Cutting a Pancake With an Exotic Knife

Experimental Math Seminar,
Rutgers University, September 11 2025

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Outline

- The Problem
- Long-legged letters
- The planar graph
- Euler's formula, the edge-vertex count, the basic equation
- Examples, k-armed V,   k-chain
- 3-armed V (Wu) = 3-chain = long-legged A
- Further shapes
- Unsolved shapes
- Circles and pentagrams

1. The Problem

What is the maximum number of regions formed by n lines, circles, V's, Wu's, X's, long-legged letters, etc. in the plane?

a Wu

Classic problem: What is the maximum number of regions formed by n lines in the plane?
That is, cuts by an infinite straight knife K ?

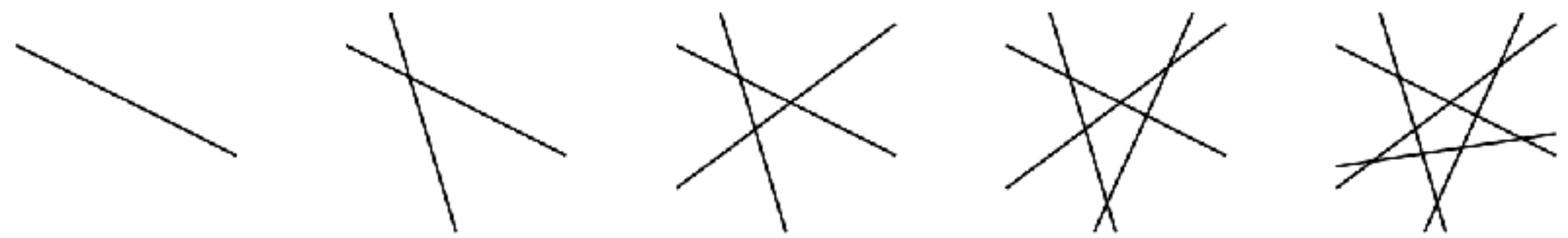


Figure 1: Cutting a pancake with 1, 2, 3, 4, or 5 cuts using an infinite straight knife produces a maximum of 2, 4, 7, 11, and 16 pieces.

a_K(n) (n >= 0) = 1, 2, 4, 7, 11, 16, 22, 29, 37, …
= n(n+1)/2 + 1
Entry A000124 in OEIS

Classic reference: Graham, Knuth, and Patashnik, Concrete Mathematics, page 7

In general, we ask:

For a shape S (a line, circle, etc) what is the value of

$a_S(n)$ = maximum number of regions formed in the plane by drawing n copies of S ?

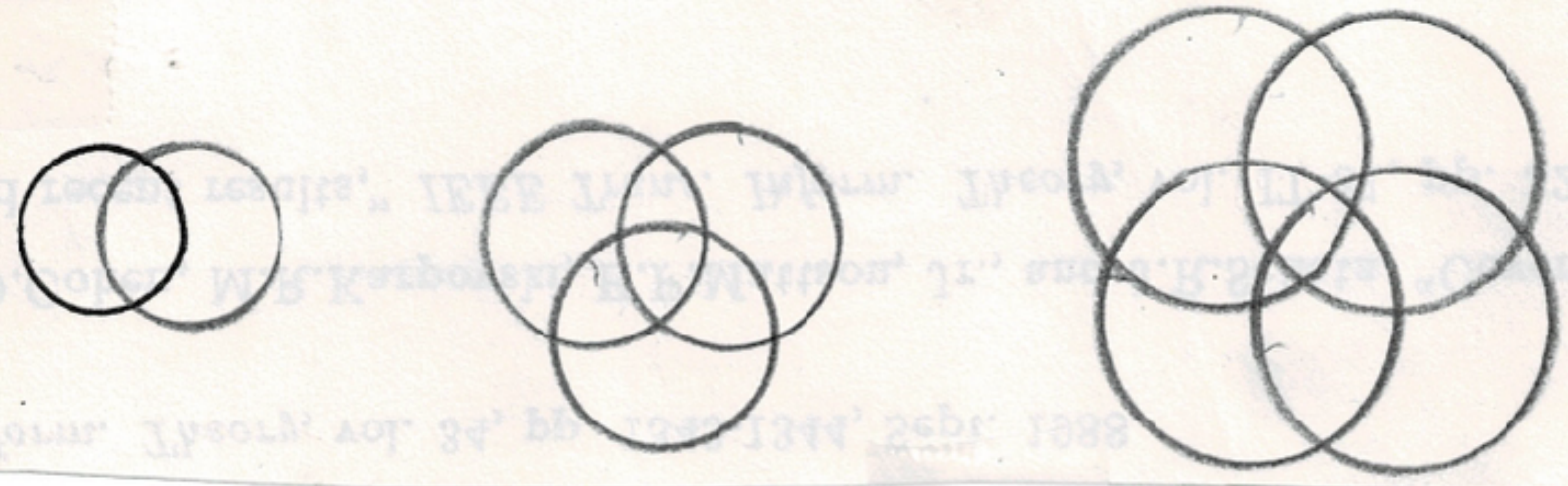
The copies may be individually magnified, rotated, reflected, etc., just as long as they look like S's.

For extra credit: How many essentially different solutions are there ?

Remark: Although these problems may sound frivolous, they originated in a book by some of the greatest names in Discrete Mathematics, there are new and surprising theorems, difficult unsolved problems, and there may even be applications. And it has been enjoyable working on a problem that can be explained to a neighbor.

Other Famous Examples

S = circle  $a_0(1) = 2$



$a_0(2) = 4$ $a_0(3) = 8$ $a_0(4) = 14$

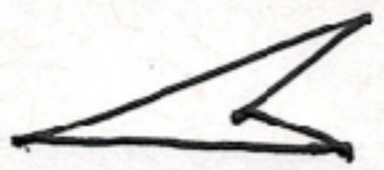
$$a_0(n) = n^2 - n + 2$$
$(n > 0)$ (A386480)

S = square (or convex quadrilateral)

2 10 26 50 82

$$a_\square(n) = 4n^2 - 4n + 2 \quad (n > 0)$$
(A069894)

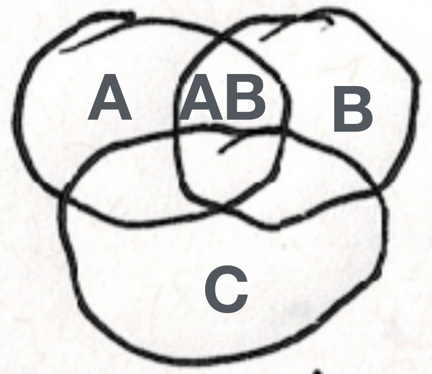
S = concave quadrilateral

 $2(2n-1)^2$ $(n>0)$ (A077591)

Other shapes (cont.)

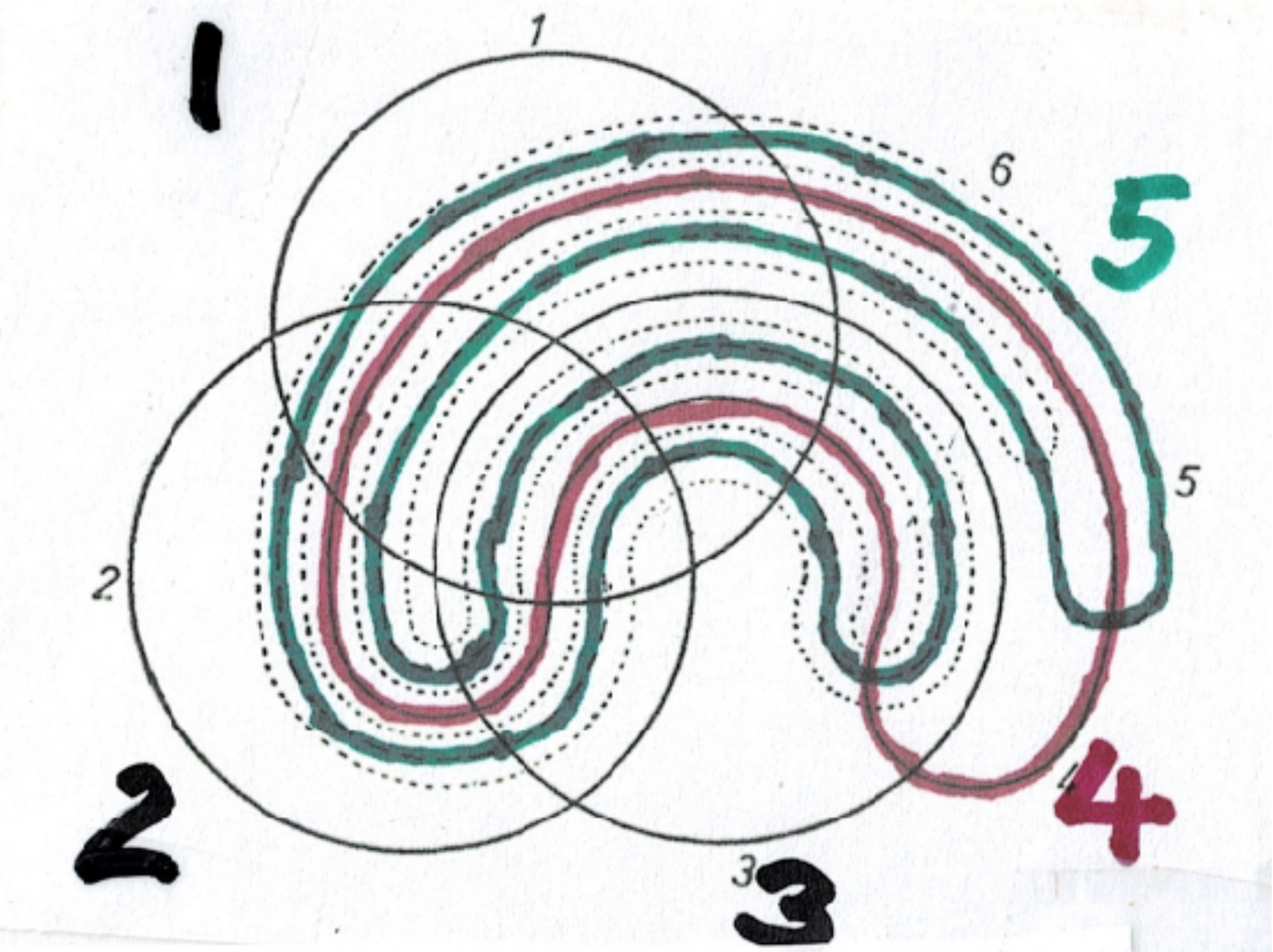
So far, $a_S(n)$ = quadratic in n. However:

Venn Diagrams



$n = 3$ sets

$n = 4$?



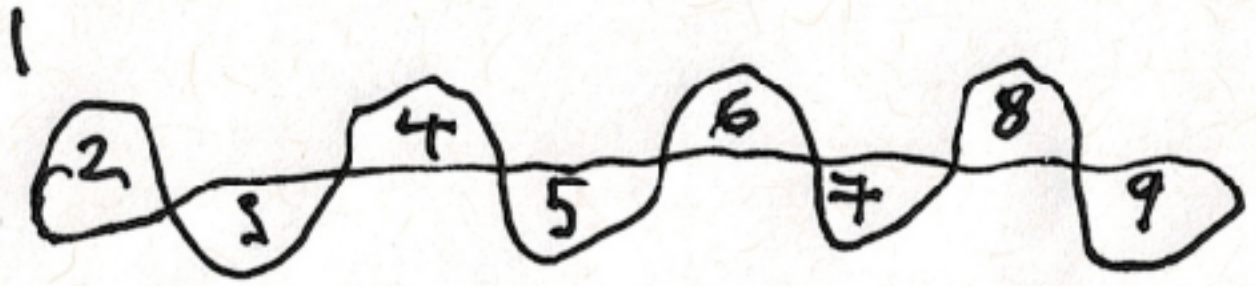
Branko Grünbaum (1975):

S = sausage (simple Jordan curve):

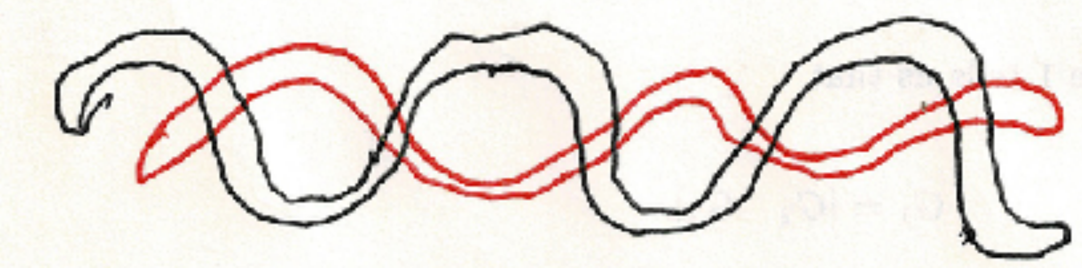
$a_{SJC}(n) \geq 2^n$

In fact, one twisted sausage:

$a_S(1)$ is unbounded:



and $a_{SJC}(2)$ is unbounded:



Therefore we assume

S = connected planar graph made from straight line segments (which may be infinite)

Three examples from "Concrete Mathematics"

