Counting Baxter Matrices

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Background

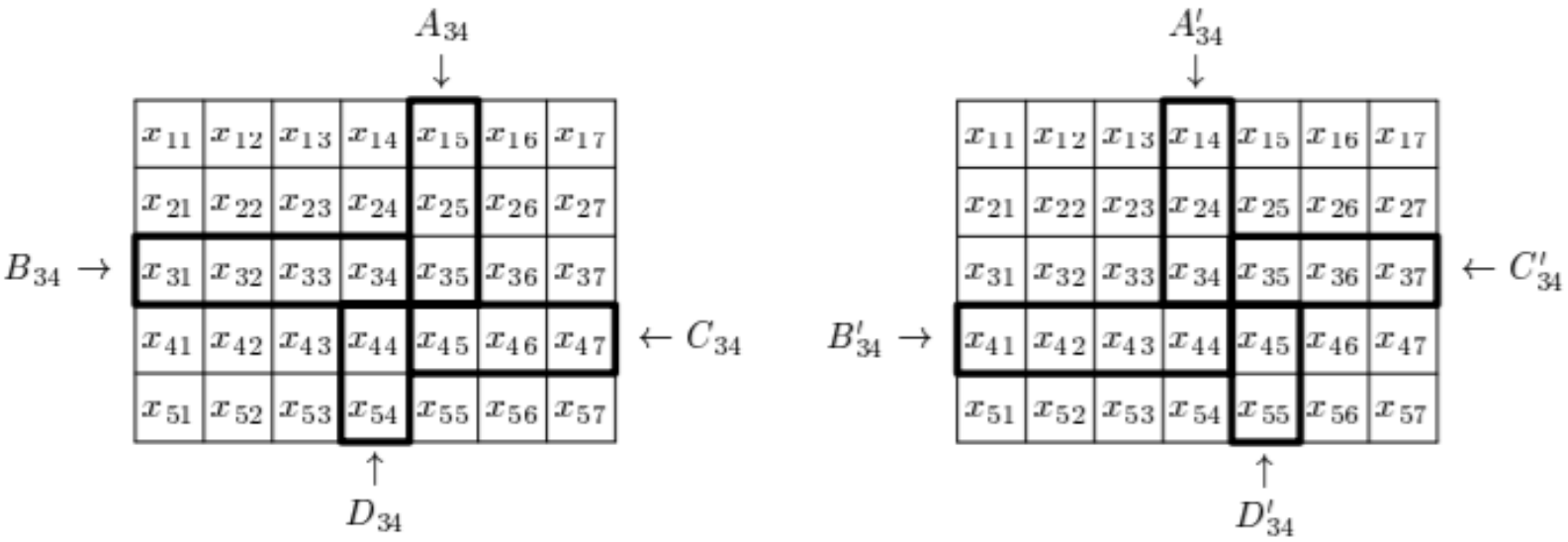
- Don Knuth
- *Baxter Matrices* – an "Unpublication"
- September 5, 2021
- Extension of Baxter Permutations

What is a Baxter Matrix?

A $m \times n$ matrix of 0's and 1's satisfying 4 conditions:

1. Each row contains a 1
2. Each column contains a 1
3. Each clockwise pinwheel contains a segment of all 0's
4. Each counterclockwise pinwheel contains a segment of all 0's

Pinwheels



- Each pinwheel requires a segment of zeroes
- Center can be on any vertex in the interior of the matrix
- $(m-1)*(n-1)$ possible centers

An Example?

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

How many 1's can we fit into a Baxter Matrix?

Try 3×3

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Putting three 1's in a corner:

$$\begin{pmatrix} 1 & 1 & \cdot \\ 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

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$$\begin{pmatrix} 1 & 1 & \cdot \\ 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & \cdot \\ 1 & 0 & 0 \\ \cdot & 0 & \cdot \end{pmatrix}$$

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$$\begin{pmatrix} 1 & 1 & \cdot \\ 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \to \begin{pmatrix} 1 & 1 & \cdot \\ 1 & 0 & 0 \\ \cdot & 0 & \cdot \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

It turns out that the maximum number of 1's in a 3×3 matrix is 5.

Conjecture (Knuth)

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The maximum number of 1's in a $m \times n$ Baxter matrix is:

$$m + n - 1$$

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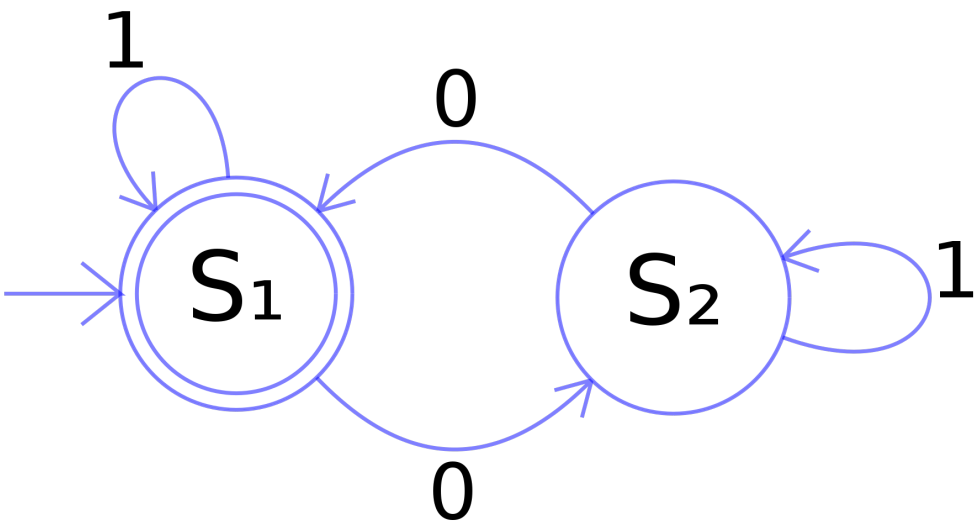
- Knuth verified this conjecture for all Baxter Matrices up to size 7×7 by enumerating them
- Proving this directly seemed hard, maybe the special case $m = 2$ is easier?

