Some positivity conjectures for symmetric functions motivated by classical theorems from the analytic theory of polynomials

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Rutgers Experimental Mathematics Seminar
April 3, 2025

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Many theorems in algebra or combinatorics assert a pointwise positivity. Can they be "upgraded" to coefficientwise positivity?

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Theorem (Laguerre 1882)

An entire function $f(t)$ with $f(0) = 1$ belongs to LP^+ iff

$$f(t) \;=\; e^{\gamma t} \prod_{i=1}^{\infty} (1 + \alpha_i t)$$

with $\gamma, \alpha_i \geq 0$ and $\sum \alpha_i < \infty$.

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Applications:

- Mechanics of oscillatory systems
- Numerical linear algebra
- Approximation theory
- Stochastic processes
- Lie theory and cluster algebras
- Representation theory of the infinite symmetric group
- Planar discrete potential theory and the planar Ising model
- Stieltjes moment problem
- Zeros of polynomials and entire functions
- Enumerative combinatorics

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Generalize the theory of total positivity from matrices of real numbers to matrices with entries in a partially ordered commutative ring.

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A *partially ordered commutative ring* is a (unital) commutative ring R together with a subset $\mathcal{P}$ (the nonnegative elements) satisfying

(i) $0, 1 \in \mathcal{P}$.

(ii) *If* $a, b \in \mathcal{P}$, *then* $a + b \in \mathcal{P}$ *and* $ab \in \mathcal{P}$.

(iii) $\mathcal{P} \cap (-\mathcal{P}) = \{0\}$.

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Total positivity is then defined in the usual way.

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Given a sequence $\boldsymbol{a} = (a_n)_{n \geq 0}$, we define its *Toeplitz matrix*

$$T_\infty(\boldsymbol{a}) \;=\; (a_{i-j})_{i,j\geq 0} \;=\; \begin{pmatrix} a_0 & & & & \\ a_1 & a_0 & & & \\ a_2 & a_1 & a_0 & & \\ a_3 & a_2 & a_1 & a_0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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- We say that the sequence $\boldsymbol{a}$ is *Toeplitz-totally positive* if its Toeplitz matrix $T_\infty(\boldsymbol{a})$ is totally positive.

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- This definition makes sense in any partially ordered commutative ring.

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Theorem (Aissen–Edrei–Schönberg–Whitney 1951–53)

A sequence $\boldsymbol{a} = (a_n)_{n \geq 0}$ of real numbers with $a_0 = 1$ is Toeplitz-TP iff

$$\sum_{n=0}^{\infty} a_n t^n \;=\; e^{\gamma t} \prod_{i=1}^{\infty} \frac{1 + \alpha_i t}{1 - \beta_i t}$$

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Corollary (cases with no denominators)

(a) A *finite* sequence $\boldsymbol{a} = (a_0, \ldots, a_N, 0, 0, \ldots)$ with $a_0 = 1$ is Toeplitz-TP iff the polynomial $P(t) = \sum_{n=0}^{N} a_n t^n$ is *negative-real-rooted*.

(b) A sequence $\boldsymbol{a} = (a_n)_{n\ge 0}$ with $a_0 = 1$ and $\lim\limits_{n\to\infty} |a_n|^{1/n} = 0$ is Toeplitz-TP if and only if the entire function $f(t) = \sum_{n=0}^{\infty} a_n t^n$ belongs to the Laguerre–Pólya class LP^+.

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Here are some partial orders on the ring of symmetric functions:

e-nonnegative $\searrow$ Schur-nonnegative

h-nonnegative $\nearrow$ Schur-nonnegative

Schur-nonnegative $\nearrow$ m-nonnegative $\longrightarrow$ pointwise nonnegative on $\boldsymbol{\alpha} \geq \mathbf{0}$

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$\longrightarrow$ (by the easy half of Aissen–Edrei–Schönberg–Whitney) the sequence $\big(e_i(\boldsymbol{\alpha})\big)_{i \ge 0}$ is Toeplitz-totally positive

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So we have a double upgrading:
from pointwise nonnegativity to m-nonnegativity to Schur-nonnegativity.

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The class of negative-real-rooted polynomials
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- The **Laguerre composition**

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Conjectures for Hadamard product and factorial Hadamard product

(a) All the Toeplitz minors of the sequence $\big(e_n(\mathbf{X})\, e_n(\mathbf{Y})\big)_{n \ge 0}$ are monomial-positive in $\mathbf{X}$ and $\mathbf{Y}$.

(b) All the Toeplitz minors of the sequence $\big(n!\, e_n(\mathbf{X})\, e_n(\mathbf{Y})\big)_{n \ge 0}$ are monomial-positive in $\mathbf{X}$ and $\mathbf{Y}$.

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(d) Etc. for k factors and $0 \le p \le k-1$.

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Conjecture for Laguerre composition

Fix $M \geq 1$. Then the Toeplitz minors of the sequence

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coefficientwise in the a_i and b_i.

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Variant conjecture

The Toeplitz minors of $\big(c_n\, e_n(\mathbf{X})\big)_{n \geq 0}$ are monomial-positive in $\mathbf{X}$, whenever $(c_n)_{n \geq 0}$ are real numbers with $\displaystyle\sum_{n=0}^{\infty} \frac{c_n}{n!}\, t^n \;\in\; LP^+$.

Here $c_n = \displaystyle\prod_{i=1}^{M} (a_i + b_i n)$ is a special case.

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Follows from the preceding conjecture on factorial Hadamard product:
approximate $c_n / n!$ by $e_n(\mathbf{Y})$ with $\mathbf{Y} \geq 0$.

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- Then the Toeplitz minors of $(c_n \, e_n)_{n \geq 0}$ are the $\phi_{\boldsymbol{c}}$-images of Schur functions.

Richard Stanley's idea for proving these conjectures

- The Toeplitz minors of $(e_n)_{n \geq 0}$ are (positive linear combinations of) Schur functions.
- For any sequence $\boldsymbol{c} = (c_n)_{n \geq 1}$, consider the homomorphism defined by $\phi_{\boldsymbol{c}}(e_n) = c_n \, e_n$ for $n \geq 1$. [If we set $c_0 = 1$, this holds also for $n = 0$.]
- Then the Toeplitz minors of $(c_n \, e_n)_{n \geq 0}$ are the $\phi_{\boldsymbol{c}}$-images of Schur functions.
- What does this give for $c_n = \prod_{i=1}^{M} (1 + b_i n)$?

Polynomial generalization of the Kostka numbers (I)

Recall that if $\lambda \vdash n$, then $s_\lambda = \sum_{\mu \vdash n} K_{\lambda\mu}\, m_\mu$, where the Kostka number $K_{\lambda\alpha}$ is the number of semistandard Young tableaux of shape λ and content α.

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So for any sequence $\boldsymbol{c} = (c_n)_{n\geq 1}$, write $\phi_{\boldsymbol{c}}(s_\lambda) \;=\; \sum_{\mu \vdash n} K^{\boldsymbol{c}}_{\lambda\mu}\, m_\mu$.

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Can we find a **combinatorial** description of these polynomials?

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  - It can be either too high or too low.

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where

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- Goal: First understand $K^{\boldsymbol{c}}_{\lambda\mu}$; then specialize to $c_n = \prod_{i=1}^{M} (1 + b_i n)$

  to prove the Conjecture.