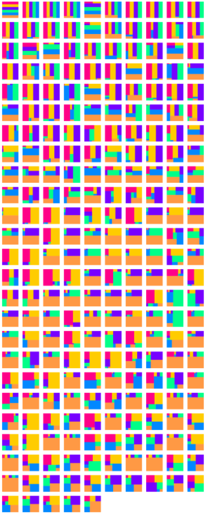
Case: $n = 5$

Ian Henderson has drawn the 51 solutions:



Case: $n = 6$

Ian Henderson has drawn the 245 solutions:



How far can we go?

- A359146 in the OEIS records our current progress on the problem.
- Ian Henderson computed the first 8 terms: 1, 1, 3, 11, 51, 245, 1372, 8522
- A couple weeks ago David Einstein computed 2 more terms using a graph theory approach that is closely connected to electrical networks. Time permitting I can talk about this at the end.

Which aspect ratios are possible?

If we allow any finite number of rectangles, which aspect ratios are possible?

Are all rational numbers possible?

An example: 14/47

Recall the continued fraction expansion of a rational number:

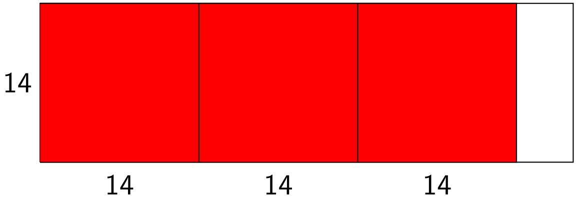
$$\frac{14}{47} = \frac{1}{\frac{47}{14}} = \frac{1}{3 + \frac{5}{14}} = \frac{1}{3 + \frac{1}{\frac{14}{5}}} = \frac{1}{3 + \frac{1}{2+\frac{4}{5}}}$$

$$= \frac{1}{3 + \frac{1}{2+\frac{1}{1+\frac{1}{4}}}}$$

This gives the sequence $3, 2, 1, 4$

An example: 14/47

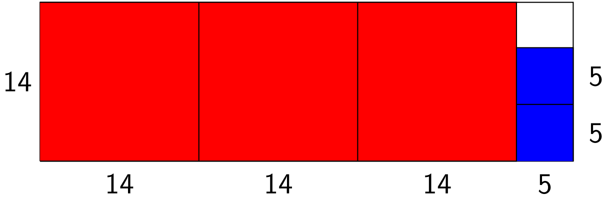
Let's now try to tile a 14×47 rectangle with squares.



Here are three 14×14 squares.

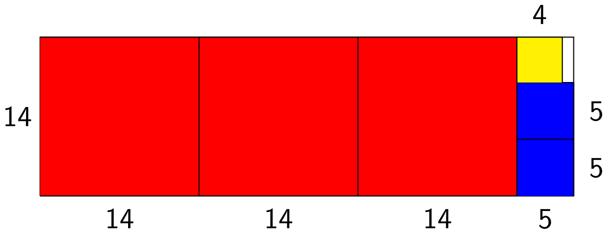
An example: 14/47

We can add two 5×5 squares.



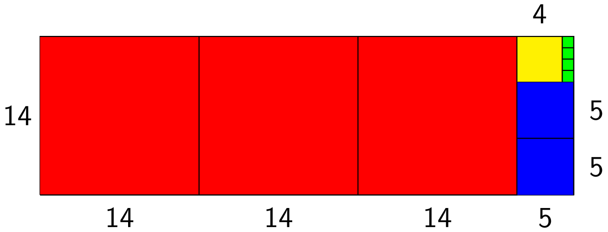
An example: 14/47

And then a 4×4 square.



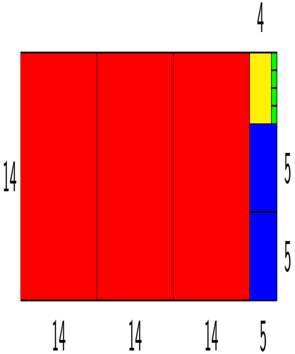
An example: 14/47

And then finally four 1×1 squares.



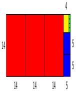
Does the sequence 3,2,1,4 look familiar?

Method of Squinty Vision



The big rectangle has become a square, and the squares have become rectangles with aspect ratio 14/47!