Finite State Machines (FSM)



- Input a string (0's and 1's in this case)
- Start from the start state and follow the arrows
- If you end up in an accept state, accept, else reject

FSM for $2 \times n$ Baxter Matrices

- Matrix as a sequence of columns
- Any sequence of columns can be "accepted" or "rejected"
- possible columns: $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Rejecting Early

Suppose we read the columns $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

This means that our input matrix starts out like $\begin{pmatrix} 0 & 1 & 1 & \dots \\ 0 & 1 & 1 & \end{pmatrix}$

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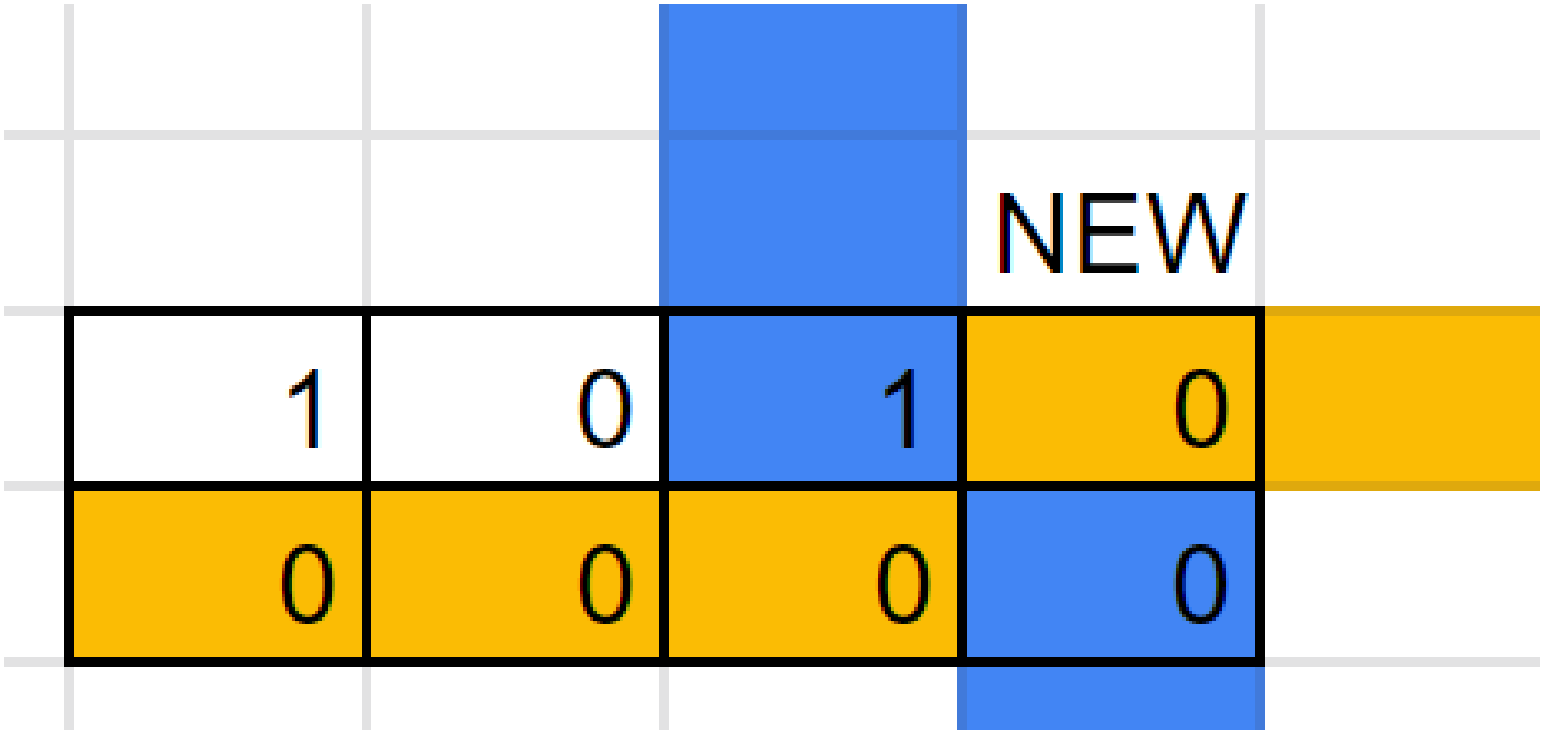
- Not possible for the pinwheels to ever be satisfied!