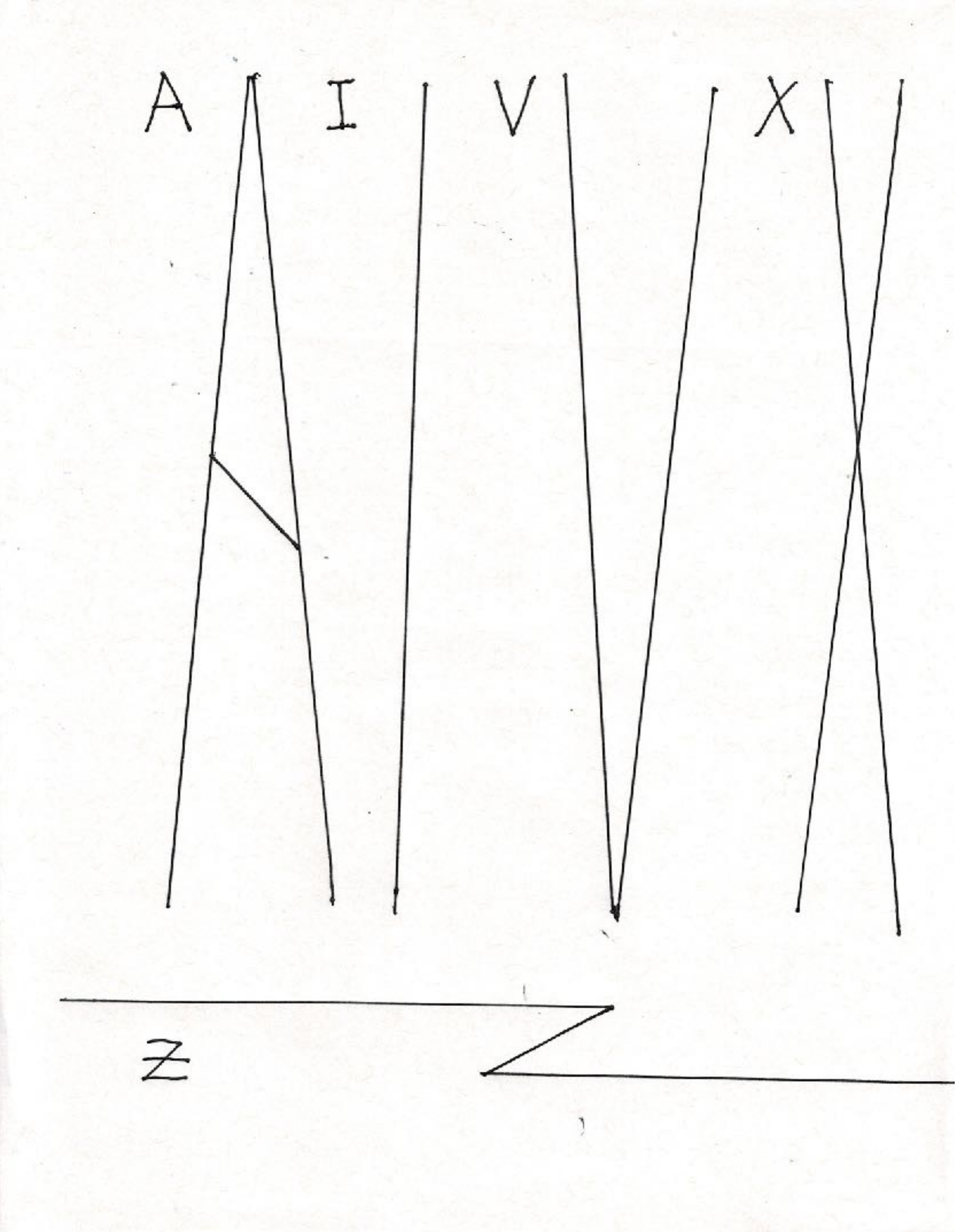
The three cases when S is a line, or a bent line in the form of a V with two very long limbs

or a "zig-zag", a doubly bent line in the form of a Z with two long parallel limbs joined by a diagonal line segment

are discussed in the first chapter of Graham, Knuth, and Patashnik's *Concrete Mathematics* [3]. In order to generalize their examples, we observe that they may be regarded as "long-legged"[4] versions of the upper-case letters I, V, and Z.

Some “long-legged” letters

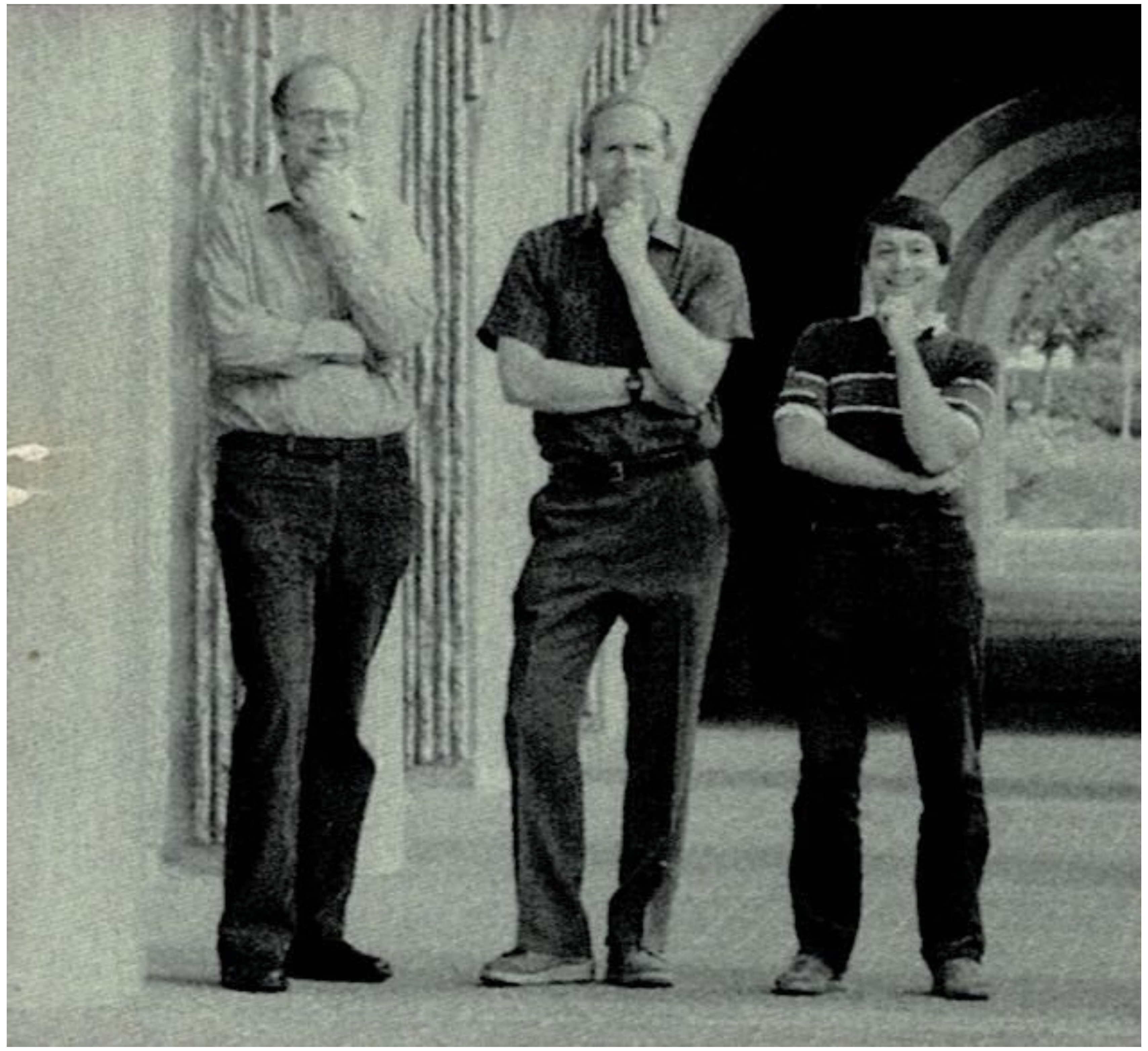
Concrete Mathematics calls
the long-legged V a “bent line”,
and the long-legged Z a “zig-zag”



Long-Legged Fly

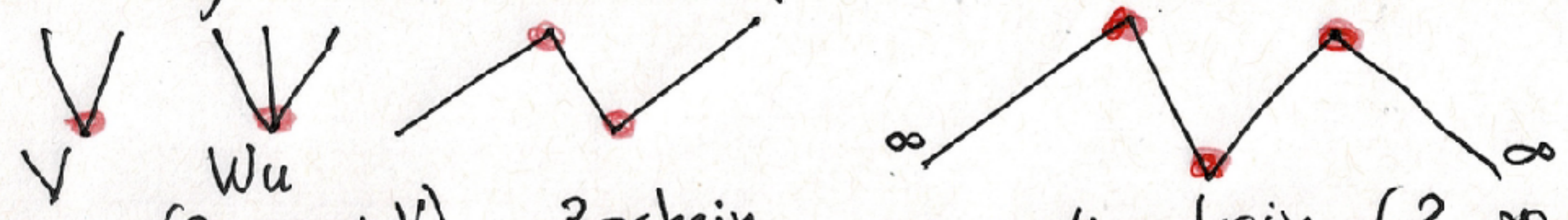
That civilisation may not sink.
Its great battle lost,
Quiet the dog, tether the pony
To a distant post:
Our master Caesar is in the tent
Where the maps are spread
His eyes fixed upon nothing,
A hand under his head.
Like a long-legged fly upon the stream
His mind moves upon silence.

W. B. Yeats, 1939



The Planar Graph $G_S(n)$ defined by n copies of S

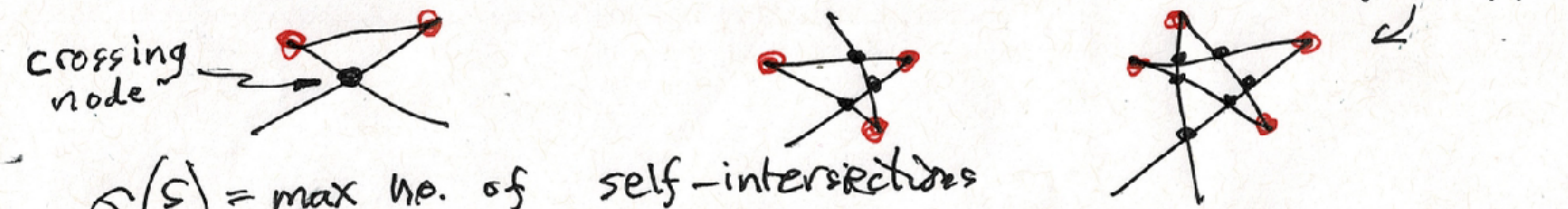
Think of S itself as a planar graph:



V   Wu (3-armed V)   3-chain   4-chain (2 ∞ segments, 2 finite ")

Nodes in S are the <u>base nodes</u> ●
Edges in S are the <u>arms</u>

Can redraw a 3-chain as   a 4-chain   5-chain

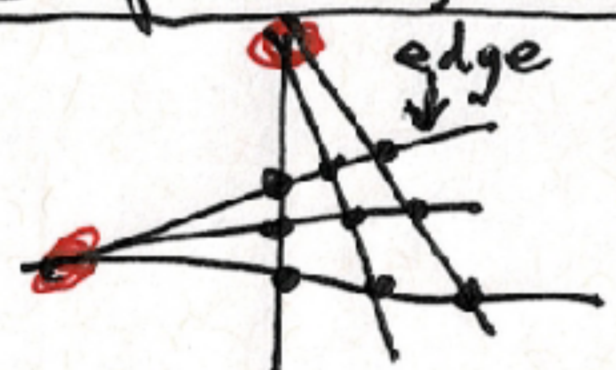


$\sigma(S)$ = max no. of self-intersections

$\sigma = 1$ $\sigma = 3$ $\sigma = 6$

$$\sigma(k\text{-chain}) = \binom{k-1}{2}$$

<u>2 copies of S</u>



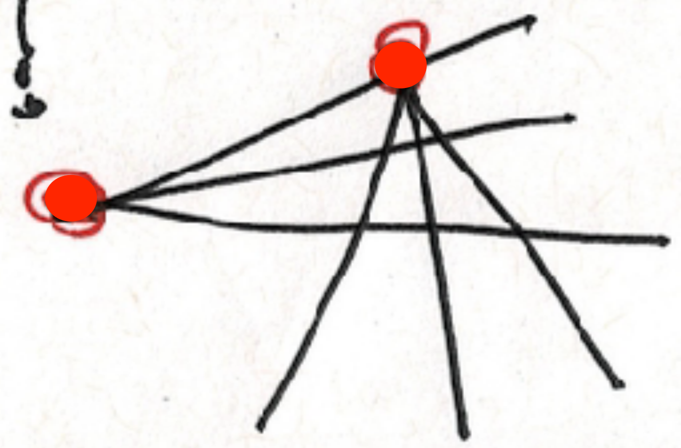
9 intersections

$\chi(S)$ = max number of intersections between 2 copies of S

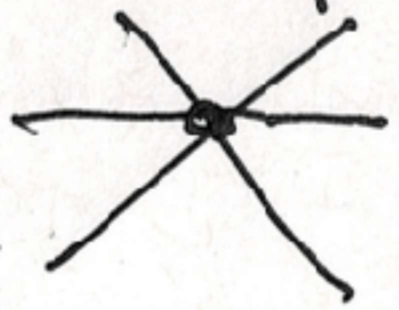
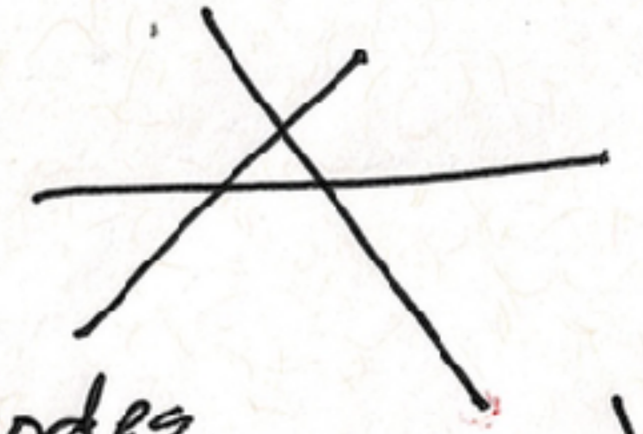
$\chi(\mathrm{Wu}) = 9$

The Planar Graph $G_S(n)$ (cont.)

