Odds and Evens
Permutation statistics of the Alternating subgroup

Experimental Math Seminar
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- Some generalisations

Descents

1. Take a permutation $\pi = \pi_1\pi_2 \ldots \pi_n \in \mathfrak{S}_n$.

Example:

$$\pi = 3712546$$

Let $\mathfrak{S}_n$ denote the set of permutations of $\{1, 2, \ldots, n\}$

Descents

1. Take a permutation $\pi = \pi_1\pi_2\ldots\pi_n \in \mathfrak{S}_n$.
2. Identify the positions i where $\pi_i > \pi_{i+1}$.

Example:

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3. $\text{des}(\pi) = |\{i : \pi_i > \pi_{i+1}\}|$

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$$\text{des}(3712546) = 2$$

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Descents

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Note that des is a function from $\mathfrak{S}_n$ to $\mathbb{N}$ for all n. Descent is an example of a permutation statistic, a map from the set $\cup_{n\geqslant 1}\mathfrak{S}_n$ to the set $\mathbb{N}$.

Let $\mathfrak{S}_n$ denote the set of permutations of $\{1, 2, \dots, n\}$

Enumerating polynomials

The set $\mathfrak{S}_n$ has $n!$ elements and each of them have a descent value between 0 and $n-1$.

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Let $A(n,d)$ be the number of permutations in $\mathfrak{S}_n$ with d descents.
For each n, we have the n-tuple

$$(A(n,0), A(n,1), \ldots, A(n,n-1)).$$

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What better to encode it than as a polynomial

$$A_n(t) := \sum_{d=0}^{n-1} A(n,d)t^d = \sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{des}(\pi)}.$$

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These polynomials are the famous Eulerian polynomials that enjoy multiple exciting properties. It is palindromic and has only real roots.

Enumerating polynomials

Here is a list of the first 5 Eulerian polynomials.

- $A_1(t) = 1$
- $A_2(t) = 1 + t$
- $A_3(t) = 1 + 4t + t^2$
- $A_4(t) = 1 + 11t + 11t^2 + t^3$
- $A_5(t) = 1 + 26t + 66t^2 + 26t^3 + t^4$

Enumerating polynomials

In fact, we can do this with any statistic. Given a statistic stat : $\cup_{n\geqslant 1}\mathfrak{S}_n \mapsto \mathbb{N}$, we have the stat-enumerating polynomial

$$\mathrm{stat}_n(t) := \sum_{\pi\in\mathfrak{S}_n} t^{\mathrm{stat}(\pi)}.$$

Enumerating polynomials

In fact, we can do this with any statistic. Given a statistic $\mathrm{stat} : \cup_{n \geqslant 1} \mathfrak{S}_n \mapsto \mathbb{N}$, we have the stat-enumerating polynomial

$$\mathrm{stat}_n(t) := \sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{stat}(\pi)}.$$

We will also use the signed stat-enumerating polynomial

$$\mathrm{SgnStat}_n(t) := \sum_{\pi \in \mathfrak{S}_n} \mathrm{sign}(\pi) t^{\mathrm{stat}(\pi)}$$

Why stop at one variable?

Given a r-tuple of permutation statistics $(\mathrm{stat}_1, \mathrm{stat}_2, \ldots, \mathrm{stat}_r)$, we have the $(\mathrm{stat}_1, \mathrm{stat}_2, \ldots, \mathrm{stat}_r)$-enumerating polynomial

$$(\mathrm{stat}_1, \mathrm{stat}_2, \ldots, \mathrm{stat}_r)_n(t_1, \ldots, t_r) := \sum_{\pi \in \mathfrak{S}_n} t_1^{\mathrm{stat}_1(\pi)} \ldots t_r^{\mathrm{stat}_r(\pi)}.$$

Similarly, the signed $(\mathrm{stat}_1\, \mathrm{stat}_2 \ldots \mathrm{stat}_r)$-enumerating polynomial is

$$(\mathrm{SgnStat}_1 \ldots \mathrm{Stat}_r)_n(t_1, \ldots, t_r) := \sum_{\pi \in \mathfrak{S}_n} \mathrm{sign}(\pi) t_1^{\mathrm{stat}_1(\pi)} \cdots t_r^{\mathrm{stat}_r(\pi)}.$$

Why study the signed enumerating polynomials?