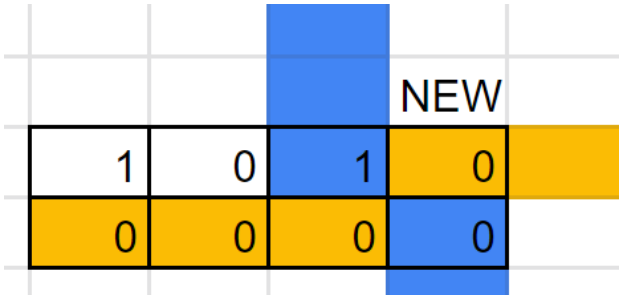
Keeping Track of Data

As we read a column, we check two new pinwheels

- They have center to the immediate left of our column

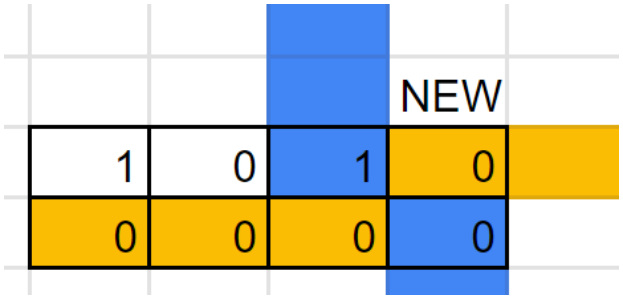


Keeping Track of Data



- In order to check the pinwheel, we need to know the contents of the previous columns
- Could be arbitrarily many columns!
- Finite State machine can only keep track of finitely many things

Keeping Track of Data



- In order to check the pinwheel, we need to know the contents of the previous columns
- Could be arbitrarily many columns!
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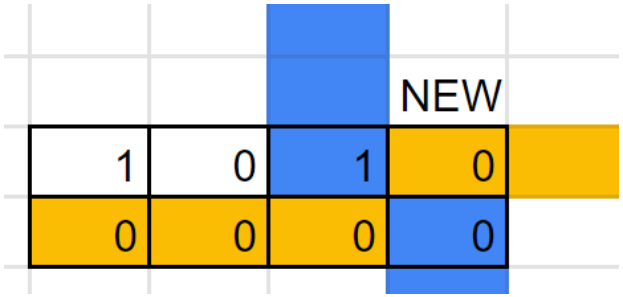
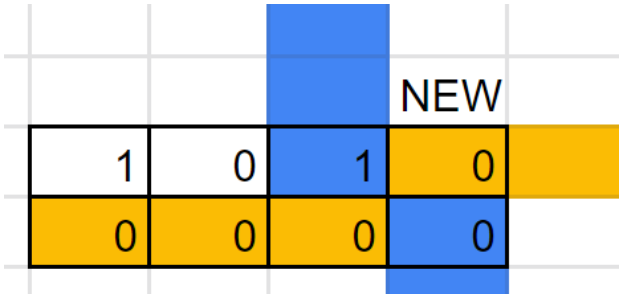
To check the leftward orange segment, only need to know whether that row has been all 0's so far

Keeping Track of Data



- Keep track of the exact contents of the previous column
- Keep track of whether each row has only 0's so far
- What about the rightward segment?

Keeping Track of Data



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- Keep track of whether each row has only 0's so far
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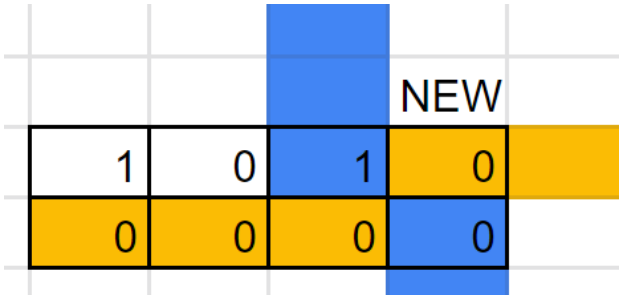
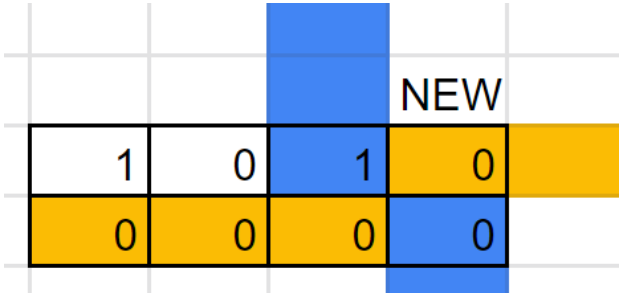
Cannot see the future

Seeing the future



- If any of the other segments are 0's, then we don't care
- Else, we have a row that must be permanently 0's in the future

The 4 rowstates



1. This row had a 1 in the previous column
2. This row has only contained 0's so far
3. This row must contain only 0's in the future
4. This row had a 0 in the previous column, but neither 2 or 3 applies

If we know which rowstate each row is in, we can check whether our new pinwheel is satisfied!

