

RUTGERS EXPERIMENTAL MATHEMATICS SEMINAR

Zoom presentation, October 23, 2025

Computer algebra tools for Feynman Integrals

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Definition: A function $f : \mathbb{K} \rightarrow \mathbb{K}$ is called **holonomic** if

there exist $b_0(x), \dots, b_\lambda(x) \in \mathbb{K}[x]$ (not all zero) with

$$b_0(x)f(x) + \dots + b_\lambda(x)D^\lambda f(x) = 0 \quad (\text{DE})$$

Definition: A power series $f(x) \in \mathbb{K}[[x]]$ is called **holonomic** if

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Definition: A formal power series $f(x) \in \mathbb{K}[[x]]$ is called **holonomic** if

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Definition: sequence $(F(n))_{n \geq 0} \in \mathbb{K}^{\mathbb{N}}$ is called **holonomic** if

there exist $a_0(x), \dots, a_\delta(x) \in \mathbb{K}[x]$ (not all zero) with

$$a_0(n)F(n) + \dots + a_\delta(n)F(n + \delta) = 0 \quad (\text{RE})$$

symbolic
summation

symbolic
integration

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Feynman
integrals

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Example: A master integral from Ladder and V -topologies

[arXiv:1509.08324]

$$F(\varepsilon, n) = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{\varepsilon/2} y^{\varepsilon/2} (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+n+1} \underbrace{(1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^n}_{f(\varepsilon, n, x, y, z)}$$

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The integrand is

► hyperexponential in x, y, z :

$$\frac{D_x f(\varepsilon, n, x, y, z)}{f(\varepsilon, n, x, y, z)} \in \mathbb{Q}(\varepsilon, n, x, y, z)$$

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The integrand is

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- ▶ hyperexponential in x, y, z :
- ▶ hypergeometric in n :

$$\frac{f(\varepsilon, n+1, x, y, z)}{f(\varepsilon, n, x, y, z)} \in \mathbb{Q}(\varepsilon, n, x, y, z)$$

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The integrand is

- ▶ hyperexponential in x, y, z :
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$$\frac{f(\varepsilon, n+1, x, y, z)}{f(\varepsilon, n, x, y, z)} \in \mathbb{Q}(\varepsilon, n, x, y, z)$$

Holonomic theory shows that there exists a **holonomic recurrence!**

[Multi-Variable Zeilberger and Almkvist-Zeilberger Algorithms and the Sharpening of Wilf-Zeilberger Theory; Moa Apagodu (formerly Mohamud Mohammed) and Doron Zeilberger]

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Ablinger's
MultiIntegrate.m

↓ (9 hours)

$$a_0(\varepsilon, n)F(\varepsilon, n) + a_1(\varepsilon, n)F(\varepsilon, n+1) + \cdots + a_5(\varepsilon, n)F(\varepsilon, n+5) = 0$$

$$\begin{aligned}
a_0(n, \varepsilon) = & (n+1)(n+2)(8\varepsilon^{10} + 104\varepsilon^9(n+3) + 4\varepsilon^8(96n^2 + 601n + 887) \\
& + 4\varepsilon^7(12n^3 + 414n^2 + 1583n + 1393) \\
& - 8\varepsilon^6(264n^4 + 2436n^3 + 8643n^2 + 14518n + 9947) \\
& - 16\varepsilon^5(156n^5 + 1690n^4 + 6847n^3 + 12661n^2 + 9537n + 717) \\
& + 32\varepsilon^4(68n^6 + 1158n^5 + 8155n^4 + 30114n^3 + 61712n^2 + 67616n + 31693) \\
& + 64\varepsilon^3(40n^7 + 560n^6 + 2755n^5 + 3729n^4 - 14194n^3 - 61920n^2 - 89140n - 46600) \\
& - 128\varepsilon^2(n+2)(12n^7 + 254n^6 + 2249n^5 + 10758n^4 + 30173n^3 + 50610n^2 \\
& + 49122n + 22706) \\
& + 256\varepsilon(n+2)^2(n+3)(n+4)(44n^4 + 501n^3 + 2044n^2 + 3455n + 1976) \\
& - 512(n+1)(n+2)^3(n+3)^2(n+4)(6n^2 + 47n + 95)),
\end{aligned}$$