Can assume: No arm through a different base node

No!  better is 

No triple intersections:

No!  better is 

d_v = degree of base node v

Let V_B = no. of base nodes

V_C = " " crossing nodes

E_∞ = " " infinite edges

E_f = " " finite edges

R_∞ = " " infinite regions

R_f = " " finite "

$V = V_B + V_C$ = vertices in $G_S(n)$

$E = E_\infty + E_f$ edges in $G_S(n)$

$R = R_\infty + R_f$ regions in $G_S(n)$

Euler: If $E_\infty > 0$, $R = E - V + 1$, $G_S(n)$ lives in $\mathbb{R}^2$

If $E_\infty = 0$, $R = E - V + 2$, $G_S(n)$ lives on S_2

Euler's Formulas

In the Euclidean plane: R = E - V + 1

Proof courtesy of Gareth McCaughan:

**The Euler characteristic depends only on the space's homology,
and homology is homotopy-invariant;
the plane is contractible and therefore has homology groups Z, 0, 0, 0, ...
with ranks 1, 0, 0, 0, ..., and hence the Euler characteristic of the plane
is 1 - 0 + 0 - 0 + 0 ... = 1.**

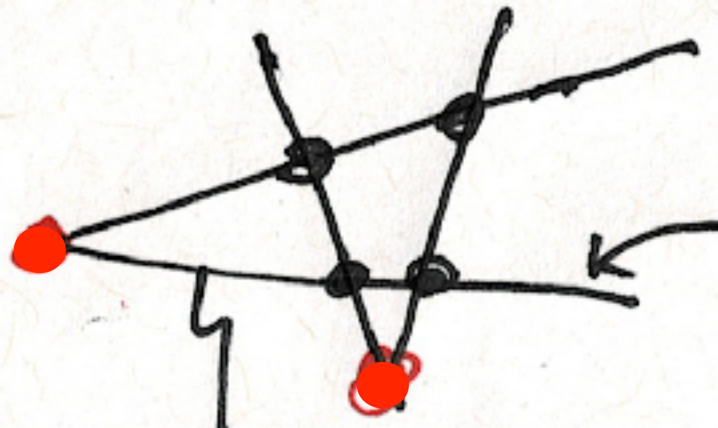
On the sphere S_2: R = E - V + 2

**For a dozen proofs, see Imre Lakatos,
"Proofs and Refutations: The Logic of Mathematical Discovery",
Cambridge, 1976.**

The Edge-Vertex Count

Assume S is infinite

Count edge-vertex incidences in two ways



$$\sum_{v \text{ base node}} d_v + 4V_c + E_\infty \quad \text{counts every edge twice}$$

$$= 2(E_\infty + E_f)$$

$$\therefore \quad E_f = 2V_c + \frac{1}{2}\sum_{v \text{ bare}} d_v - \frac{1}{2}E_\infty$$

But $\quad R = (E_f + E_\infty) - (V_c + V_B) + 1$

$$\therefore \quad R = V_c + \frac{1}{2}\sum_{\text{base nodes}} d_v - V_B + \frac{1}{2}E_\infty + 1$$

BASIC EQN. 1

Maximize R by maximizing number of crossings fixed

If S finite:

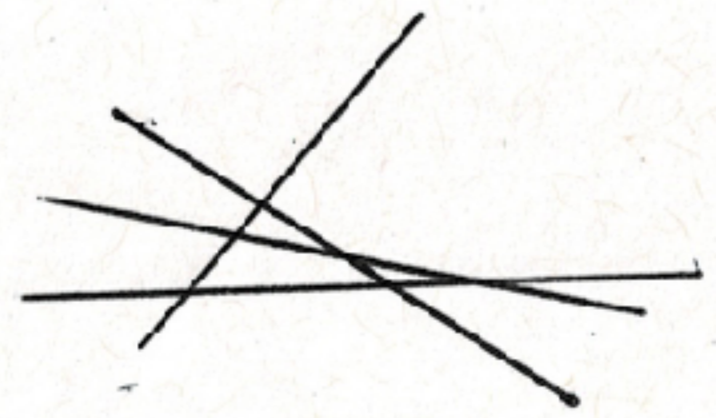
$$R = V_c + \frac{1}{2}\sum_{v \text{ base node}} d_v - V_B + 2$$

BASIC EQN. 2

Note: $V_c \leq n\,\sigma(S) + \binom{n}{2}\chi(S)$

EXAMPLES 1

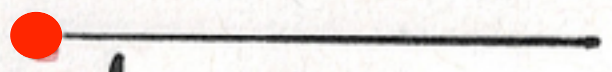
S = line (or knife)   Pancake graph $G_K(n)$.



Assume $n \geq 2$, & each line cuts at least one other line.

$V_B = 0$, $E_\infty = 2n$, $\chi = 1$, $\sigma = 0$, $V_C \leq \binom{n}{2} \cdot 1$

$R \leq \binom{n}{2} + n + 1$, =? Yes ✓   A000124

S = Hat pin  head arm $V_B = n$, $d_v = 1$, $E_\infty = n$, $\chi = 1$,

$V_c \leq \binom{n}{2}$, $R \leq \binom{n}{2} + \frac{1}{2}n - n + \frac{1}{2}n + 1 = \binom{n}{2} + 1$ Yes ✓   A000124

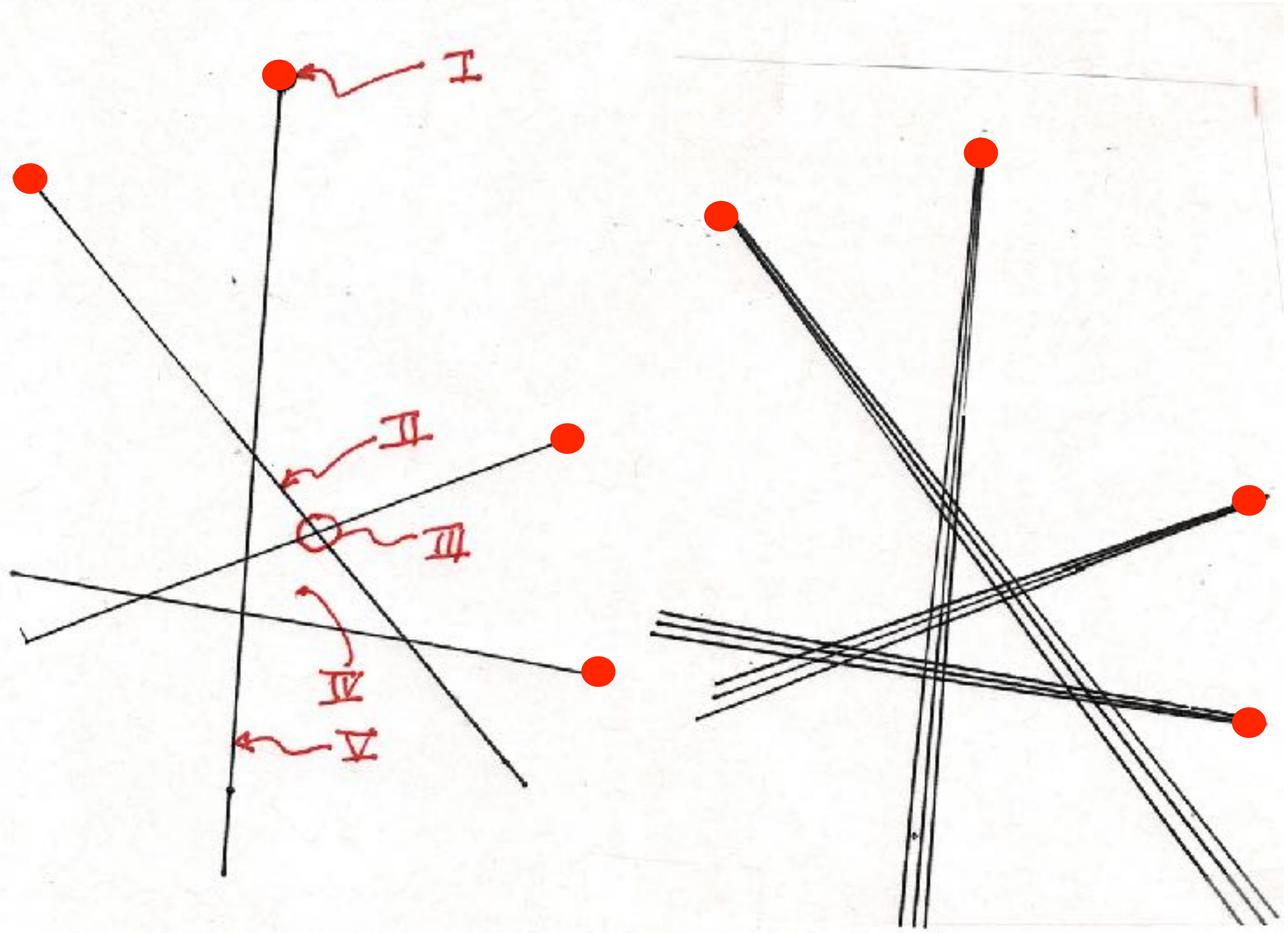
S = k-armed V = , $k \geq 1$

$V_B = n$, $d_v = k$, $\sigma = 0$, $\chi = k^2$

$V_C \leq k^2 \binom{n}{2}$

$R \leq \frac{k^2 n^2}{2} - \frac{k^2 - 2k + 2}{2} n + 1$

=? Yes, use hat pin graph



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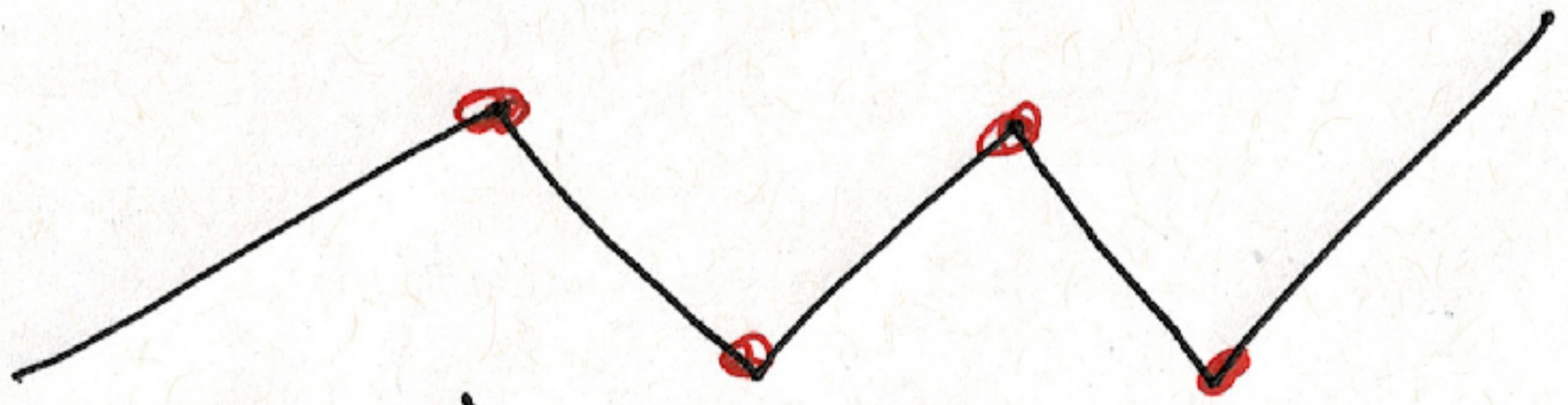
Case $k=3$ found by
Edward Xiong, Jonathan Pei,
and David Cutler
June 24 2025