Notes

- Reject when we see this column: $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

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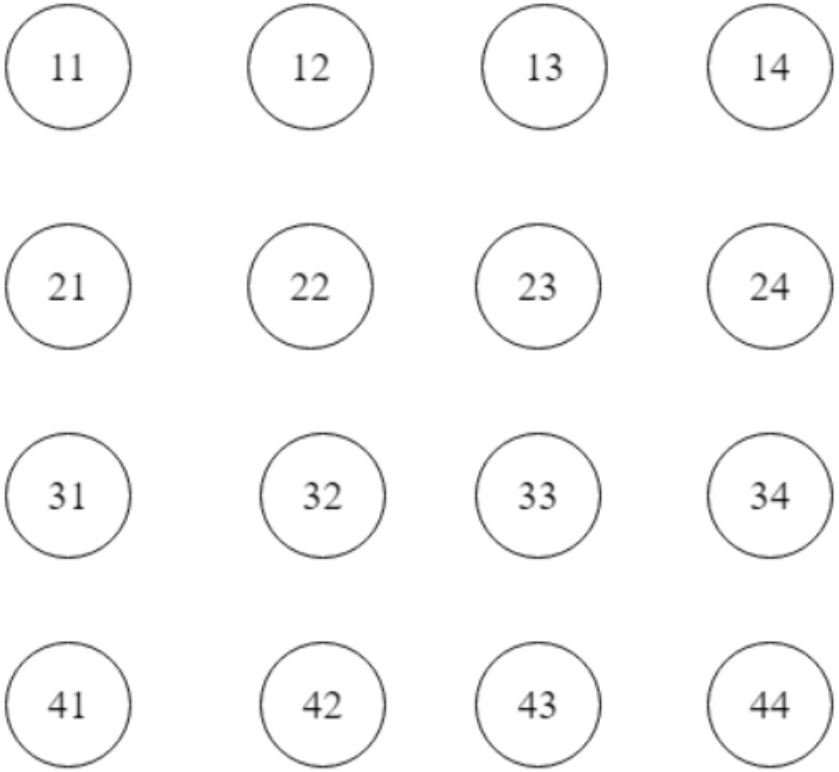
- If a row enters rowstate 3 it can never leave 3.
- A row may not transition from 2 to 3 directly.

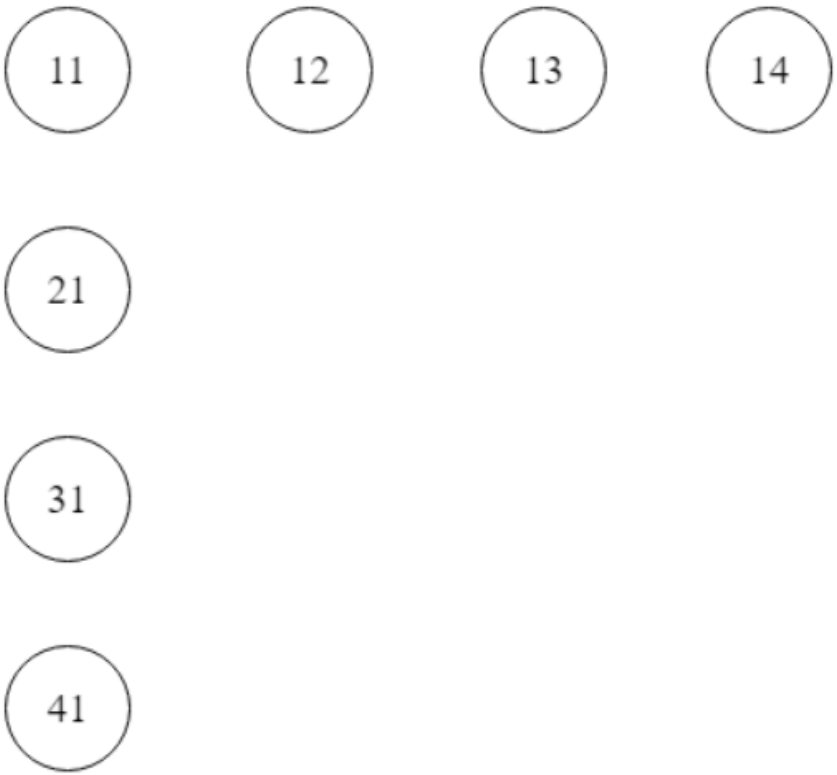
Constructing the FSM

- $4 \times 4 = 16$ states, to allow for all combinations of rowstates between the two rows.