Because knowing $\text{stat}_n(t)$ and $\text{SgnStat}_n(t)$ allows us to do $\text{stat}_n(t) + \text{Sgnstat}_n(t) = \sum_{\pi \in \mathfrak{S}_n} \left(t^{\text{stat}(\pi)} + \text{sign}(\pi) t^{\text{stat}(\pi)} \right)$ $= 2 \sum_{\pi \in \mathcal{A}_n} t^{\text{stat}(\pi)}$. Therefore, we can get the distribution of the statistic over $\mathcal{A}_n$.

Here, $\mathcal{A}_n$ is the alternating subgroup in $\mathfrak{S}_n$.

Let us look at the signed descent enumerating polynomial.

Signed Eulerian Numbers

Loday, when studying the cyclic homology of commutative algebras, came across a sequence of numbers ($\mathrm{SgnDes}_{n,k}$) related to the classical Eulerian numbers.

$$\mathrm{SgnDes}_{n,k} := \sum_{\sigma\in\mathfrak{S}_n:\mathrm{des}(\sigma)=k} \mathrm{sign}(\sigma)$$

$$\mathrm{SgnDes}_n(t) := \sum_{k=0}^{n-1} \mathrm{SgnDes}_{n,k} t^k = \sum_{\sigma\in\mathfrak{S}_n} \mathrm{sign}(\sigma) t^{\mathrm{des}(\sigma)}.$$

Theorem (Foata, Desarmenien [4])

The $\mathrm{SgnDes}_n(t)$ *satisfy*

$$\mathrm{SgnDes}_{2n}(t) = (1-t)^n \,\mathrm{des}_n(t),$$

$$\mathrm{SgnDes}_{2n+1}(t) = (1-t)^n \,\mathrm{des}_{n+1}(t).$$

An involution?

Let's look at $\mathrm{SgnDes}_n(t)$ for $n = 3$.

- $\mathrm{sign}(123) = 1, \mathrm{des}(123) = 0$
- $\mathrm{sign}(132) = -1, \mathrm{des}(132) = 1$
- $\mathrm{sign}(213) = -1, \mathrm{des}(213) = 1$
- $\mathrm{sign}(231) = 1, \mathrm{des}(231) = 1$
- $\mathrm{sign}(312) = 1, \mathrm{des}(312) = 1$
- $\mathrm{sign}(321) = -1, \mathrm{des}(321) = 2$

So, $\mathrm{SgnDes}_3(t) = 1 - t - t + t + t - t^2 = (1-t)(1+t)$.

An involution?

Is there a sign reversing involution that explains all the cancellations and the terms that will be left at the end?

Wachs' involution

1. Take a permutation $\pi = \pi_1\pi_2 \ldots \pi_n \in \mathfrak{S}_n$.

Example:

$$\pi = 3712546$$

Wachs' involution

1. Take a permutation $\pi = \pi_1\pi_2 \dots \pi_n \in \mathfrak{S}_n$.
2. Find the smallest $2i - 1$ and $2i$ such that they are not adjacent in the permutation.

Example:

$$\pi = \underline{3}71254\underline{4}6$$

Wachs' involution

1. Take a permutation $\pi = \pi_1\pi_2\ldots\pi_n \in \mathfrak{S}_{2n}$.
2. Find the smallest $2i-1$ and $2i$ such that they are not adjacent in the permutation.
3. If i exists, $f(\pi) = (2i-1, 2i).\pi$. Else, $f(\pi) = \pi$.

Example:

$$\pi = \underline{3}7125\underline{4}6$$

$$f(3712546) = 4712536$$

Fixed points

A barred permutation on $\mathfrak{S}_n$ is a permutation written in one line notation along with some bars over some symbols of the permutation.