EXAMPLES 2

S = k-chain



$k \geq 1$

5-chain
4 base nodes
5 arms

$\sigma(k\text{-chain}) = \binom{k-1}{2}$ self-crossings

$V_B = (k-1)n$ base nodes, deg $d_v = 2$

$V_C \leq \binom{k-1}{2} n + k^2 \binom{n}{2}$

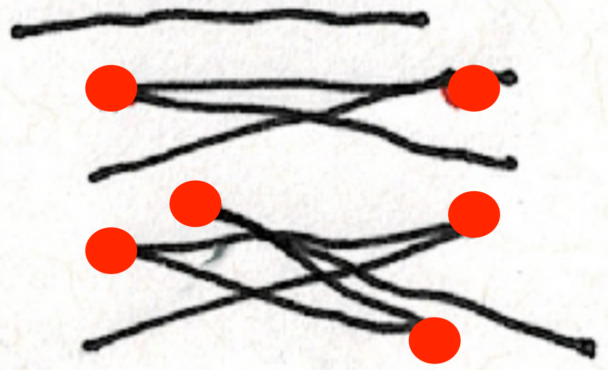
$\chi = k^2$

$$\boxed{R \leq \frac{k^2 n^2}{2} - (3k-4)\frac{n}{2} + 1}$$ = ? Yes

A386478

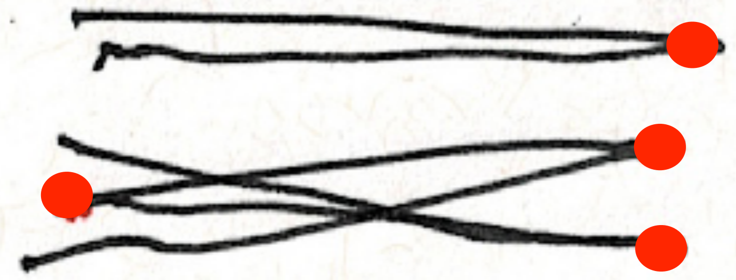
If k odd, use a pancake graph

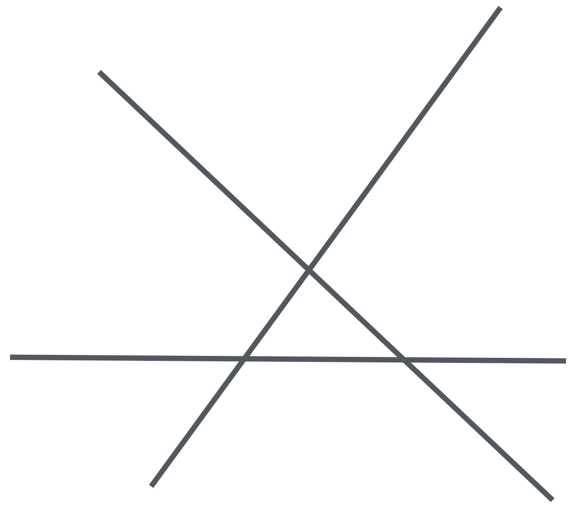
$k=1$
$k=3$
$k=5$

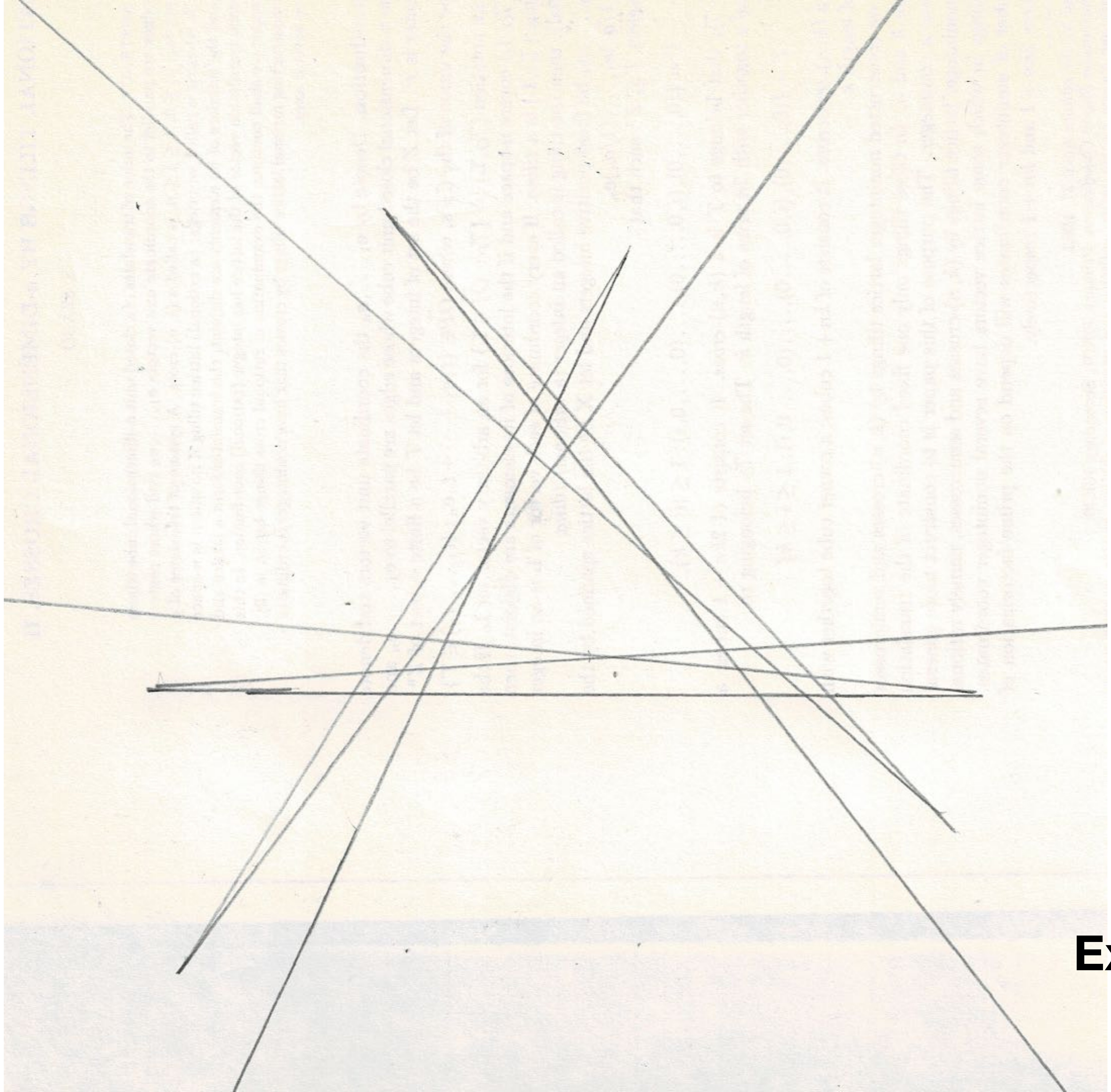


If k even, use a hat pin graph

$k=2$
$k=4$







Transforming a pancake graph with three cuts into a graph with three 3-chains and 34 regions

Examples 2 (continued)

Examples 3

Classic problem:

A386481

k\n	0	1	2	3	4	5
0	1	1	1	1	1	1
1	1	1	2	4	7	11
2	1	2	7	16	29	46
3	1	3	14	34	63	101
4	1	4	23	58	109	176
5	1	5	34	88	167	271
6	1	6	47	124	237	386

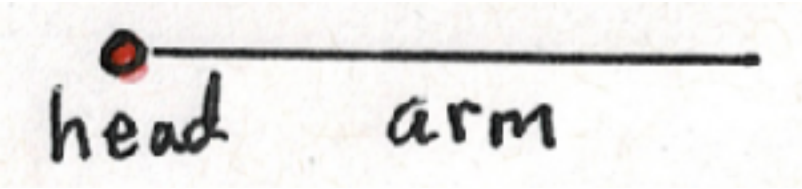
= = = = !

A386478

k\n	0	1	2	3	4	5
0	1	1	1	1	1	1
1	1	2	4	7	11	16
2	1	2	7	16	29	46
3	1	3	14	34	63	101
4	1	5	25	61	113	181
5	1	8	40	97	179	286

Max number of regions in the plane formed by n copies of a k-armed V

k = 1: a hat pin



k = 2: a long-legged V

k = 3: a long-legged Wu  = (9 n^2- 5 n + 2) / 2 =

Max number of regions in the plane formed by n copies of a k-chain

k = 1: a line

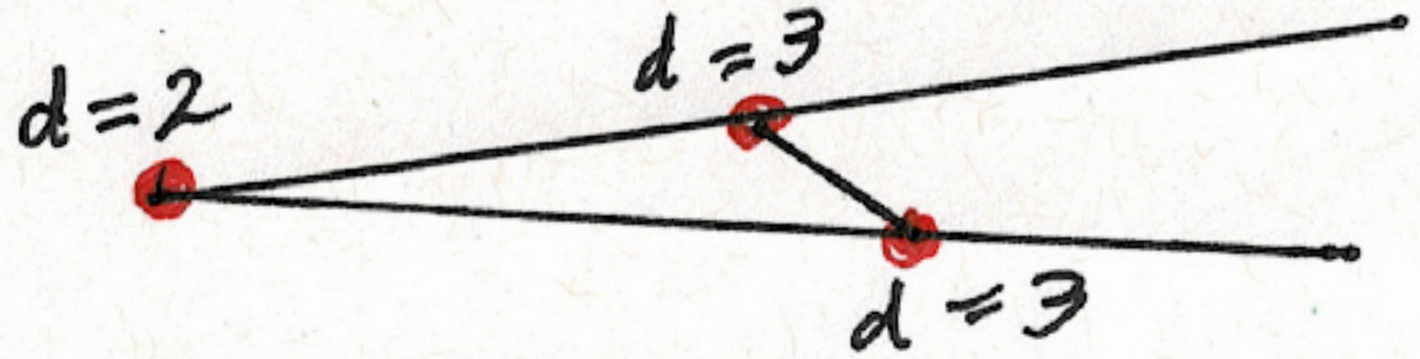
k = 2: a long-legged V

k = 3: a long-legged 3-chain (or picnic table)