$$\begin{aligned}
a_1(n, \varepsilon) = & (n+2)(-22\varepsilon^{11} - 2\varepsilon^{10}(157n+435) - \varepsilon^9(1500n^2+8611n+11745) \\
& - \varepsilon^8(2548n^3+22936n^2+63597n+54229) \\
& + 4\varepsilon^7(266n^4+1857n^3+6065n^2+14351n+15987) \\
& + 8\varepsilon^6(994n^5+12961n^4+67246n^3+174692n^2+226821n+116092) \\
& + 16\varepsilon^5(336n^6+5348n^5+33569n^4+104918n^3+165290n^2+108259n+6100) \\
& - 16\varepsilon^4(404n^7+7578n^6+61778n^5+284762n^4+802660n^3+1382074n^2 \\
& + 1340455n+560287) \\
& - 64\varepsilon^3(94n^8+1823n^7+14305n^6+55870n^5+96299n^4-37256n^3 \\
& - 447044n^2-704959n-379338) \\
& + 128\varepsilon^2(n+3)(30n^8+715n^7+7667n^6+48253n^5+194086n^4+507439n^3 \\
& + 835393n^2+785327n+320382) \\
& - 256\varepsilon(n+2)(n+3)^2(107n^6+2070n^5+16342n^4+67226n^3+151557n^2 \\
& + 176932n+83196) \\
& + 256(n+2)^3(n+3)^3(n+4)(30n^3+331n^2+1193n+1386)),
\end{aligned}$$

$$\begin{aligned}
a_2(n, \varepsilon) = & (12\varepsilon^{12} + 12\varepsilon^{11}(17n + 45) + 2\varepsilon^{10}(620n^2 + 3553n + 4795) \\
& + 2\varepsilon^9(1504n^3 + 14190n^2 + 41901n + 38907) \\
& + 4\varepsilon^8(172n^4 + 4983n^3 + 30942n^2 + 69119n + 50850) \\
& - 4\varepsilon^7(1996n^5 + 24056n^4 + 113313n^3 + 269119n^2 + 337198n + 185290) \\
& - 16\varepsilon^6(450n^6 + 8210n^5 + 59749n^4 + 227386n^3 + 486841n^2 + 563176n + 275664) \\
& + 16\varepsilon^5(340n^7 + 4314n^6 + 19137n^5 + 25532n^4 - 55105n^3 - 206516n^2 - 191528n \\
& - 23458) \\
& + 32\varepsilon^4(140n^8 + 2940n^7 + 26550n^6 + 139926n^5 + 493839n^4 + 1240186n^3 \\
& + 2161699n^2 + 2304248n + 1100084) \\
& + 64\varepsilon^3(4n^9 + 506n^8 + 8651n^7 + 63510n^6 + 236215n^5 + 395334n^4 - 105413n^3 \\
& - 1551017n^2 - 2362944n - 1217770) \\
& - 128\varepsilon^2(n + 3)(12n^9 + 314n^8 + 3782n^7 + 29105n^6 + 160727n^5 + 640273n^4 \\
& + 1750874n^3 + 3052505n^2 + 3017094n + 1276604) \\
& + 256\varepsilon(n + 2)(n + 3)^2(n + 4)(26n^6 + 825n^5 + 8967n^4 + 46529n^3 + 125411n^2 \\
& + 168628n + 88652) \\
& - 512(n + 1)(n + 2)^2(n + 3)^3(n + 4)^2(6n^3 + 98n^2 + 459n + 655)),
\end{aligned}$$

$$\begin{aligned}
a_3(n, \varepsilon) = & (-64\varepsilon^{12} - 8\varepsilon^{11}(113n + 298) - 8\varepsilon^{10}(519n^2 + 2948n + 3896) \\
& - 4\varepsilon^9(1444n^3 + 13839n^2 + 39746n + 34305) \\
& + 4\varepsilon^8(1948n^4 + 17868n^3 + 63837n^2 + 112966n + 84655) \\
& + 16\varepsilon^7(1456n^5 + 20460n^4 + 112365n^3 + 304963n^2 + 412258n + 221769) \\
& - 8\varepsilon^6(320n^6 + 2050n^5 + 4192n^4 + 27408n^3 + 174901n^2 + 411759n + 324872) \\
& - 16\varepsilon^5(1756n^7 + 33154n^6 + 265889n^5 + 1186719n^4 + 3218059n^3 + 5349388n^2 \\
& + 5071913n + 2113696) \\
& + 32\varepsilon^4(188n^8 + 4802n^7 + 59527n^6 + 439922n^5 + 2025336n^4 + 5813984n^3 \\
& + 10076450n^2 + 9621283n + 3878602) \\
& + 64\varepsilon^3(140n^9 + 2768n^8 + 22500n^7 + 99545n^6 + 287700n^5 + 723136n^4 \\
& + 1854572n^3 + 3714620n^2 + 4272517n + 2031600) \\
& - 128\varepsilon^2(24n^{10} + 830n^9 + 14362n^8 + 152630n^7 + 1053620n^6 + 4834279n^5 \\
& + 14824351n^4 + 29964399n^3 + 38244797n^2 + 27875896n + 8824032) \\
& + 256\varepsilon(n+2)(n+3)(n+4)(118n^7 + 2639n^6 + 24247n^5 + 118311n^4 + 329565n^3 \\
& + 520306n^2 + 426076n + 136854) \\
& - 512(n+1)(n+2)^2(n+3)^2(n+4)^2(n+5)(12n^3 + 97n^2 + 230n + 144)),
\end{aligned}$$