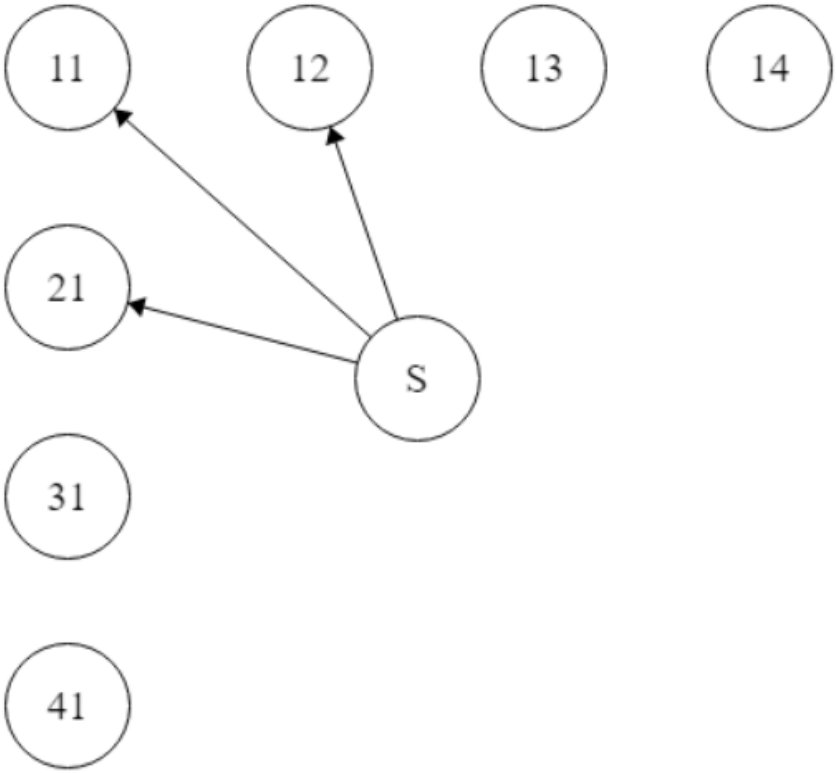
Unused States

- Remove the states which do not have a row in rowstate 1
- $16 - 9 = 7$ states



Start State

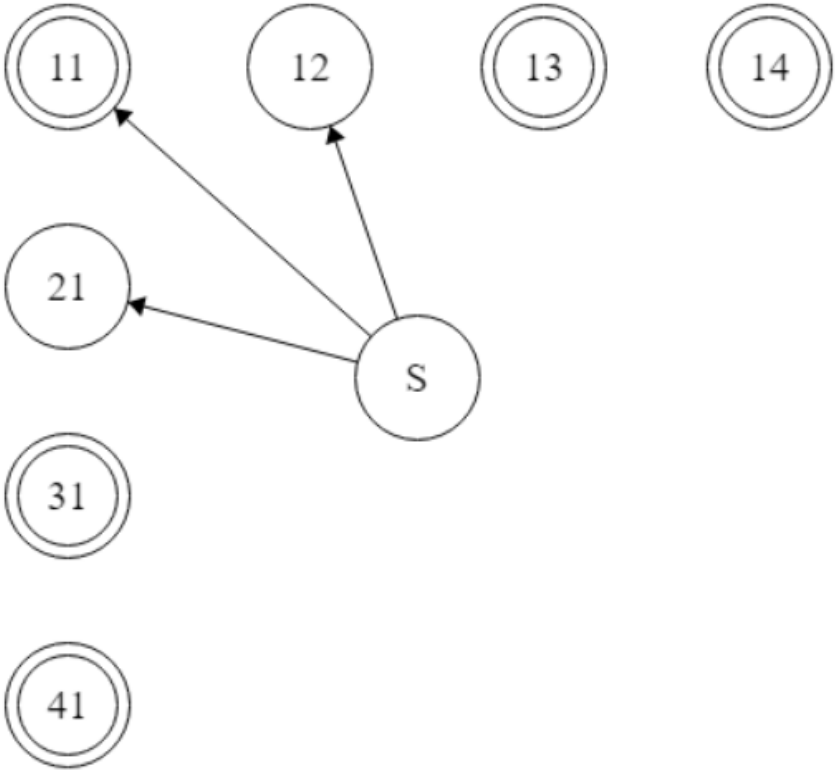
After a single column, each row must be in either rowstate 1 or rowstate 2