EXAMPLES 4

The long-legged A

$a_A(1) = 3$



R = 3

$$R = V_X + \tfrac{1}{2}\Sigma d_v - V_B + \tfrac{1}{2}E_\infty + 1$$

$\sigma(A) = 0 \qquad \chi(A) = 9$

$E_\infty = 2n \qquad \Sigma d_v = 2n + 3n + 3n$

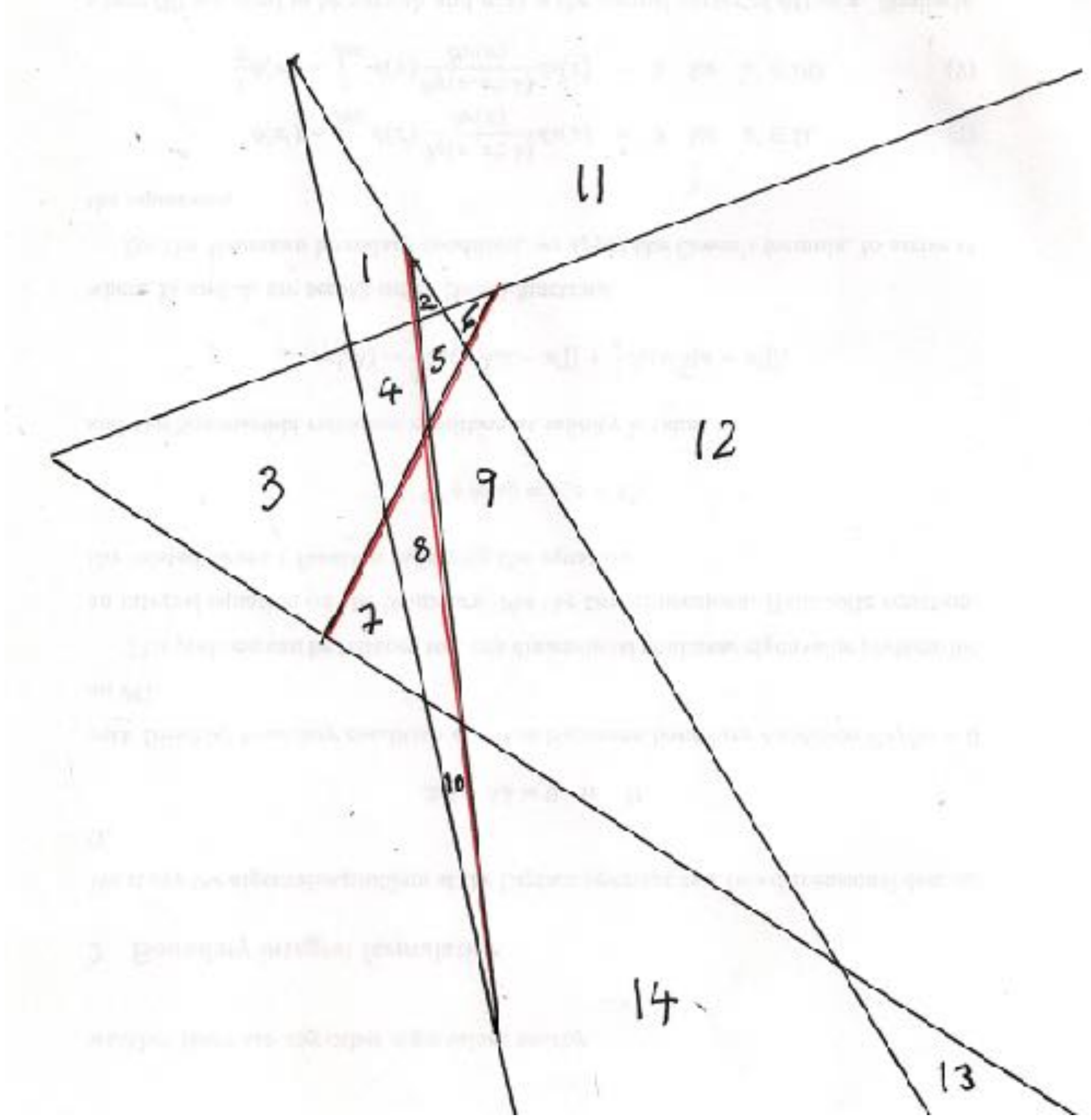
$\qquad\qquad\qquad\quad = 8n$

$V_B = 3n \qquad V_X \le 9\binom{n}{2}$

$$R \le 9\binom{n}{2} + 4n - 3n + n + 1$$

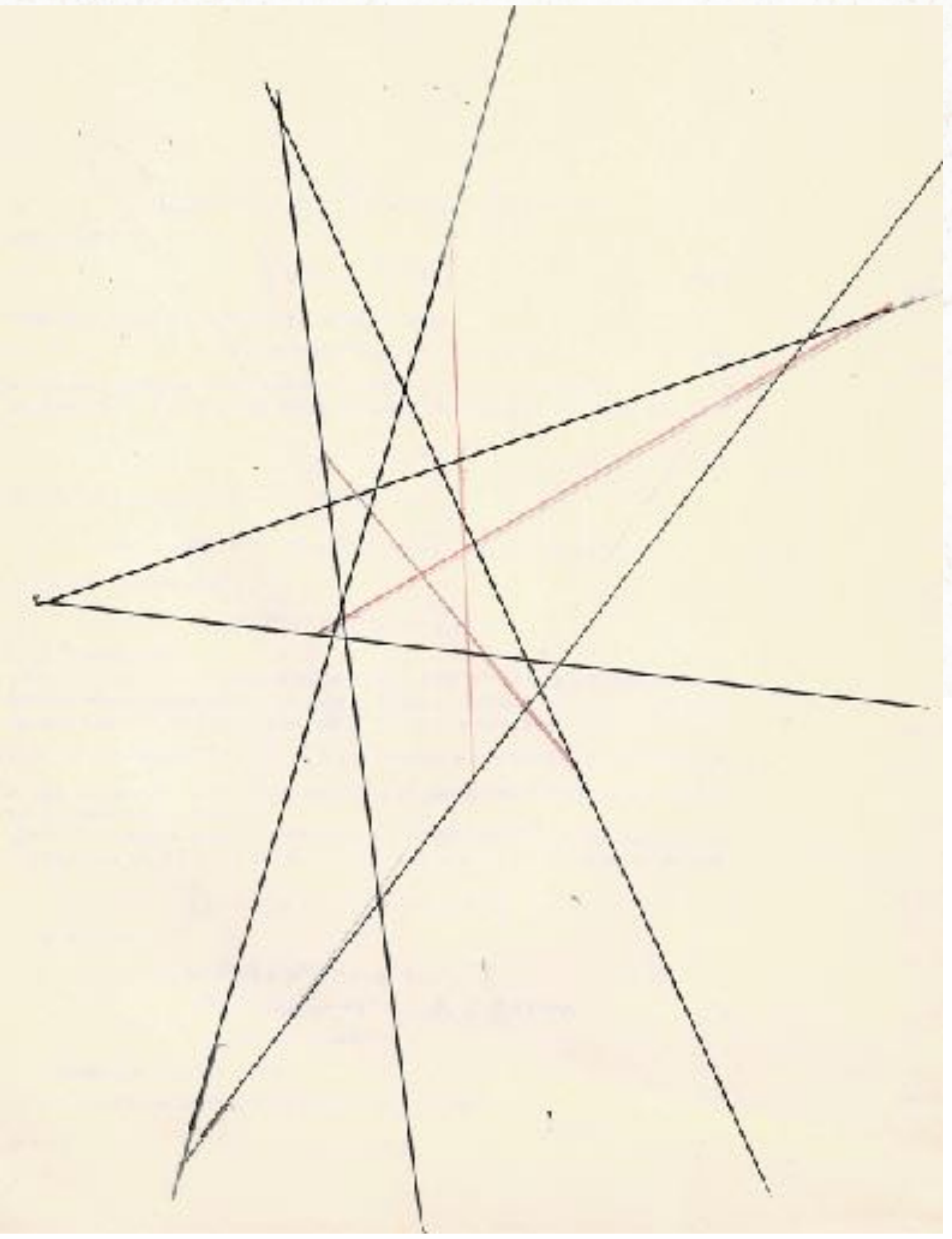
$$= \frac{9n^2 - 5n + 2}{2} \quad =? \ \text{Yes}$$

A140064



$a_A(2) = 14$

$\chi(A) = 9$

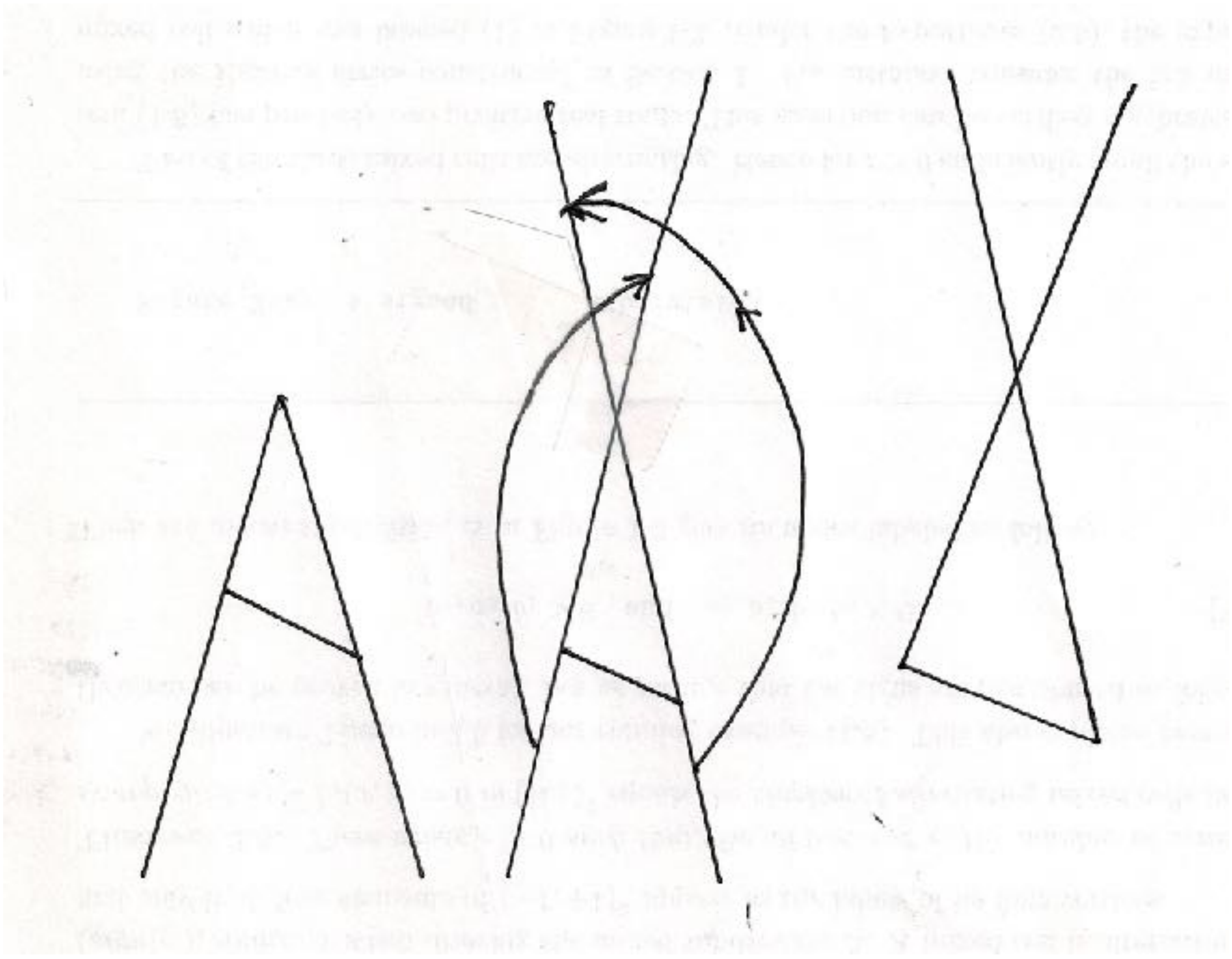


$a_A(3) = 34$

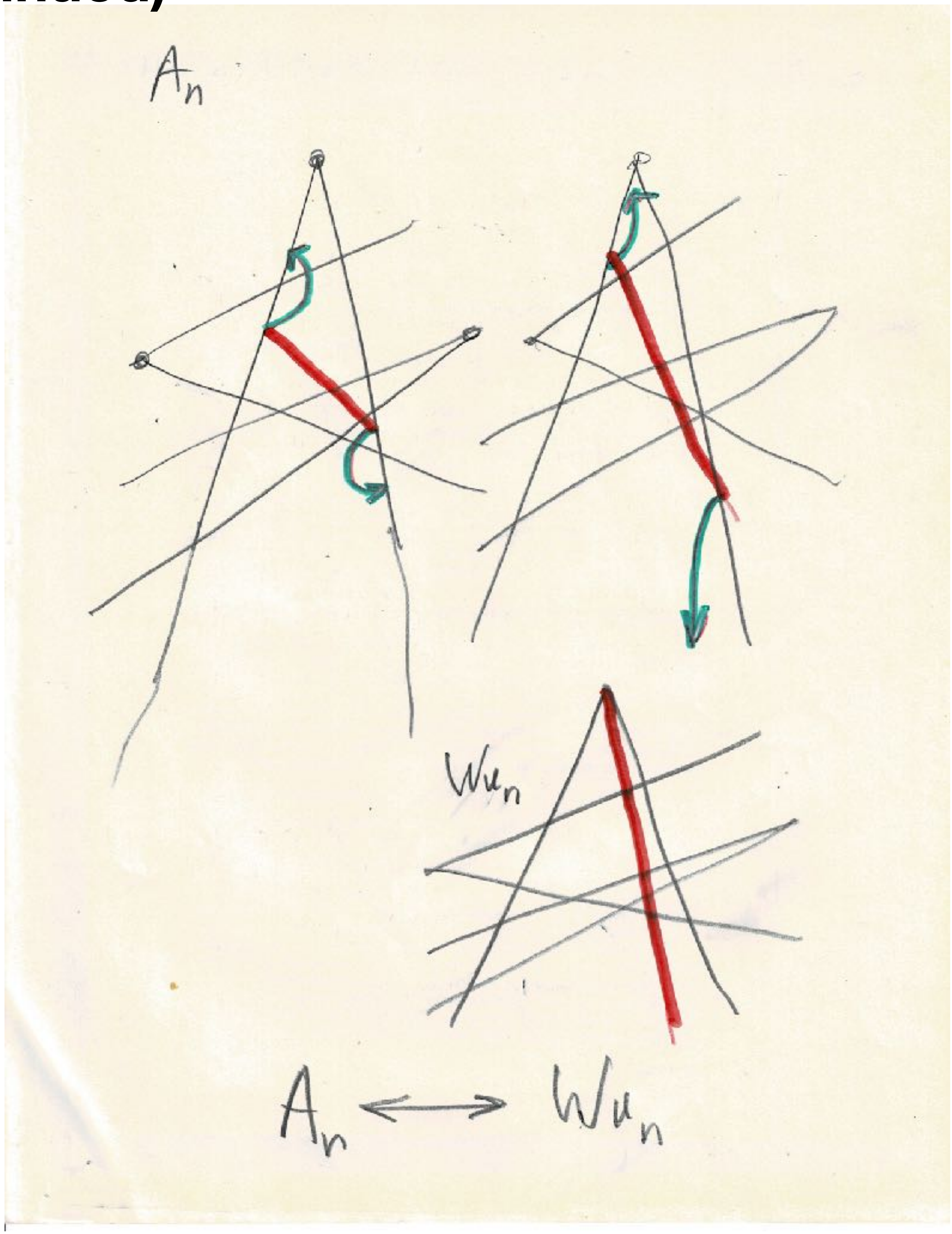
The cross-bars are colored red.

Examples 4 (continued)

Theorem: Maximum number of regions formed by n (long-legged) A's, n Wu's, and n 3-chains are all equal.



A ⟷ 3-chain



Examples 4 (continued)

A Mystery:
Why is this table almost symmetrical?

$k\backslash n$	0	1	2	3	4	5
0	1	1	1	1	1	1
1	1	1	2	4	7	11
2	1	2	7	16	29	46
3	1	3	14	34	63	101
4	1	4	23	58	109	176
5	1	5	34	88	167	271
6	1	6	47	124	237	386

Max number of regions in the plane formed by n copies of a k-armed V

$k\backslash n$	0	1	2	3	4	5
0	1	1	1	1	1	1
1	1	2	4	7	11	16
2	1	2	7	16	29	46
3	1	3	14	34	63	101
4	1	5	25	61	113	181
5	1	8	40	97	179	286

Max number of regions in the plane formed by n copies of a k-chain is essentially the same as the

Max number of regions in the plane formed by k copies of an n-chain

The difference is 2|n-k|, which is exactly the difference in the numbers of infinite regions.

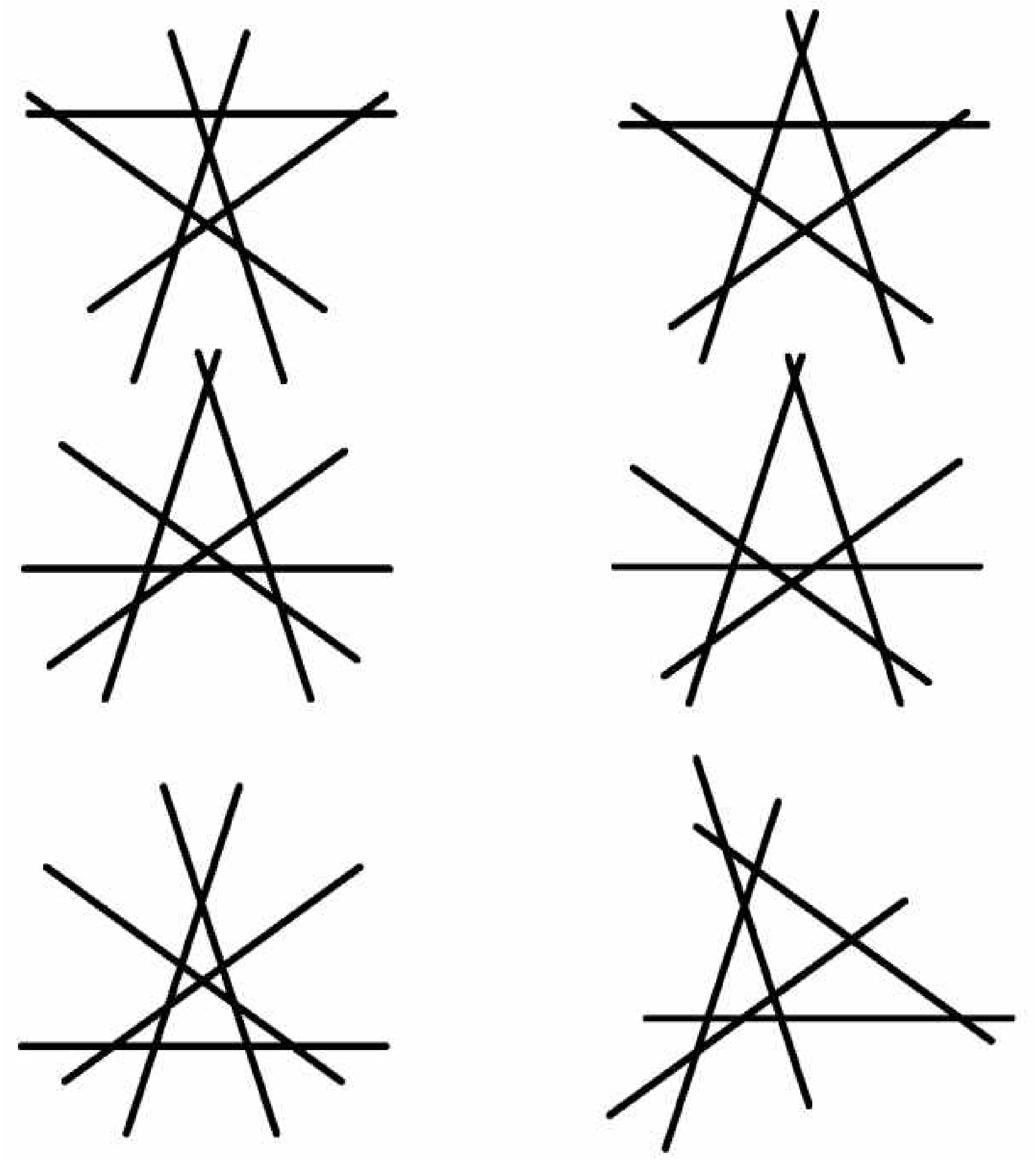
It is true, but what is the geometric explanation?