$$\begin{aligned}
a_4(n, \varepsilon) = & (64\varepsilon^{12} + 192\varepsilon^{11}(5n + 14) + 16\varepsilon^{10}(297n^2 + 1769n + 2451) \\
& + 16\varepsilon^9(453n^3 + 4462n^2 + 13094n + 11244) \\
& - 8\varepsilon^8(1084n^4 + 11117n^3 + 47258n^2 + 103981n + 94650) \\
& - 8\varepsilon^7(3304n^5 + 51138n^4 + 311957n^3 + 948722n^2 + 1440105n + 858544) \\
& + 16\varepsilon^6(420n^6 + 5507n^5 + 36275n^4 + 169650n^3 + 536911n^2 + 952507n + 694370) \\
& + 16\varepsilon^5(1828n^7 + 38868n^6 + 353301n^5 + 1801014n^4 + 5604391n^3 + 10664390n^2 \\
& + 11433064n + 5260048) \\
& - 32\varepsilon^4(316n^8 + 8356n^7 + 105800n^6 + 802421n^5 + 3836854n^4 + 11588223n^3 \\
& + 21401558n^2 + 22066744n + 9745752) \\
& - 64\varepsilon^3(116n^9 + 2424n^8 + 19923n^7 + 82966n^6 + 208191n^5 + 530980n^4 + 1847484n^3 \\
& + 4687014n^2 + 6120858n + 3111104) \\
& + 128\varepsilon^2(24n^{10} + 826n^9 + 14897n^8 + 172000n^7 + 1314686n^6 + 6710299n^5 \\
& + 22873183n^4 + 51298261n^3 + 72551278n^2 + 58573022n + 20544948) \\
& - 256\varepsilon(n + 2)(n + 3)(106n^8 + 3278n^7 + 42903n^6 + 310942n^5 + 1366350n^4 \\
& + 3729418n^3 + 6173159n^2 + 5657732n + 2191212) \\
& + 512(n + 1)(n + 2)^2(n + 3)^2(n + 4)(n + 5)(n + 6)(12n^3 + 121n^2 + 396n + 431)),
\end{aligned}$$

$$\begin{aligned}
a_5(n, \varepsilon) = & (n+5)(-128\varepsilon^{11} - 128\varepsilon^{10}(11n+26) - 32\varepsilon^9(115n^2+592n+647) \\
& + 32\varepsilon^8(63n^3+430n^2+1665n+2384) \\
& + 16\varepsilon^7(714n^4+7881n^3+33802n^2+66225n+47654) \\
& - 16\varepsilon^6(234n^5+2444n^4+13989n^3+50862n^2+104083n+87848) \\
& - 16\varepsilon^5(580n^6+10181n^5+76586n^4+319207n^3+772120n^2+1012046n+547832) \\
& + 16\varepsilon^4(244n^7+5456n^6+61605n^5+401216n^4+1536277n^3+3408574n^2 \\
& + 4066436n+2026928) \\
& + 64\varepsilon^3(26n^8+357n^7+583n^6-11139n^5-65193n^4-120264n^3+11864n^2 \\
& + 272830n+222624) \\
& - 64\varepsilon^2(n+3)(12n^8+298n^7+4684n^6+49024n^5+306907n^4+1122441n^3 \\
& + 2350650n^2+2607576n+1185072) \\
& + 256\varepsilon(n+2)(n+3)(25n^7+743n^6+8856n^5+55358n^4+197497n^3+404131n^2 \\
& + 439902n+196128) \\
& - 256(n+1)(n+2)^2(n+3)^2(n+4)(n+6)(n+7)(6n^2+35n+54)).
\end{aligned}$$

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Ablinger's
MultiIntegrate.m

↓ (9 hours)

$$a_0(\varepsilon, n)F(\varepsilon, n) + a_1(\varepsilon, n)F(\varepsilon, n+1) + \cdots + a_5(\varepsilon, n)F(\varepsilon, n+5) = 0$$

symbolic
summation

symbolic
integration
RE finding

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Feynman
integrals

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symbolic
summation

recurrence
solving

symbolic
integration
RE finding

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Feynman
integrals

A recurrence solver (Sigma.m)

GIVEN a recurrence

$a_0(n), \dots, a_\delta(n)$: polynomials in n

$h(n)$: expression in indefinite nested sums
defined over hypergeometric products.

$$a_0(n)F(n) + \dots + a_\delta(n)F(n + \delta) = h(n);$$

together with initial values $F(0), \dots, F(\delta - 1) \in \mathbb{K}$

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DECIDE constructively if $F(n)$ can be expressed in terms **indefinite nested sums** defined over **hypergeometric products**.