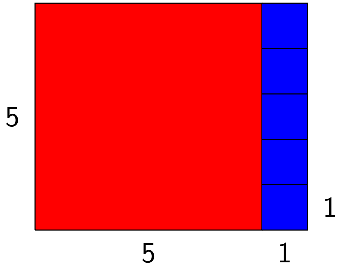
Method of Squinty Vision



- This shows that there will be solutions for any rational number.
- We can also go backward – if all the rectangles in our tiling of a square are oriented the same way, we can produce a tiling of a rectangle with squares.
- In 1903 Max Dehn proved that in any tiling of a rectangle with finitely many squares, the aspect ratio of the rectangle must be rational.

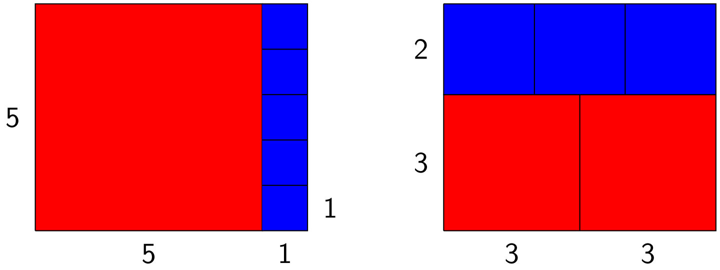
Tiling a Rectangle with the fewest number of squares

Does this method of continued fractions give the minimal number of squares to tile a rectangle?



A 6×5 rectangle

Tiling a Rectangle with the fewest number of squares



$$\frac{5}{6} = \frac{1}{1 + \frac{1}{5}}$$

so we needed $1 + 5$ squares using the continued fractions method.

Tiling a Rectangle with the fewest number of squares

Given two natural numbers n and m, find the minimum number of integer sided squares required to tile the $n \times m$ rectangle. Call this number $h(n, m)$.

- Extensive research has been done on this problem.
- We have a large table of results, and some upper and lower bounds.

It is conjectured that $h(dn, dm) = h(n, m)$.

Does this help us?

A210517 in the OEIS gives the number of rectangles dissectable into n squares, unique up to aspect ratio, but we only know 8 terms.

If the conjecture is true, we can probably compute many more terms.

But there is still a problem:

Open Question

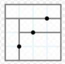
What if sometimes we *need* to put the rectangles in different orientations?

Does there exist a rational ratio r, such that in any minimum partition of the square into rectangles with aspect ratio r, there exists two rectangles in different orientations?

Which irrational numbers are possible?

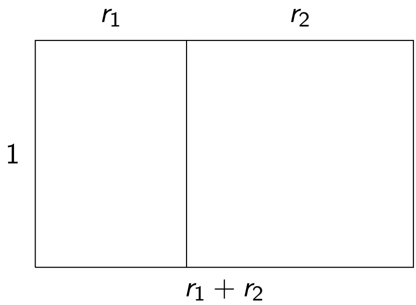
- Let's first consider the case of tilings obtainable from guillotine cuts.
- Let T_r be the rectangle with ratio r.
- Let S_r be set of all possible ratios of rectangles that are buildable using rectangles similar to T_r.
- We are then interested in determining for which r, is $1 \in S_r$.

Building up a Rectangle

- Guillotine cut tilings can be built up from smaller pieces in a nice way. 
- On each side of a cut, we must have a rectangle with ratio in S_r.
- Say the ratios are r_1 and r_2.
- Then the ratio of the rectangle being divided by the cut is either

$$r_1 + r_2, \quad \frac{1}{r_1} + r_2, \quad r_1 + \frac{1}{r_2}, \quad \frac{1}{r_1} + \frac{1}{r_2}$$

Building up a Rectangle



A guillotine cut diagram is built up from repeated joins and rotations.

We conclude S_r must be closed under addition and multiplicative inverses.

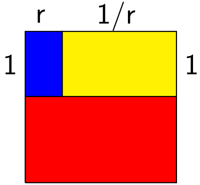
Continued Fractions

This fits with what we know about continued fractions:

- It is possible to get from any rational to any other rational with just using the operations of addition and reciprocal!

