

Some positivity conjectures for symmetric functions motivated by classical theorems from the analytic theory of polynomials

Alan Sokal

University College London

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Work in collaboration with Yusra Naqvi

With early help from Richard Stanley and Lauren Williams

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Many theorems in algebra or combinatorics assert a **pointwise** positivity.
Can they be “upgraded” to **coefficientwise** positivity?

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Theorem (Laguerre 1882)

An entire function $f(t)$ with $f(0) = 1$ belongs to LP^+ iff

$$f(t) = e^{\gamma t} \prod_{i=1}^{\infty} (1 + \alpha_i t)$$

with $\gamma, \alpha_i \geq 0$ and $\sum \alpha_i < \infty$.

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Applications:

- Mechanics of oscillatory systems
- Numerical linear algebra
- Approximation theory
- Stochastic processes
- Lie theory and cluster algebras
- Representation theory of the infinite symmetric group
- Planar discrete potential theory and the planar Ising model
- Stieltjes moment problem
- Zeros of polynomials and entire functions
- Enumerative combinatorics

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A **partially ordered commutative ring** is a (unital) commutative ring R together with a subset \mathcal{P} (the **nonnegative elements**) satisfying

- (i) $0, 1 \in \mathcal{P}$.
- (ii) If $a, b \in \mathcal{P}$, then $a + b \in \mathcal{P}$ and $ab \in \mathcal{P}$.
- (iii) $\mathcal{P} \cap (-\mathcal{P}) = \{0\}$.

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Total positivity is then defined in the usual way.

Toeplitz-total positivity

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Given a sequence $\mathbf{a} = (a_n)_{n \geq 0}$, we define its *Toeplitz matrix*

$$T_\infty(\mathbf{a}) = (a_{i-j})_{i,j \geq 0} = \begin{pmatrix} a_0 & & & & \\ a_1 & a_0 & & & \\ a_2 & a_1 & a_0 & & \\ a_3 & a_2 & a_1 & a_0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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- We say that the sequence \mathbf{a} is *Toeplitz-totally positive* if its Toeplitz matrix $T_\infty(\mathbf{a})$ is totally positive.

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- This definition makes sense in any *partially ordered commutative ring*.

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Theorem (Aissen–Edrei–Schönberg–Whitney 1951–53)

A sequence $\mathbf{a} = (a_n)_{n \geq 0}$ of **real numbers** with $a_0 = 1$ is **Toeplitz-TP** iff

$$\sum_{n=0}^{\infty} a_n t^n = e^{\gamma t} \prod_{i=1}^{\infty} \frac{1 + \alpha_i t}{1 - \beta_i t}$$

with $\gamma, \alpha_i, \beta_i \geq 0$, $\sum \alpha_i < \infty$ and $\sum \beta_i < \infty$.

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Corollary (cases with no denominators)

- (a) A **finite** sequence $\mathbf{a} = (a_0, \dots, a_N, 0, 0, \dots)$ with $a_0 = 1$ is **Toeplitz-TP** iff the polynomial $P(t) = \sum_{n=0}^N a_n t^n$ is **negative-real-rooted**.
- (b) A sequence $\mathbf{a} = (a_n)_{n \geq 0}$ with $a_0 = 1$ and $\lim_{n \rightarrow \infty} |a_n|^{1/n} = 0$ is **Toeplitz-TP** if and only if the entire function $f(t) = \sum_{n=0}^{\infty} a_n t^n$ belongs to the **Laguerre–Pólya class** LP^+ .

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Symmetric functions come into the game because

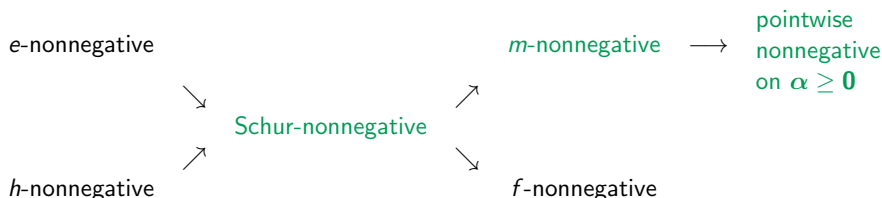
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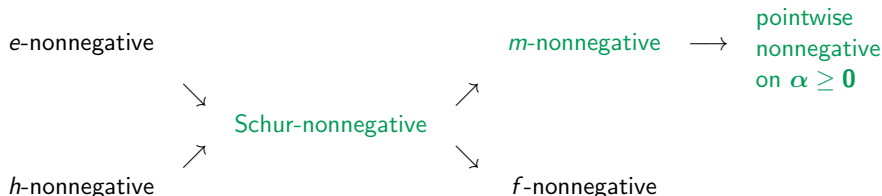


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So we have a **double** upgrading:

from pointwise nonnegativity to **m-nonnegativity** to **Schur-nonnegativity**.

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- The **Laguerre composition**

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Conjectures for Hadamard product and factorial Hadamard product

- (a) All the Toeplitz minors of the sequence $(e_n(\mathbf{X}) e_n(\mathbf{Y}))_{n \geq 0}$ are **monomial-positive** in \mathbf{X} and \mathbf{Y} .
- (b) All the Toeplitz minors of the sequence $(n! e_n(\mathbf{X}) e_n(\mathbf{Y}))_{n \geq 0}$ are **monomial-positive** in \mathbf{X} and \mathbf{Y} .

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- (d) Etc. for k factors and $0 \leq p \leq k - 1$.

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Conjecture for Laguerre composition

Fix $M \geq 1$. Then the Toeplitz minors of the sequence

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Variant conjecture

The Toeplitz minors of $(c_n e_n(\mathbf{X}))_{n \geq 0}$ are monomial-positive in \mathbf{X} , whenever $(c_n)_{n \geq 0}$ are real numbers with $\sum_{n=0}^{\infty} \frac{c_n}{n!} t^n \in LP^+$.

Here $c_n = \prod_{i=1}^M (a_i + b_i n)$ is a special case.

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Follows from the preceding conjecture on factorial Hadamard product:
approximate $c_n/n!$ by $e_n(\mathbf{Y})$ with $\mathbf{Y} \geq 0$.

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- Then the Toeplitz minors of $(c_n e_n)_{n \geq 0}$ are the $\phi_{\mathbf{c}}$ -images of Schur functions.
- What does this give for $c_n = \prod_{i=1}^M (1 + b_i n)$?

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Recall that if $\lambda \vdash n$, then $s_\lambda = \sum_{\mu \vdash n} K_{\lambda\mu} m_\mu$, where the Kostka number $K_{\lambda\alpha}$ is the number of semistandard Young tableaux of shape λ and content α .

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So for any sequence $\mathbf{c} = (c_n)_{n \geq 1}$, write $\phi_{\mathbf{c}}(s_\lambda) = \sum_{\mu \vdash n} K_{\lambda\mu}^{\mathbf{c}} m_\mu$.

What can we say about the generalized Kostka numbers $K_{\lambda\mu}^{\mathbf{c}}$?

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Can we find a **combinatorial** description of these polynomials?

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 - It can be either too high or too low.

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- We believe that $K_{\lambda\mu}^{\mathbf{c}}$ has the general form

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- **Goal:** First understand $K_{\lambda\mu}^{\mathbf{c}}$; then specialize to $c_n = \prod_{i=1}^M (1 + b_i n)$ to prove the Conjecture.