Example: $\bar{2}4\bar{1}53$ is a barred permutation. $2\bar{4}153$ also a barred permutation.

Fixed points

For $2n$, the fixed points are in bijection with barred permutations on $\mathfrak{S}_n$. Replace i by $2i-1, 2i$ and $\bar{i}$ by $2i, 2i-1$.

$$24\overline{15}3 \longleftrightarrow 3478\underline{21(10)9}56$$

For $2n+1$, the fixed points are in bijection with barred permutations on $\mathfrak{S}_{n+1}$ where the element $n+1$ is unbarred. Replace $i \neq n+1$ by $2i-1, 2i$, replace $\bar{i}$ by $2i, 2i-1$ and finally, $n+1$ by $2n+1$.

$$2\overline{41}53 \longleftrightarrow 34\underline{8721}956$$

Let us look at another permutation statistic.

Excedance statistic

1. Take a permutation $\pi = \pi_1\pi_2\ldots\pi_n \in \mathfrak{S}_n$.

Example:

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 7 & 1 & 2 & 5 & 4 & 6 \end{pmatrix}$$

Excedance statistic

1. Take a permutation $\pi = \pi_1\pi_2\ldots\pi_n \in \mathfrak{S}_n$.
2. Identify the positions i where $\pi_i > i$.

Example:

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \underline{3} & \underline{7} & 1 & 2 & 5 & 4 & 6 \end{pmatrix}$$

Excedance statistic

1. Take a permutation $\pi = \pi_1\pi_2 \ldots \pi_n \in \mathfrak{S}_n$.
2. Identify the positions i where $\pi_i > \pi_{i+1}$.
3. $\mathrm{exc}(\pi) = |\{i : \pi_i > i\}|$.

Example:

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \underline{3} & \underline{7} & 1 & 2 & 5 & 4 & 6 \end{pmatrix}$$

$$\mathrm{exc}(3712546) = 2$$

Descent and Excedance have the same distribution over $\mathfrak{S}_n$

It is one of the first results in permutation statistics proven nearly 100 years ago, by MacMahon, that

$$\sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{des}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{exc}(\sigma)}.$$

A bijective proof of this, i.e. one that a permutation with k descents to one with k excedances, is the first fundamental transformation.

Let’s look at the signed excedance enumerating polynomial.

Signed Excedance numbers

Theorem (Mantaci [8])

For $n \in \mathbb{N}$ $\mathrm{SgnExc}_n(t) = \sum_{\pi \in \mathfrak{S}_n} \mathsf{sign}(\pi) t^{\mathsf{exc}(\pi)} = (1-t)^{n-1}$.

We will look at a sign reversing involution that gives us this result. But before that, we will also look at a simpler proof of this fact.

Determinantal proof (due to Sivasubramanian [9])

Consider the $n \times n$ matrix $M_n = \begin{pmatrix} 1 & t & \dots & t & t \\ 1 & 1 & \dots & t & t \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & 1 & t \\ 1 & 1 & \dots & 1 & 1 \end{pmatrix}$.

Then, $\det(M_n) = \sum_{\sigma \in \mathfrak{S}_n} \text{sign}(\sigma) m_{1,\sigma(1)} \dots m_{n,\sigma(n)}$.

Now, $m_{i,\sigma(i)} = 1$ if i is not an excedance position and $m_{i,\sigma(i)} = t$ if it is. Therefore, $\det(M) = \text{SgnExc}_n(t) = \sum_{\sigma \in \mathfrak{S}_n} \text{sign}(\sigma) t^{\text{exc}(\sigma)}$. This determinant can be explicitly computed (say, using recursion) to be $(1-t)^{n-1}$. This completes the proof.

Canonical reduced words

Let s_i be the transposition $(i, i+1)$. We know that $\mathfrak{S}_n$ is generated by $s_1, \ldots, s_{n-1}$.

We will look at a canonical way to write any permutation as a product of these generators.

How to get it?