A Mystery: (continued)

Why is this table almost symmetric?

Is there a 1-1-correspondence between optimal graphs with n k-chains and k n-chains? No!

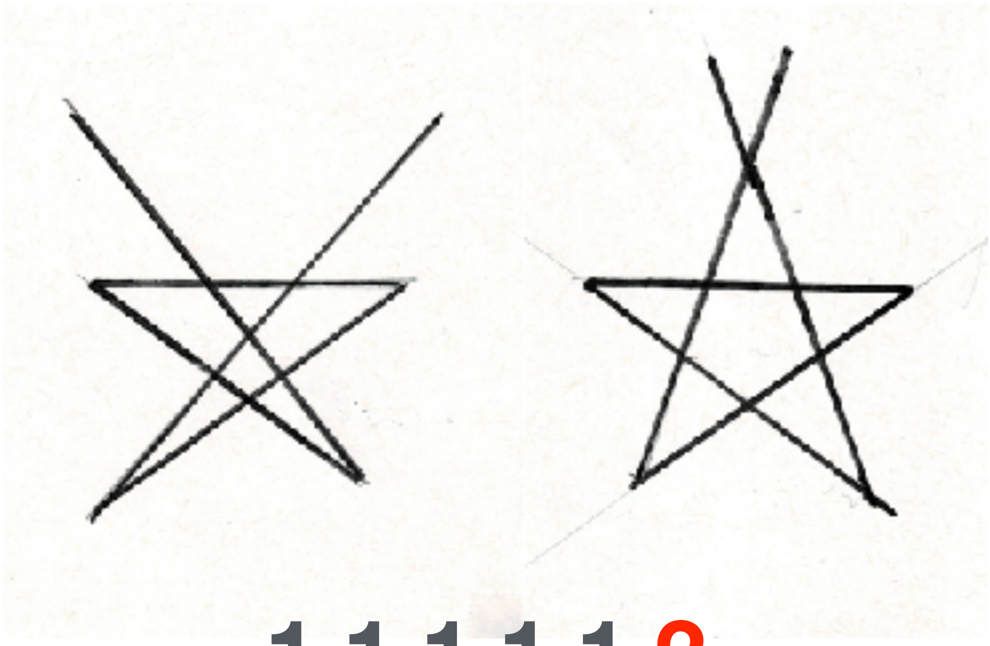
Five 1-chains = 5 lines in general position: 6 ways



J. Wild and L. Reeves, from A090338

1 1 1 1 1 6 43 922 38609 ...

One 5-chain 2 ways



1 1 1 1 1 2 ...

(needs more terms!)

The Circle

The basic equation says $R = V_C + (\frac{1}{2} d_B - 1) V_B + 2$.

One circle: $V_C = V_B = 0$, $R = 2$ ✓

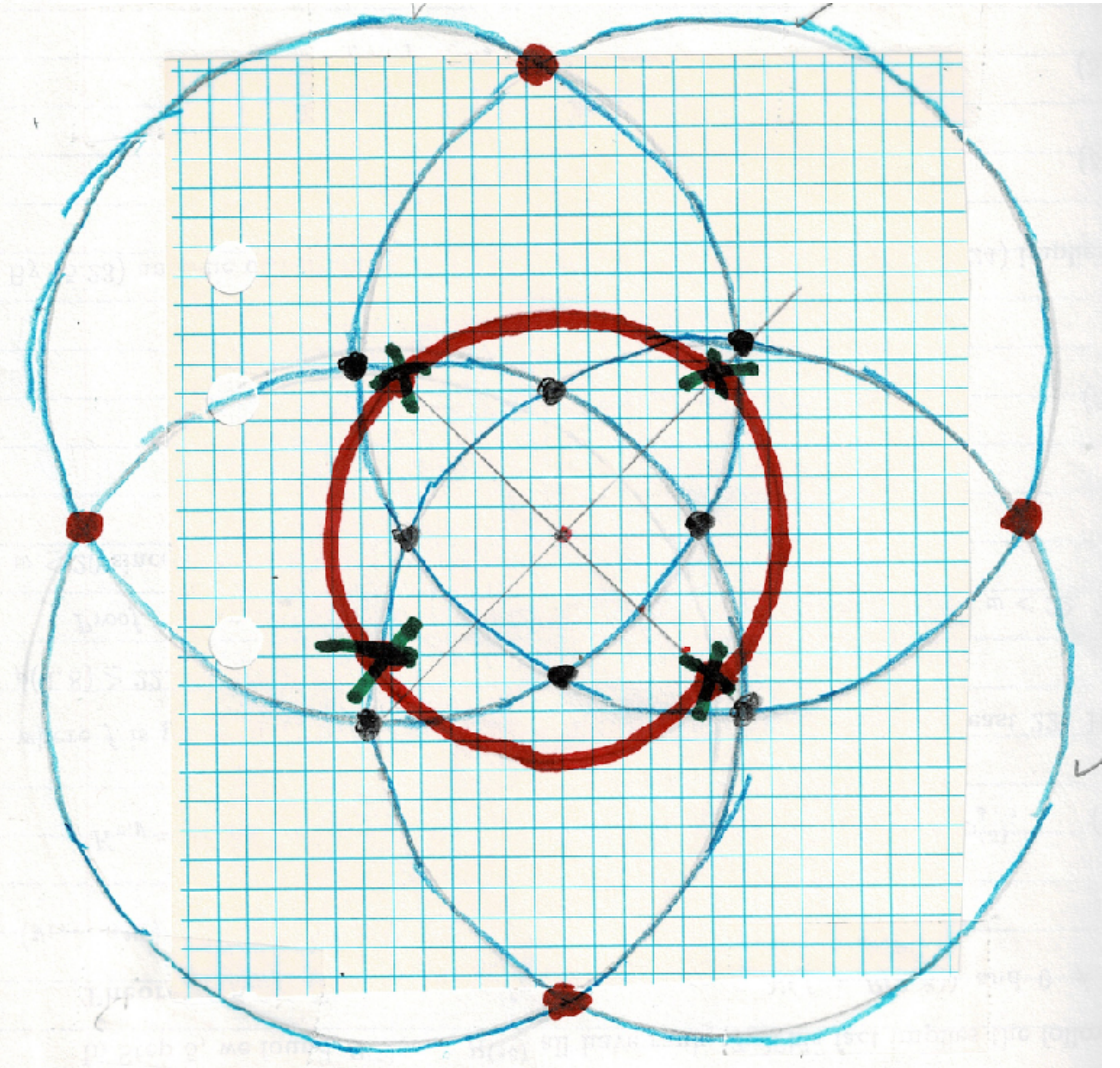
$n \geq 2$ circles: $d_B = 4$, $R = V_C + V_B + 2 = V + 2$,

and $X(0) = 2$, $V_C \leq 2\binom{n}{2}$, $R \leq n^2 - n + 2$

(A386480)

<u>Construction</u>

Draw a temporary circle of radius ρ (say) (red), mark n equally spaced points, draw n (blue) circles of radius $8\rho/5$.



The red circle and
the green X's are used
for the construction,
but are not part of the graph.

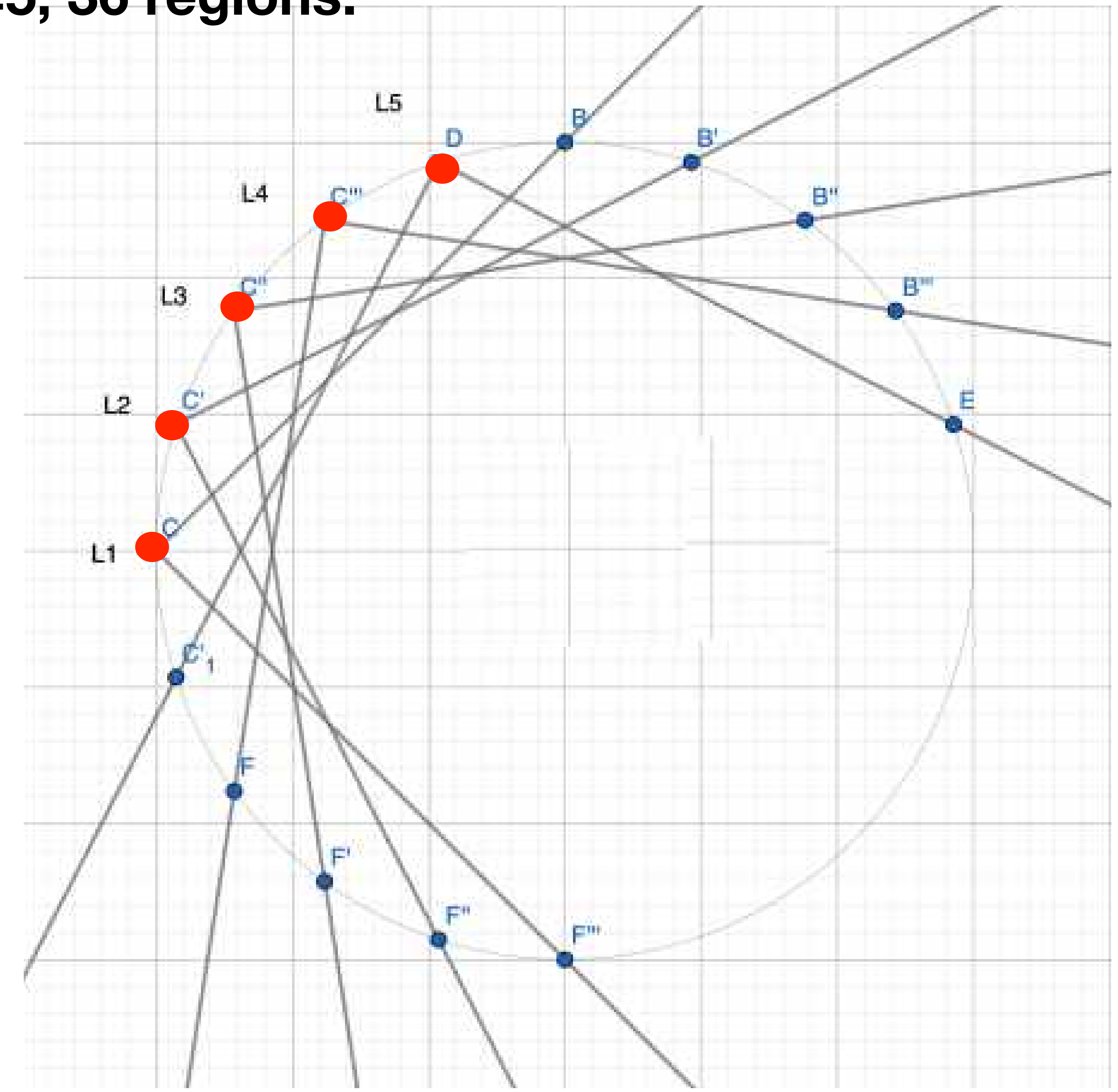
Further Shapes

See D. O. H. Cutler and N. J. A. Sloane, "Cutting a Pancake with an Exotic Knife" for more examples, open problems, etc.

Definition: A "constrained long-legged letter" means extra conditions.