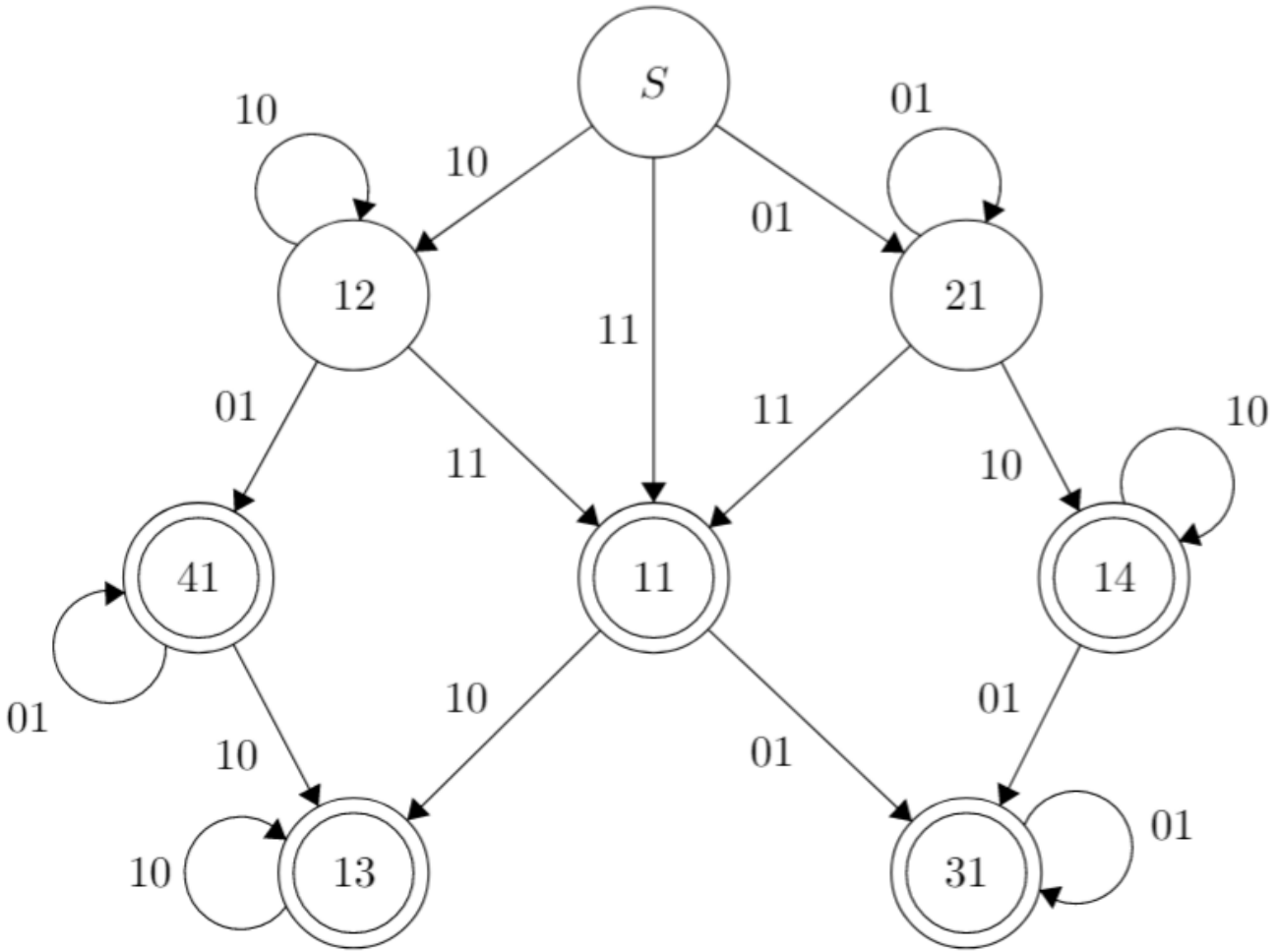
Accept States

In a Baxter Matrix each row must contain a 1.

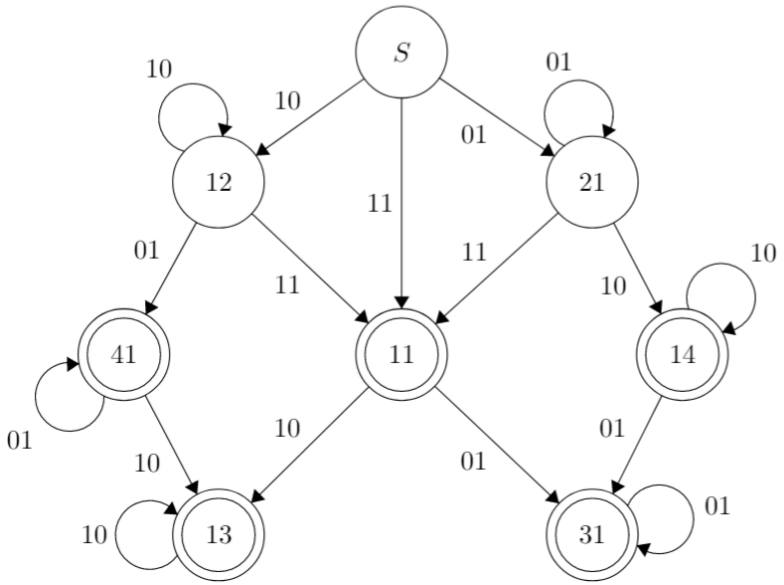
To accept, we must require that every row has left rowstate 2.