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Special cases of indefinite nested sums over hypergeometric products:

$$S_{2,1}(n) = \sum_{i=1}^n \frac{1}{i^2} \sum_{j=1}^i \frac{1}{j} \quad (\text{harmonic sums})$$

J. Blümlein and S. Kurth, Phys. Rev. D **60** (1999) 014018 [arXiv:hep-ph/9810241];

J.A.M. Vermaseren, Int. J. Mod. Phys. A **14** (1999) 2037 [arXiv:hep-ph/9806280].

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$$\sum_{k=1}^n \frac{2^k}{k} \sum_{i=1}^k \frac{2^{-i}}{i} \sum_{j=1}^i \frac{S_1(j)}{j} \quad (\text{generalized harmonic sums})$$

S. Moch, P. Uwer and S. Weinzierl, J. Math. Phys. **43** (2002) 3363 [hep-ph/0110083];

J. Ablinger, J. Blümlein and CS, J. Math. Phys. **54** (2013) 082301 [arXiv:1302.0378].

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Special cases of indefinite nested sums over hypergeometric products:

$$\sum_{k=1}^n \frac{1}{(1+2k)^2} \sum_{j=1}^k \frac{1}{j^2} \sum_{i=1}^j \frac{1}{1+2i} \quad (\text{cyclotomic harmonic sums})$$

A recurrence solver (Sigma.m)

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$$\sum_{j=1}^n \frac{4^j S_1(j-1)}{\binom{2j}{j} j^2} \quad (\text{binomial sums})$$

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$$\sum_{h=1}^n 2^{-2h} (1 - \eta)^h \binom{2h}{h} \sum_{k=1}^h \frac{2^{2k}}{k^2 \binom{2k}{k}} \quad (\text{generalized binomial sums})$$

J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, CS, K. Schönwald. Nucl.Phys.B 932. 2018. [arXiv:1804.02226].

J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, M. Saragnese, CS, K. Schönwald. Nucl.Phys.B 955. 2020. [arXiv:2004.08916]

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defined over hypergeometric products.

$$a_0(n)F(n) + \dots + a_\delta(n)F(n + \delta) = h(n);$$

together with initial values $F(0), \dots, F(\delta - 1) \in \mathbb{K}$

DECIDE constructively if $F(n)$ can be expressed in terms **indefinite nested sums** defined over **hypergeometric products**.

A more general example:

$$\sum_{k=1}^n \left(\prod_{i=1}^k \frac{1+i+i^2}{i+1} \right) \sum_{j=1}^k \frac{1}{j \binom{4j}{3j}^2}$$

Example: A master integral from Ladder and V -topologies

[arXiv:1509.08324]

$$F(\varepsilon, n) = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{\varepsilon/2} y^{\varepsilon/2} (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+n+1} \underbrace{(1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^n}_{f(\varepsilon, n, x, y, z)}$$

Ablinger's
MultiIntegrate.m

↓ (9 hours)

$$a_0(\varepsilon, n)F(\varepsilon, n) + a_1(\varepsilon, n)F(\varepsilon, n+1) + \cdots + a_5(\varepsilon, n)F(\varepsilon, n+5) = 0$$

recurrence solver ↓

$F(\varepsilon, n)$ = expression in terms of special functions

Example: A master integral from Ladder and V -topologies

[arXiv:1509.08324]

$$F(\varepsilon, n) = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{\varepsilon/2} y^{\varepsilon/2} (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+n+1} \underbrace{(1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^n}_{f(\varepsilon, n, x, y, z)}$$

Ablinger's
MultiIntegrate.m

↓ (9 hours)

$$a_0(\varepsilon, n)F(\varepsilon, n) + a_1(\varepsilon, n)F(\varepsilon, n+1) + \dots + a_5(\varepsilon, n)F(\varepsilon, n+5) = 0$$

refined
recurrence solver

↓

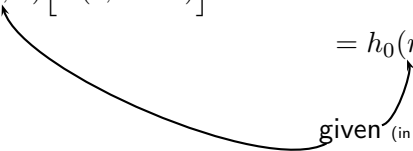
$$F(\varepsilon, n) = F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + \dots$$