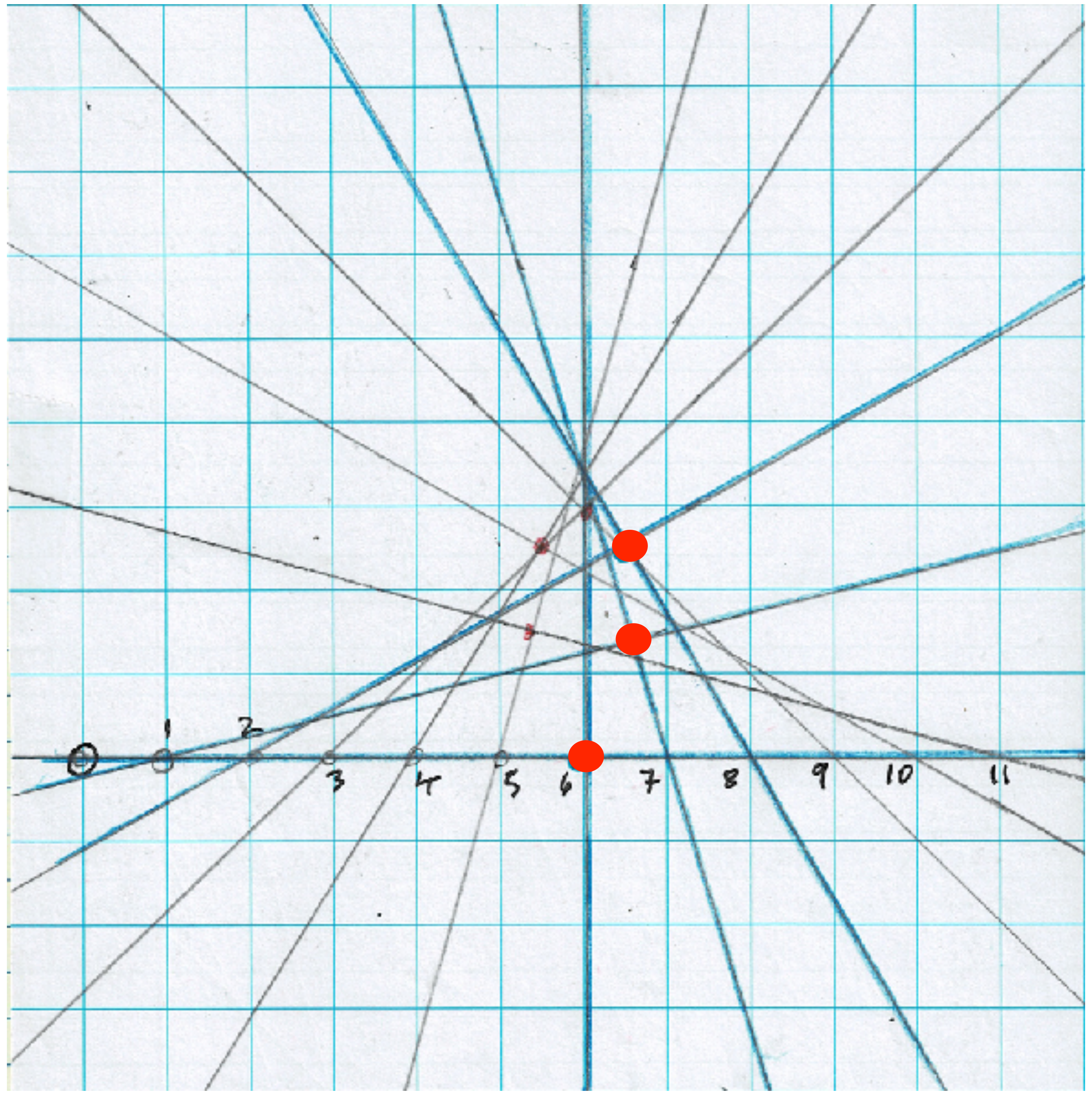
A constrained long-legged L: angle is fixed at 90 degrees.

a(n) = (3n^2-n+2)/2, A143689, based on solution for squares. Here n=5, 36 regions:

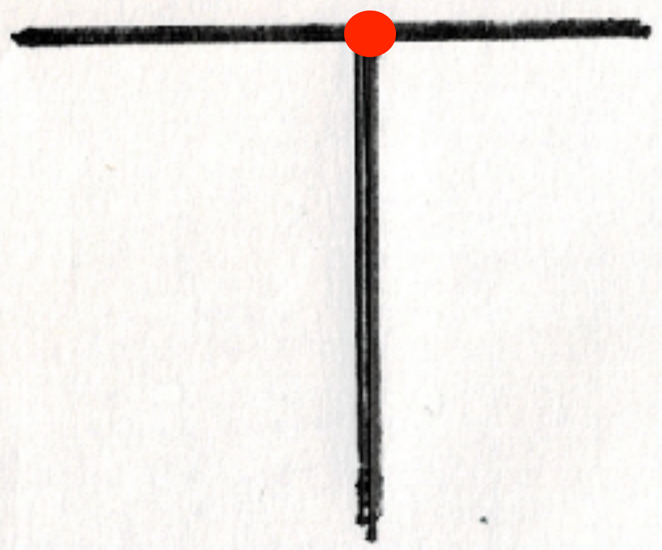


A constrained long-legged X: angle is fixed at 90 degrees.

a(n) = 2n^2+n+1, A084849, based on a pancake construction, here n = 3, 22 regions:



UNSOLVED

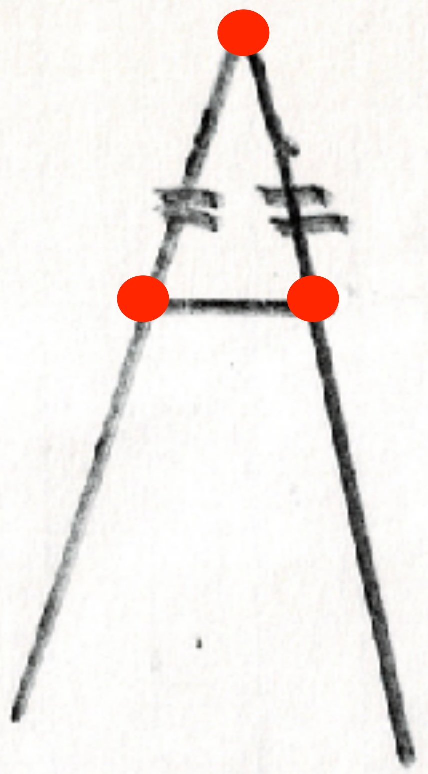


Basic eq. gives

$a_{cT}(n) \le 2n^2+1$

but $\ne$ for $n \geqslant 4$.

What is $a_{cT}(n)$?



Conjecture:

$$a_{cA}(n) \stackrel{?}{=} \frac{7n^2-3n+2}{2}$$

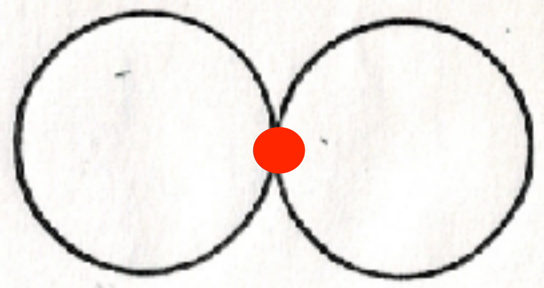
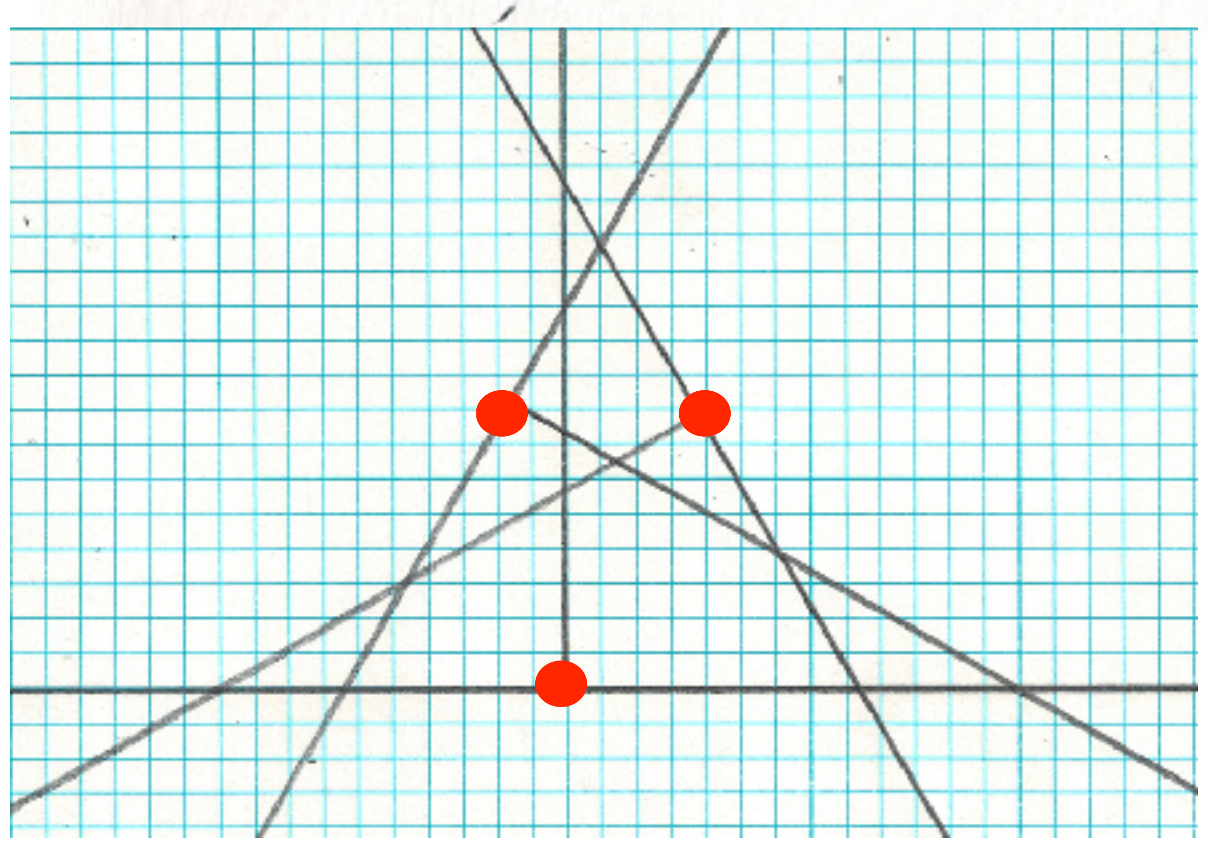


Figure-8

Conjecture:

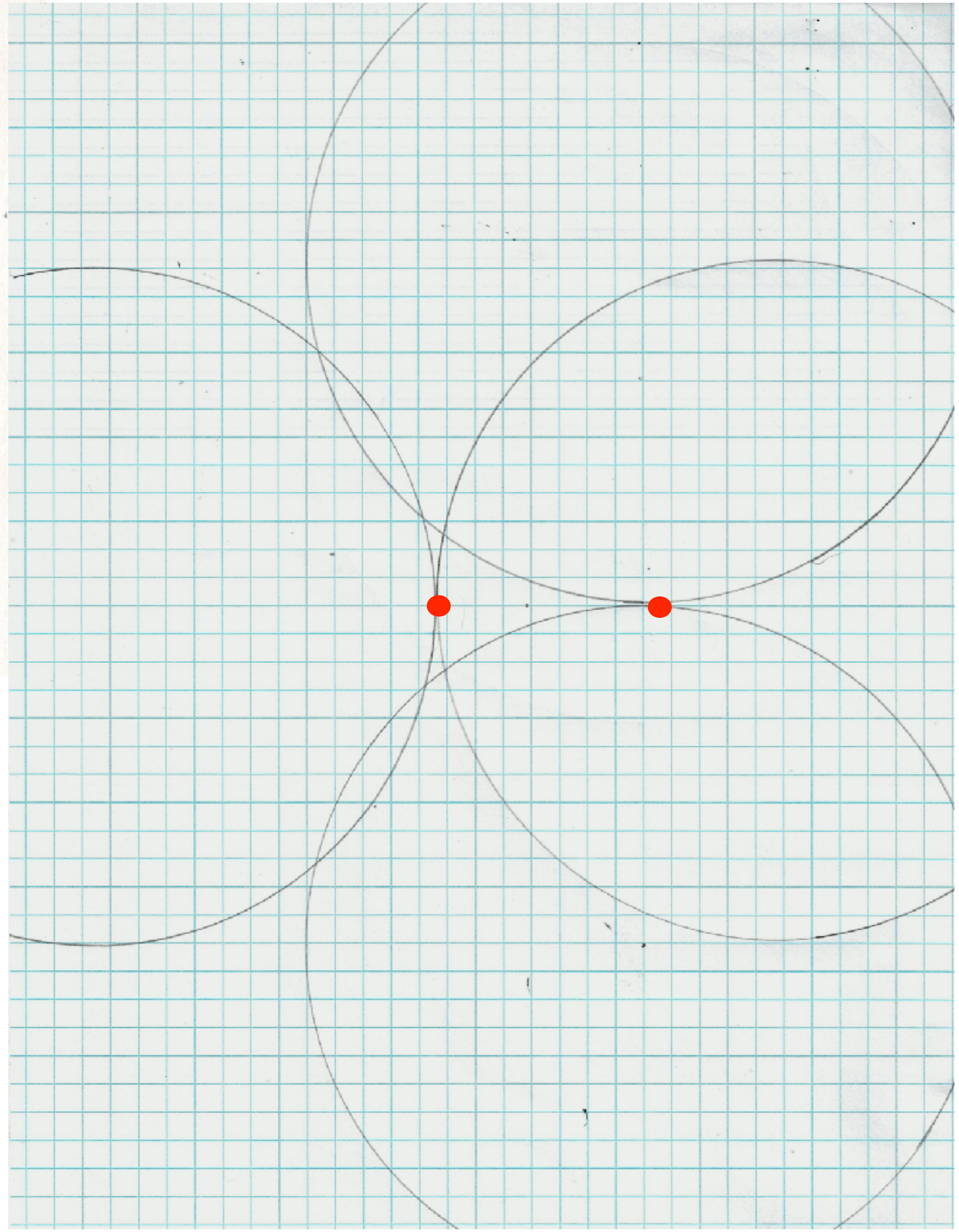
$4n^2-3n+2$??



Three T's, 19 regions,
essentially unique, and it isimpossible
to add a 4th T and get
33 regions. Please check!
Need 6 crossings per arm for the 4th T.