Drawing in the Arrows



Correspondence

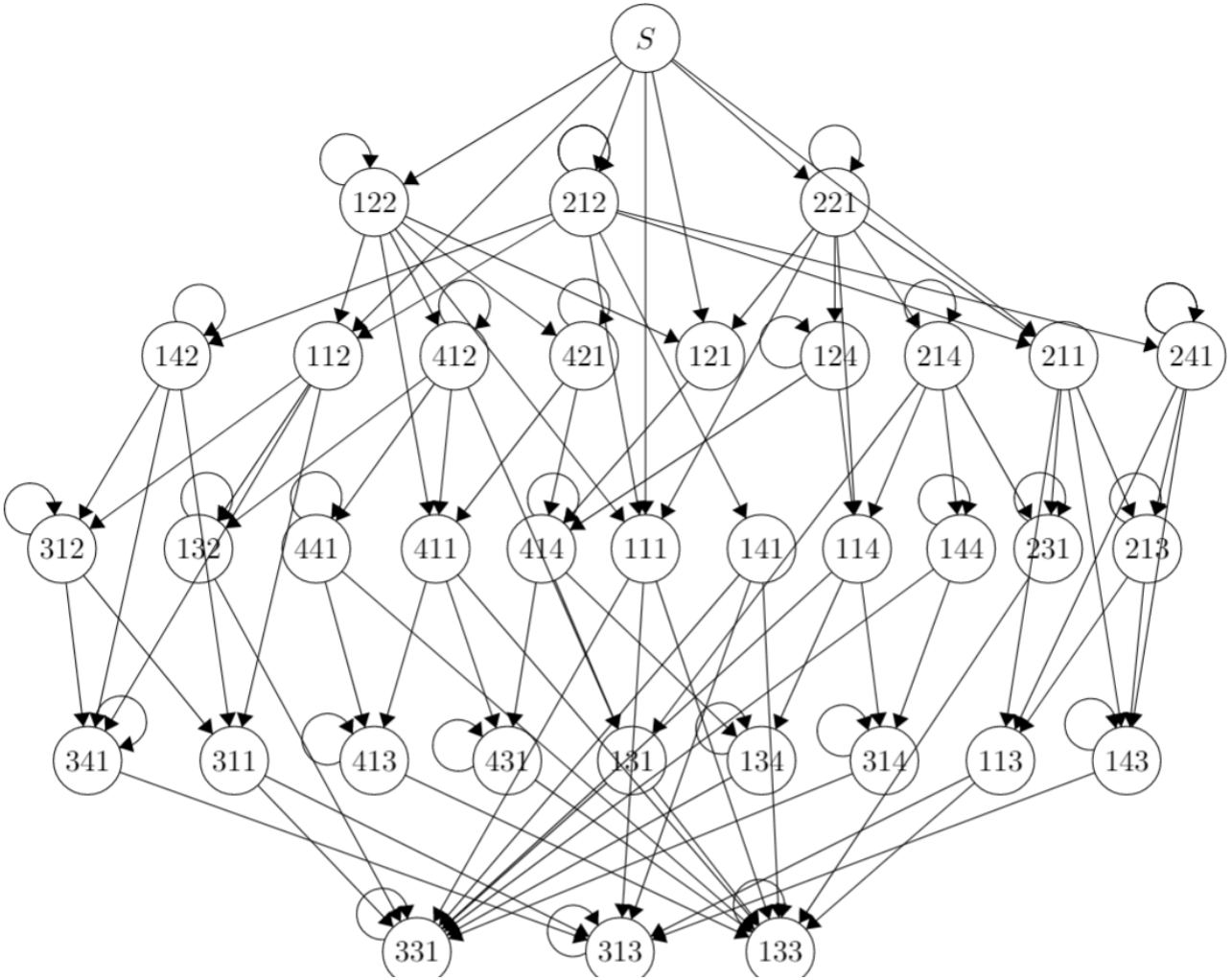


We have a 1-1 correspondence between Baxter Matrices with 2 rows and paths from the start state to an accept state

More rows!

- We can do a similar process for 3 rows, 4 rows, etc.
- Let's fix r as the number of rows
- FSM will have 2^r symbols (one for each column) and $4^r - 3^r$ used states
- Denote the FSM for r rows as A_r

A_3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Depth of a state

Definition:

The **depth** *of a state is number of 1's plus the number of 4's plus twice the number of 3's that can be found in the rowstates of the rows.*

Min Depth $= 0$ (Start). Max Depth $= 2r - 1$

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Lemma:

Any transition in A_r must either be a self arrow or increase depth.

Counting Baxter Matrices

- How many Baxter Matrices of size $r \times k$?
- How many paths of length k from S to a final state in A_r?

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- If we ignore self-arrows, the lemma forces there to be only finitely many paths in A_r.
- A self-arrow corresponds to a repeated column in the Baxter Matrix
- Let's say a Baxter Matrix with no repeated columns is "interesting"

Counting Baxter Matrices

- Only finitely many interesting Baxter Matrices with r rows.
- Each non-interesting Baxter Matrix can classified according to the interesting matrix that remains after removing the self-arrows.
- To count the total number of $r \times k$ Baxter Matrices, just need to count the number of non-interesting Baxter Matrices with k columns that correspond to each interesting Baxter Matrix with r rows.

Counting non-Interesting Baxter Matrices

- Fix a particular path with no self-arrows from S to a final state in A_r
- Say it has length l and has m nodes with self arrows on it
- Max Depth is $2r - 1$, therefore $m \leq 2r - 1$
- We want to count the number of ways to add self arrows at any of the m spots, to make a path of length k.
- How many ways to choose m distinguishable natural numbers that sum to $k - l$?

Counting non-Interesting Baxter Matrices

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Counting non-Interesting Baxter Matrices

How many ways to choose m distinguishable natural numbers that sum to $k - l$?

$O(k)$ choices for each number, except the last one.

We get a polynomial in k of degree $m - 1$!

Counting Baxter Matrices with r rows

1. Enumerate the finitely many interesting Baxter Matrices with r rows.
2. Receive a polynomial in k of degree at most $2r - 2$ from each.
3. For any specific k, plug it in to each polynomial. If the output would be negative, set it to 0.
4. Add up the results!

Counting Baxter Matrices with r rows

- For $k \geq r$, the polynomials won't be negative, so we can add up the polynomials before plugging in, to get a single polynomial of degree $2r - 2$

Theorem:

For a fixed number of rows, r, the number of Baxter matrices with r rows and k columns eventually satisfies a polynomial in k of degree $2r - 2$.

Maple Code

I have maple code that does the above process to compute the polynomial for any r.

rows	formula	works for
2	$k^2 + 3k - 4$	$k \geq 2$
3	$(1/3)k^4 + 3k^3 - (16/3)k^2 + 2k + 3$	$k \geq 3$
4	$(1/18)k^6 + (21/20)k^5 - (5/18)k^4 - \ldots$	$k \geq 4$
5	$(23/4032)k^8 + (937/5040)k^7 + \ldots$	$k \geq 5$
6	$(361/907200)k^{10} + (403/20160)k^9 + \ldots$	$k \geq 6$

Returning to Knuth's conjecture

Conjecture:

The maximum number of 1's in a $m \times n$ Baxter matrix is:

$$m + n - 1$$

Recall from the definition that each column of a Baxter Matrix must contain a 1.

- Let's say each column with more than one 1 contains extra 1's.