A Rational Expression for each Guillotine Tiling

Another way we could have solved the $n = 3$ case:

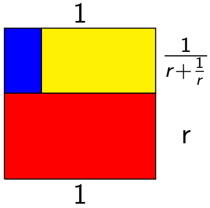


The ratio of the blue + yellow rectangle is thus

$$\frac{1}{r + \frac{1}{r}}$$

A Rational Expression for each Guillotine Tiling

Rescaling:



So the ratio of the big rectangle is thus

$$r + \frac{1}{r + \frac{1}{r}}$$

A Rational Expression for each Guillotine Tiling

We can simplify:

$$r + \frac{1}{r + \frac{1}{r}} = \frac{r^3 + 2r}{r^2 + 1}$$

Since we want the rectangle to be actually be a square, we can set the fraction equal to 1 and get

$$r^3 + 2r = r^2 + 1$$

This gives the polynomial $r^3 - r^2 + 2r - 1$ that we encountered before.

The polynomials for the $n = 4$ case

Recall in the $n = 4$ case we found 11 solutions:

The rational solutions were $1, 1/4, 3/4, 2/5, 3/5$. Except for 1, these are the rational numbers in $(0, 1]$ that have continued fraction expansion sequence adding up to 4!

The irrational solutions are the real positive roots to the following 3^{rd} degree polynomials:

$$x^3 - x^2 + 3x - 2 \qquad x^3 - x^2 + 4x - 2 \qquad 2x^3 - 2x^2 + 2x - 1$$

$$3x^3 - 2x^2 + 2x - 2 \qquad 2x^3 - x^2 + 3x - 1 \qquad 2x^3 - 2x^2 + 3x - 1$$

Alternating Coefficients!

$$x^3 - x^2 + 3x - 2 \qquad x^3 - x^2 + 4x - 2 \qquad 2x^3 - 2x^2 + 2x - 1$$

$$3x^3 - 2x^2 + 2x - 2 \qquad 2x^3 - x^2 + 3x - 1 \qquad 2x^3 - 2x^2 + 3x - 1$$

These polynomials are generated by setting the numerator and denominator of a rational expression equal to each other.

It is not too hard to see that the numerator and denominator are always polynomials with positive coefficients, and that they are always of the form even function over odd function or odd function over even function.

Alternating Coefficients!

The two operations we had for generating the rational functions was addition and taking the reciprocal. Clearly reciprocal preserves the property. For addition:

$$\frac{\text{odd}}{\text{even}} + \frac{\text{even}}{\text{odd}} = \frac{\text{odd*odd} + \text{even*even}}{\text{even*odd}} = \frac{\text{even}}{\text{odd}}$$

$$\frac{\text{odd}}{\text{even}} + \frac{\text{odd}}{\text{even}} = \frac{\text{odd*even} + \text{odd*even}}{\text{even*even}} = \frac{\text{odd}}{\text{even}}$$

$$\frac{\text{even}}{\text{odd}} + \frac{\text{even}}{\text{odd}} = \frac{\text{even*odd} + \text{even*odd}}{\text{odd*odd}} = \frac{\text{odd}}{\text{even}}$$

Which irrational ratios are possible?

So we know that if $1 \in S_r$, then r had better be the root of some polynomial with alternating coefficients.

But we also were limiting ourselves to looking at guillotine partitions.

It turns out that an amazing theorem settles this question once and for all.

Powerful Theorem

Proved in 1994 by Laczkovick and Szekeres, and also independently by Freiling and Rinne in the same year.

Theorem:

A square can be partitioned into finitely many rectangles with ratio r if and only if r is algebraic and all of its conjugate roots have positive real part.

Wall's Theorem

Used in the proof is another theorem by Wall from 1945 that enlightens the situation a little bit.

Theorem:

Let $P(x) = x^n + p_{n-1}x^{n-1} + ... + p_0$, *and let*

$Q(x) = x^n + p_{n-2}x^{n-2} + p_{n-4}x^{n-4} + ...$, *and*

$R(x) = p_{n-1}x^{n-1} + p_{n-3}x^{n-3} + ...$

Then all roots of $P(x)$ *have positive real part if and only if*

$$\frac{-R(x)}{Q(x)} = \frac{1}{c_n x + \frac{1}{c_{n-1}x + ... \frac{1}{c_1 x}}}$$

Guillotine cuts generate all possible ratios!

All roots of $P(x)$ have positive real part if and only if

$$\frac{-R(x)}{Q(x)} = \frac{1}{c_n x + \frac{1}{c_{n-1}x + \dots \frac{1}{c_1 x}}}$$

These polynomials are exactly the polynomials that we were generating with guillotine cuts!

The End

Thank you for listening!