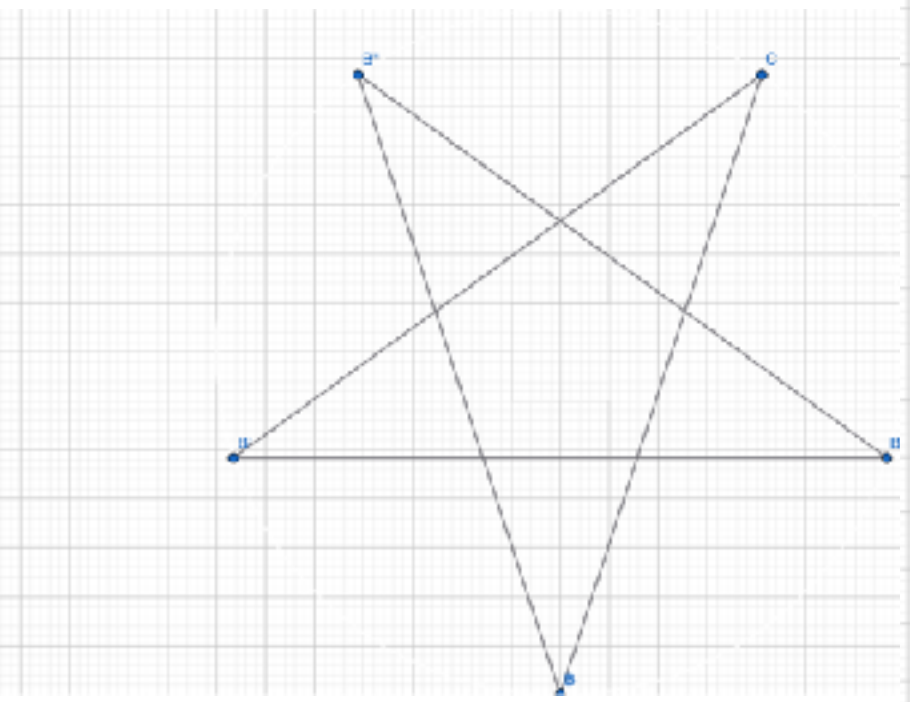
$$a_8(2) = 12$$

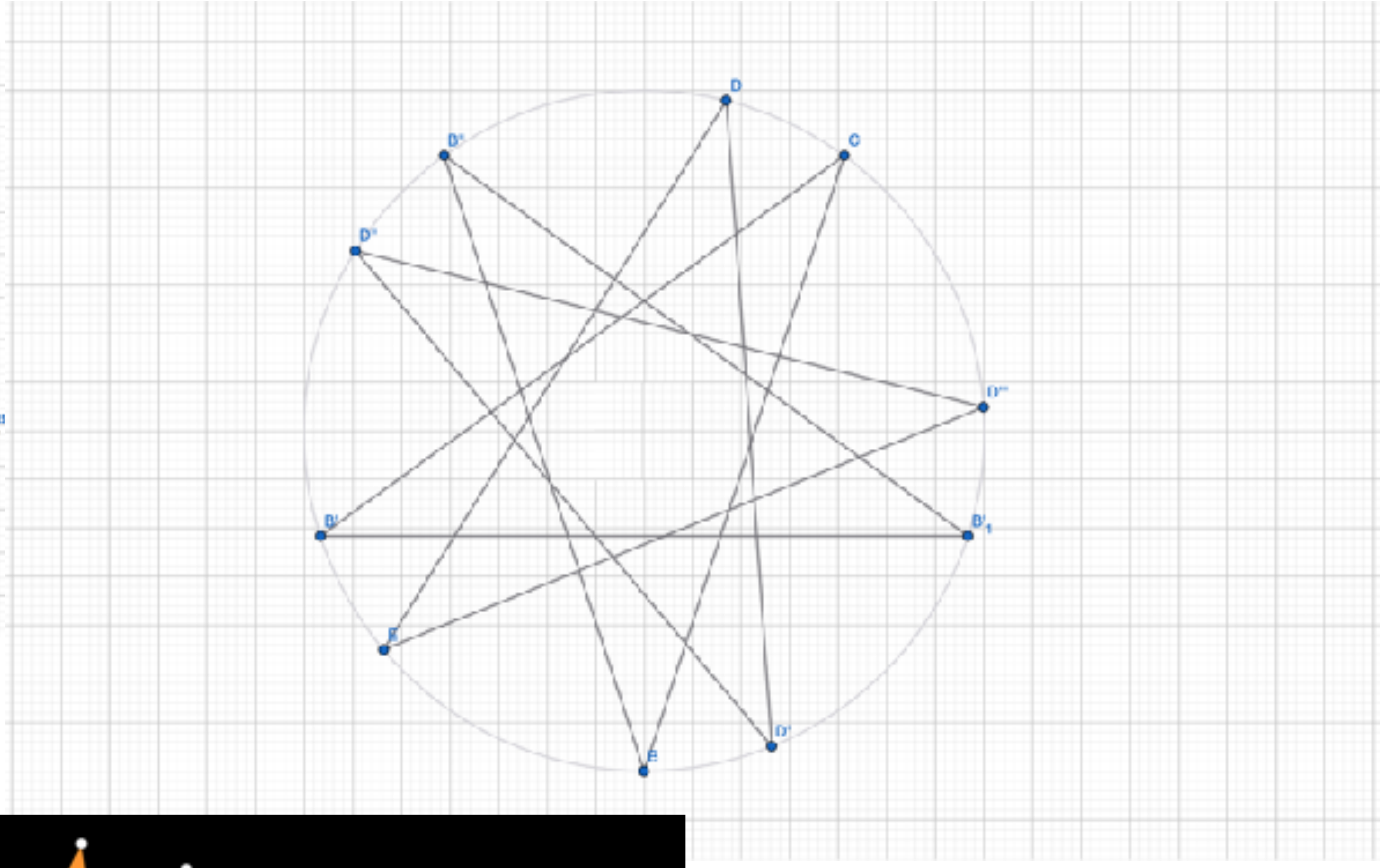
Conjecture: like 2n circles, but
must lose one intersection
per copy of 8

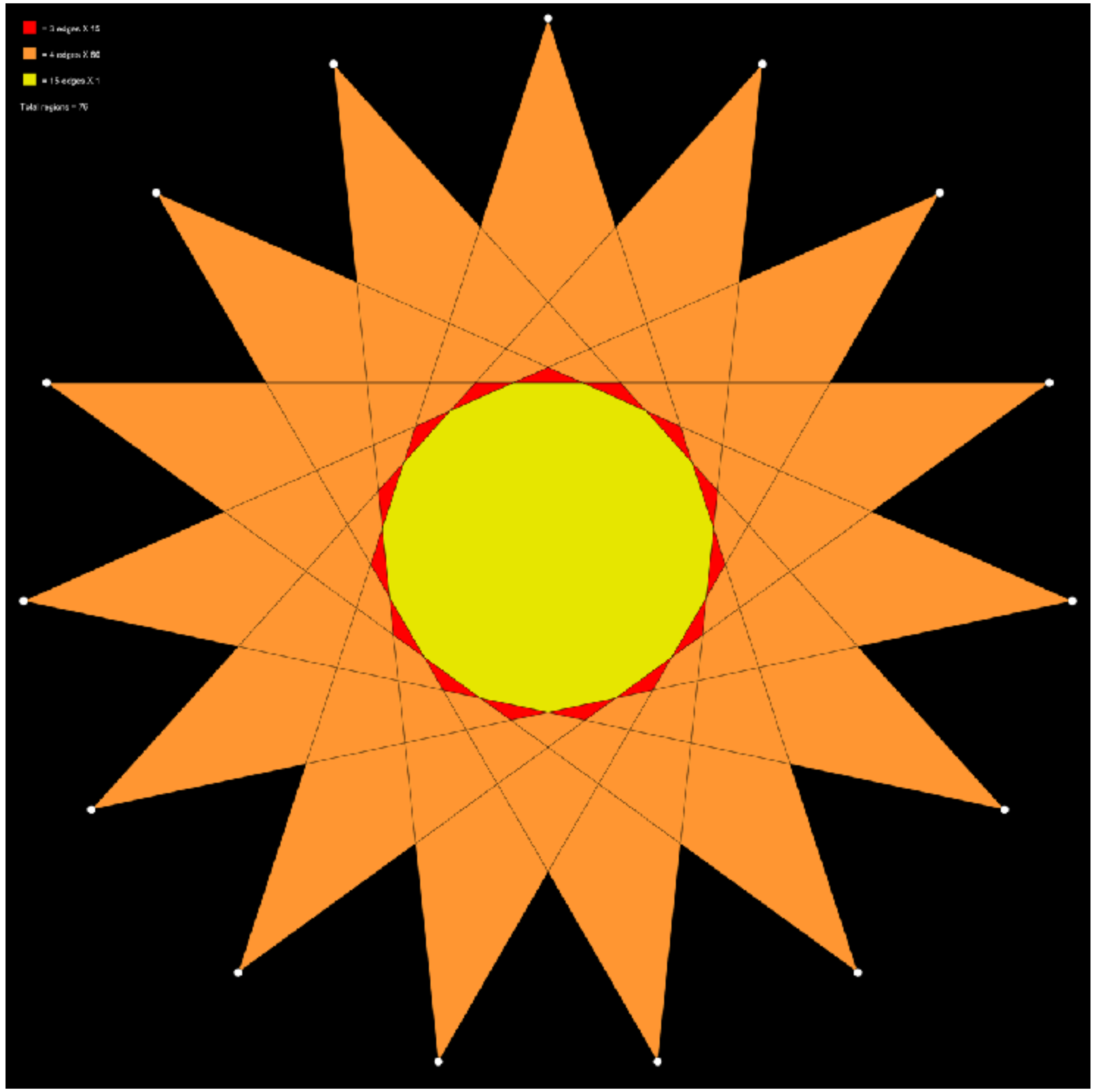


The Pentagram (with Scott Shannon):

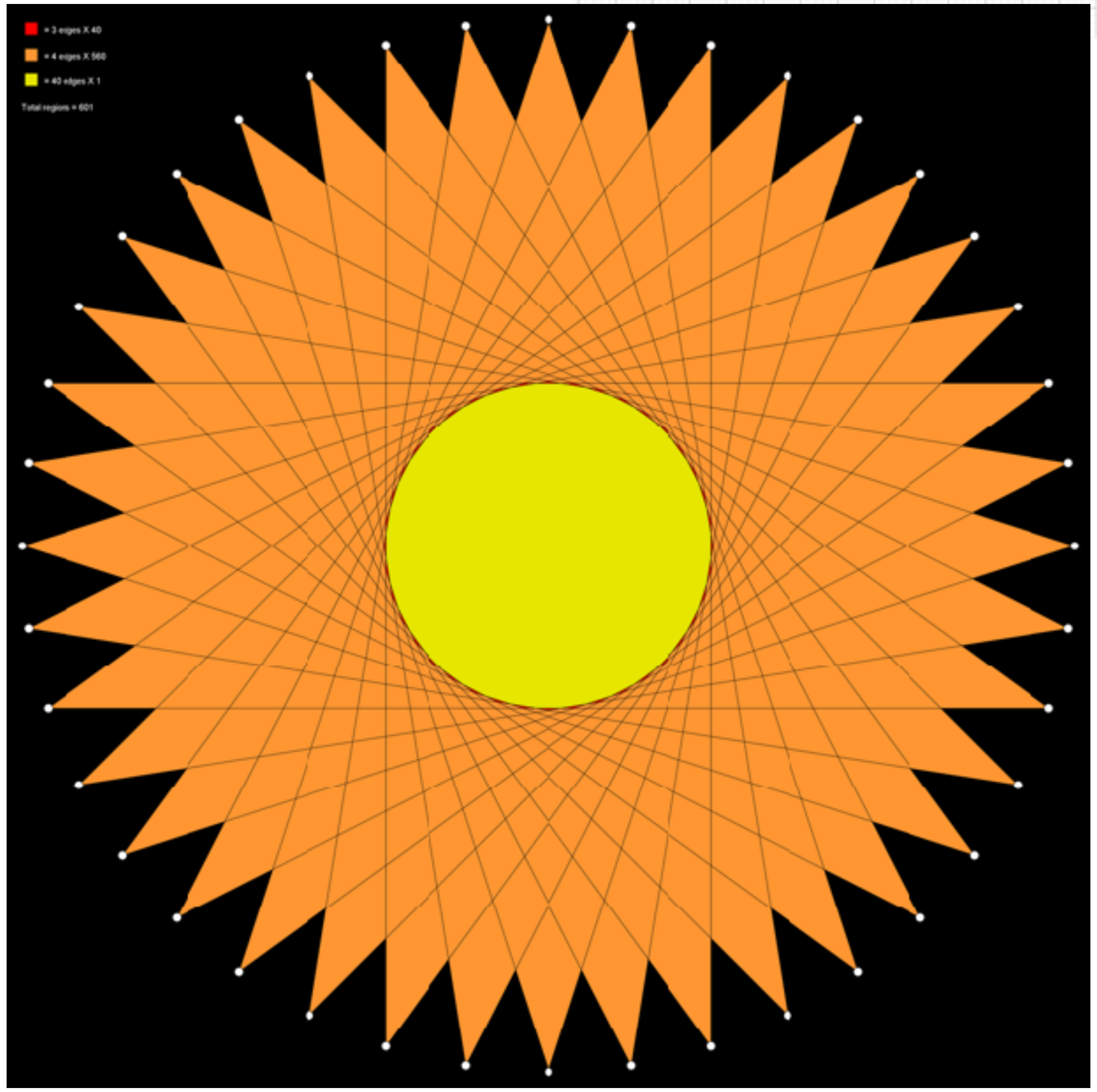
$a(n) = 10 n^2 - 5 n + 2, \quad n>0,$ A383466







a(3) = 77



a(8) = 682