1. To get $\pi_1 \ldots \pi_n$, we start from the permutation $12 \ldots n$.
2. We will multiply identity with generators on the right to move the element π_n to the last position. We will use r_n to keep track of the generators we multiplied.
3. Multiply generators to get π_{n-1} to its correct position. Keep track of what we multiplied with r_{n-1}.
4. Repeat this process till we have arrived at our required permutation.

Let's try to get the canonical reduced word decomposition of '41523'.

$$
\begin{array}{c}
12345 \\
\big\downarrow r_5 = s_3 s_4 \\
12453 \\
\big\downarrow r_4 = s_2 s_3 \\
14523 \\
\big\downarrow r_3 = 1 \\
14523 \\
\big\downarrow r_2 = s_1 \\
41523
\end{array}
$$

Therefore, we get a canonical reduced word decomposition for $\pi = 41523$ namely $r_5 r_4 r_3 r_2 r_1 = [s_3 s_4][s_2 s_3][1][s_1][1]$.

Observations

1. $r_n \dots r_i$, by definition, will get π_i to its correct spot.
2. r_i will look like $s_j s_{j+1} \dots s_{i-1}$. It has to end with s_{i-1}. Therefore, it can be at most of length $i-1$.
3. If r_i has length 1, the previous line implies that $r_i = s_{i-1}$.

An involution on canonical reduced words

For a permutation, find the rightmost r_{i+1}, say r_{i_0+1}, which has length $\geqslant 2$. The one to its right, r_{i_0}, is forced to have length 1 or 0. If it was length 1, it had to be s_{i_0-1}. If it was length 0, it was just 1.

If it was s_{i_0-1}, flip it to 1. If it was 1, flip it to s_{i_0-1}.

If such there is no component of length 2 or more, leave it fixed. This is our involution.

Example: $\pi = 41523 = r_5r_4r_3r_2r_1 = [s_3s_4][s_2s_3][1][s_1][1]$. We will flip r_3. Our involution yields $f_A(41523) = [s_3s_4][s_2s_3][s_2][s_1][1] = 51423$.

We will look at some of the interesting properties that the Eulerian distribution over $\mathfrak{S}_n$ satisfies and compare them with the $\mathcal{A}_n$ counterparts.

γ-positivity for the Eulerian distribution

A sequence is said to be γ-positive if the sequence can be written as a $\mathbb{Z}$-linear combination of rows of the Pascal triangle with the same center of symmetry.

Example:

1	4	6	4	1	
0	1	2	1	0	$\times 22$
0	0	1	0	0	$\times 16$
1	26	66	26	1	

Every palindromic sequence of positive integers (one that reads the same from front and back) can be written in this way.

γ-positivity for palindromic polynomials

A polynomial is said to be γ-positive if the polynomial can be written as a $\mathbb{Z}$-linear combinations of polynomials of the form $t^j(1+t)^{n-2j}$. (t^j shifts the center of symmetry)

Theorem (Foata, Schutzenberger [5])

There exists $\gamma_{n,j} \geqslant 0$ such that the Eulerian polynomial $A_n(t)$ can be written as

$$A_n(t) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,j} t^j (1+t)^{n-1-2j}.$$

γ-positivity for descents over $\mathcal{A}_n$

The polynomials

$$A_n^+(t) = \sum_{\pi \in \mathcal{A}_n} t^{\mathrm{des}(\pi)},$$

$$A_n^-(t) = \sum_{\pi \in \mathfrak{S}_n - \mathcal{A}_n} t^{\mathrm{des}(\pi)}$$

are not palindromic for all n. This is slightly disappointing. However, whenever they are palindromic, they are γ-positive.

Theorem (Dey,Sivasubramanian [2])

The polynomials $A_n^+(t), A_n^-(t)$ are γ-positive iff $n \equiv 0, 1 \pmod 4$ with the same center of symmetry.

γ-positivity for excedances over $\mathcal{A}_n$

The polynomials

$$AE_n^+(t) = \sum_{\pi \in \mathcal{A}_n} t^{\mathrm{exc}(\pi)},$$

$$AE_n^-(t) = \sum_{\pi \in \mathfrak{S}_n - \mathcal{A}_n} t^{\mathrm{exc}(\pi)}$$

are not palindromic for all n. They are palindromic for all odd positive integers n.