Returning to Knuth's conjecture

Rephrasing the conjecture,

Conjecture:

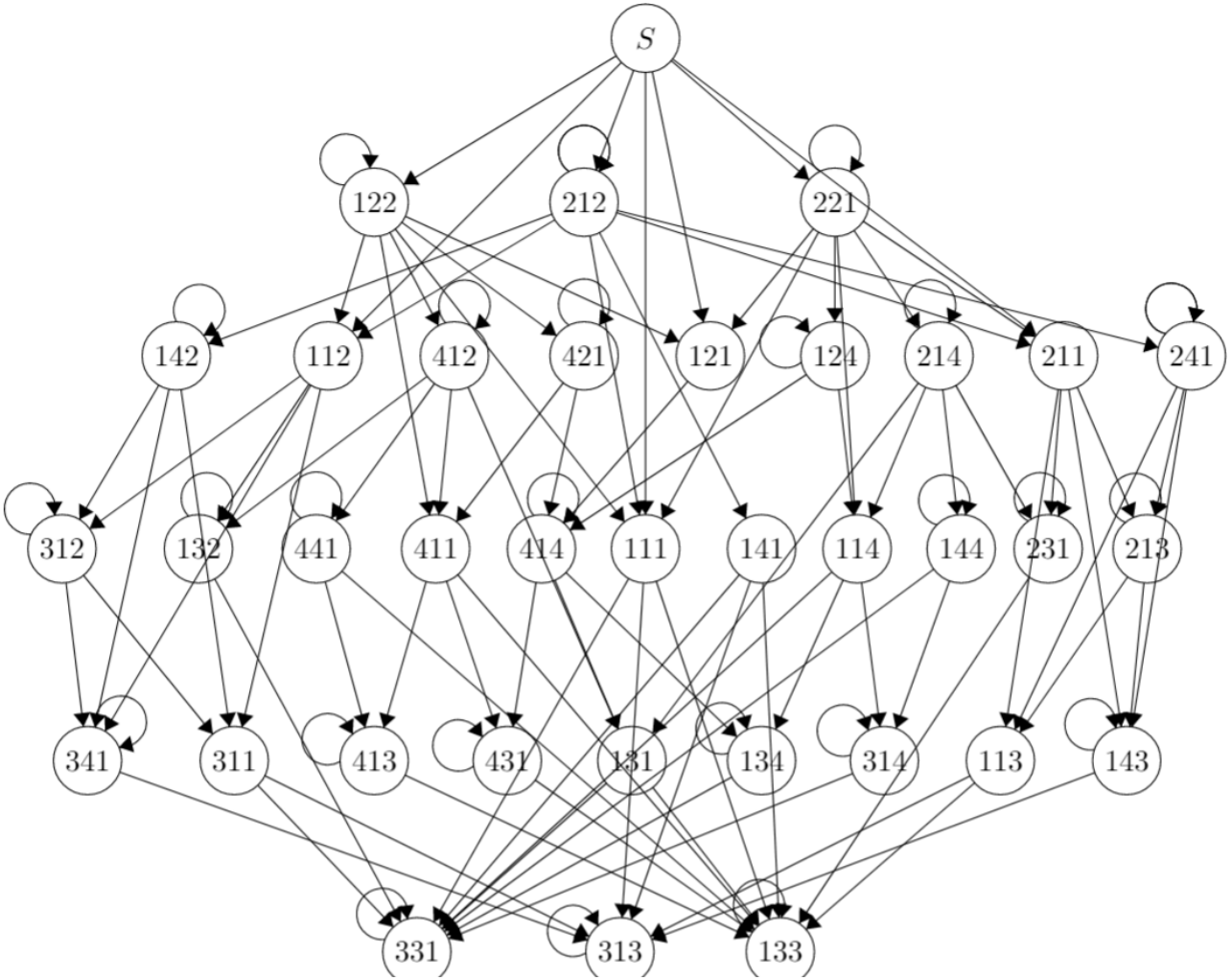
The number of extra 1's in a Baxter Matrix with r rows is less than r.

A discovery

Lemma:

The total number of extra 1's that appear in two consecutive columns is at most the change in depth of the corresponding state transition in A_r.

A discovery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Using the Discovery

Let M be a $r \times k$ Baxter Matrix, p be its corresponding path in A_r, and T be the set of transitions in p.

$$(\#\text{ of extra 1's in } M) = \frac{1}{2}\left(\sum_{\tau \in T}(\#\text{ of extra 1's in the columns associated with } \tau)\right)$$

- This assumes the first and last states do not have extra 1's.

Using the Discovery

$$(\#\text{ of extra 1's in } M) = \tfrac{1}{2}\left(\textstyle\sum_{\tau\in T}(\#\text{ of extra 1's in the columns associated with } \tau)\right)$$

$$\begin{aligned}
&\leq \frac{1}{2}\left(\sum_{\tau\in T}(\text{depth increase of } \tau)\right)\\
&\leq \frac{1}{2}(2r-1)\\
&< r
\end{aligned}$$

Done!

Number of Baxter Matrices with t 1's

$r = 3$, correct for $k \geq 3$

extra 1's	total weight	formula
0	k	$(1/3)k^4 - k^3 + (2/3)k^2$
1	k+1	$4k^3 - 12k^2 + 15k - 8$
2	k+2	$6k^2 - 13k + 11$

Number of Baxter Matrices with t 1's

$r = 4$, correct for $k \geq 4$

extra 1's	total weight	formula
0	k	$(1/18)k^6 - (3/10)k^5 + (2/9)k^4 + \ldots$
1	k+1	$(27/20)k^5 - (47/6)k^4 + (235/12)k^3 - \ldots$
2	k+2	$(22/3)k^4 - (121/3)k^3 + (335/3)k^2 - \ldots$
3	k+3	$(20/3)k^3 - 32k^2 + (238/3)k - 76$

Proving the Lemmas (Time Permitting)

The End

Thank you for listening!