Theorem (Dey,Sivasubramanian [3])

The polynomials $AE_n^+(t), AE_n^-(t)$ are γ-positive iff $n \equiv 1 \pmod 2$ with the same center of symmetry $(n-1)/2$.

Log-concavity of a sequence

A sequence $(a_k)_{k=1}^{n}$ of reals is said to be log-concave if for each $2 \leqslant i \leqslant n-1$, we have $a_i^2 \geqslant a_{i+1}a_{i-1}$.

A nice approach to proving log-concavity is the following theorem due to Newton.

Theorem

Let

$$P(x) = \sum_{k=0}^{n} \binom{n}{k} a_k x^k$$

be a (real) polynomial with only real roots. Then, the sequence $(a_k)_{k=0}^{n}$ *is log-concave.*

Ultra-log-concavity

Suppose the polynomial

$$P'(x) = \sum_{k=0}^{n} a_k x^k$$

is real-rooted. Then the sequence $\frac{a_k}{\binom{n}{k}}$ is log-concave (this follows from Newton's theorem). A sequence $(a_k)_{k=0}^n$ is called ultra-log-concave if the sequence $\left(\frac{a_k}{\binom{n}{k}}\right)_{k=0}^n$ is log-concave. So, sequences that appear as coefficients of real rooted polynomials are ultra-log-concave.

Real rootedness

The Eulerian polynomials are always real-rooted.

Theorem (Frobenius [6])

The Eulerian polynomials $A_n(t)$ have only real and simple roots for all natural $n \geqslant 1$.

However, we do not know counterparts of this result for the descent and excedance distributions over $\mathcal{A}_n$.

Conjecture (Fulman, Lee, Petersen, Kim [7])

The polynomials $A_n^+(t)$ and $A_n^-(t)$ are real-rooted for all $n \in \mathbb{N}$.

Conjecture (Shankar)

The polynomials $AE_n^+(t)$ and $AE_n^-(t)$ are real-rooted for all $n > 5$.

Back to log-concavity: Synchronicity

Let $A = (a_k)_{k=1}^n$ and $B = (b_k)_{k=1}^n$ be two sequences. Let $S(A, B)$ be the set of 2^n sequences $(c_k)_{k=1}^n$ such that $c_k \in \{a_k, b_k\}$.

Two sequences A, B are said to be strongly synchronised if all sequences in $S(A, B)$ are log-concave. This implies that the sequences A, B themselves are log-concave.

Synchronicity over $\mathcal{A}_n$

$B_{n,k}$-number of permutations in $\mathcal{A}_n$ with k descents
$C_{n,k}$-number of permutations in $\mathfrak{S}_n - \mathcal{A}_n$ with k descents
$P_{n,k}$-number of permutations in $\mathcal{A}_n$ with k excedances
$Q_{n,k}$-number of permutations in $\mathfrak{S}_n - \mathcal{A}_n$ with k excedances

Theorem (Shankar)

All 4^n sequences in $S((B_{n,k})_{k=0}^{n-1}, (C_{n,k})_{k=0}^{n-1}, (P_{n,k})_{k=0}^{n-1}, (Q_{n,k})_{k=0}^{n-1})$ are ultra log-concave.

Pattern Containment

Given a permutation $\tau = \tau_1 \ldots \tau_k \in \mathfrak{S}_k$, a permutation $\pi = \pi_1 \ldots \pi_n \in \mathfrak{S}_n$ is said to be contain the pattern τ if there is some sequence $i_1 < i_2 < \cdots < i_k$ such that $\pi_{i_1} \ldots \pi_{i_k}$ is order isomorphic to τ.

Example: 24153 contains the subsequence 413 which is order isomorphic to 312. Therefore, it contains the pattern 312. However, the permutation avoids the pattern 321.

Pattern Avoidance

If π does not contain τ, then we say π avoids τ. We will denote the set of permutations in $\mathfrak{S}_n$ that avoid τ by $\mathfrak{S}_n(\tau)$.

For example, the number of permutations in $\mathfrak{S}_{20}$ that avoid the pattern 213 is the password of our Zoom meeting.

Wilf Equivalence

If for two patterns $\sigma, \tau \in \mathfrak{S}_k$, the cardinalities $|\mathfrak{S}_n(\sigma)| = |\mathfrak{S}_n(\tau)|$ are the same for every natural n, then the two patterns σ, τ are said to be Wilf-equivalent. We denote it by $\sigma \equiv_{\mathfrak{S}_n} \tau$.

For example, all three length patterns are Wilf-equivalent.

$$123 \equiv_{\mathfrak{S}_n} 132 \equiv_{\mathfrak{S}_n} 213 \equiv_{\mathfrak{S}_n} 231 \equiv_{\mathfrak{S}_n} 312 \equiv_{\mathfrak{S}_n} 321$$

.

even-Wilf-equivalence

We can imitate the definition of Wilf-equivalence for even permutations.

We call two patterns σ, τ to be even-Wilf-equivalent if

$$\mathfrak{S}_n(\sigma) \cap \mathcal{A}_n = \mathfrak{S}_n(\tau) \cap \mathcal{A}_n.$$

We will denote $\mathfrak{S}_n(\sigma) \cap \mathcal{A}_n$ by $\mathcal{A}_n(\sigma)$. We will denote even-Wilf-equivalence of two patterns σ, τ by $\sigma \equiv_{\mathcal{A}_n} \tau$.

For patterns of length 3 and 4

For length 3 patterns, we have the following theorem.

Theorem (Simion, Schmidt)

There are two distinct even-Wilf-equivalence classes for patterns of length 3.

1. $123 \equiv_{\mathcal{A}_n} 312 \equiv_{\mathcal{A}_n} 213$
2. $132 \equiv_{\mathcal{A}_n} 321 \equiv_{\mathcal{A}_n} 231$

For length 4 *patterns, we have the following theorem.*

Theorem (Jaggard, Baxter [1])

There are 11 *even-Wilf-equivalence classes for patterns of length* 4.

For patterns of length 5 and above

As far as we know, there can be anywhere between 35 and 39 even-Wilf-equivalence classes.

Conjecture (Jaggard, Baxter [1])

The following equivalences hold:

1. $12345 \equiv_{\mathcal{A}_n} 45312$
2. $54321 \equiv_{\mathcal{A}_n} 21354$
3. $12354 \equiv_{\mathcal{A}_n} 45321$
4. $13524 \equiv_{\mathcal{A}_n} 42531$

A more surprising conjecture

Conjecture (Jaggard, Baxter [1])

If $\sigma \equiv_{\mathcal{A}_n} \tau$, *then* $\sigma \equiv_{\mathfrak{S}_n} \tau$.

The conjecture asserts that if the number of even permutations avoiding σ and τ are the same, then the number of odd permutations avoiding σ and τ are the same.

Can we generalise these results somehow?

One way to generalise

The permutations in $\mathcal{A}_n$ are the permutations in $\mathfrak{S}_n$ whose number of inversions are $0 \pmod 2$.

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What about the permutations whose number of inversions are $0 \pmod k$?

One way to generalise

The permutations in $\mathcal{A}_n$ are the permutations in $\mathfrak{S}_n$ whose number of inversions are $0 \pmod 2$.

What about the permutations whose number of inversions are $0 \pmod k$?

Are the descent enumerating polynomials over these classes palindromic? γ-positive? Real rooted?

A conjecture?

I would like to end the talk with this conjecture.

Conjecture

Let $\mathfrak{S}_{n,3} := \{\sigma \in \mathfrak{S}_n : \mathrm{inv}(\sigma) \equiv 0 \pmod 3\}$. *For* $n \equiv 0, 1 \pmod 3$ *and* $n \geqslant 4$, *then the polynomial*

$$\sum_{\sigma \in \mathfrak{S}_{n,3}} t^{\mathrm{des}(\pi)}$$

is γ*-positive.*

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