Ansatz (for power series)

$$\begin{aligned} & a_0(\varepsilon, n) \left[F(\varepsilon, n) \right] \\ & + a_1(\varepsilon, n) \left[F(\varepsilon, n + 1) \right] \\ & + \\ & \vdots \\ & + a_\delta(\varepsilon, n) \left[F(\varepsilon, n + \delta) \right] \end{aligned}$$

$= h_0(n) + h_1(n)\varepsilon + h_2(n)\varepsilon^2 + \dots$

given (in terms of indefinite nested sums and products)

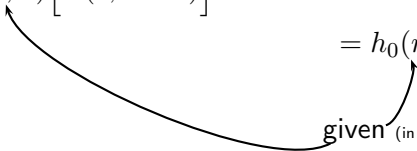


Ansatz (for power series)

$$\begin{aligned} & a_0(\varepsilon, n) \left[F_0(n) + F_1(n)\varepsilon + F_2(n)\varepsilon^2 + \dots \right] \\ & + a_1(\varepsilon, n) \left[F(\varepsilon, n+1) \right] \\ & + \\ & \vdots \\ & + a_\delta(\varepsilon, n) \left[F(\varepsilon, n+\delta) \right] \end{aligned}$$

$= h_0(n) + h_1(n)\varepsilon + h_2(n)\varepsilon^2 + \dots$

given (in terms of indefinite nested sums and products)



Ansatz (for power series)

$$\begin{aligned}
 & a_0(\varepsilon, n) \left[F_0(n) + F_1(n)\varepsilon + F_2(n)\varepsilon^2 + \dots \right] \\
 & + a_1(\varepsilon, n) \left[F_0(n+1) + F_1(n+1)\varepsilon + F_2(n+1)\varepsilon^2 + \dots \right] \\
 & + \\
 & \vdots \\
 & + a_\delta(\varepsilon, n) \left[F(\varepsilon, n + \delta) \right]
 \end{aligned}$$

$= h_0(n) + h_1(n)\varepsilon + h_2(n)\varepsilon^2 + \dots$

given (in terms of indefinite nested sums and products)

Ansatz (for power series)

$$\begin{aligned} & a_0(\varepsilon, n) \left[F_0(n) + F_1(n)\varepsilon + F_2(n)\varepsilon^2 + \dots \right] \\ & + a_1(\varepsilon, n) \left[F_0(n+1) + F_1(n+1)\varepsilon + F_2(n+1)\varepsilon^2 + \dots \right] \\ & + \\ & \vdots \\ & + a_\delta(\varepsilon, n) \left[F_0(n+\delta) + F_1(n+\delta)\varepsilon + F_2(n+\delta)\varepsilon^2 + \dots \right] \\ & \hspace{15em} = h_0(n) + h_1(n)\varepsilon + h_2(n)\varepsilon^2 + \dots \end{aligned}$$

given (in terms of indefinite nested sums and products)



Ansatz (for power series)

$$\begin{aligned}
& a_0(\varepsilon, n) \left[F_0(n) + F_1(n)\varepsilon + F_2(n)\varepsilon^2 + \dots \right] \\
& + a_1(\varepsilon, n) \left[F_0(n+1) + F_1(n+1)\varepsilon + F_2(n+1)\varepsilon^2 + \dots \right] \\
& + \\
& \vdots \\
& + a_\delta(\varepsilon, n) \left[F_0(n+\delta) + F_1(n+\delta)\varepsilon + F_2(n+\delta)\varepsilon^2 + \dots \right] \\
& \qquad \qquad \qquad = h_0(n) + h_1(n)\varepsilon + h_2(n)\varepsilon^2 + \dots
\end{aligned}$$

$\Downarrow$ constant terms must agree

$$a_0(0, n)F_0(n) + a_1(0, n)F_0(n+1) + \dots + a_\delta(0, n)F_0(n+\delta) = h_0(n)$$

Ansatz (for power series)

$$\begin{aligned}
& a_0(\varepsilon, n) \left[F_0(n) + F_1(n)\varepsilon + F_2(n)\varepsilon^2 + \dots \right] \\
& + a_1(\varepsilon, n) \left[F_0(n+1) + F_1(n+1)\varepsilon + F_2(n+1)\varepsilon^2 + \dots \right] \\
& + \\
& \vdots \\
& + a_\delta(\varepsilon, n) \left[F_0(n+\delta) + F_1(n+\delta)\varepsilon + F_2(n+\delta)\varepsilon^2 + \dots \right] \\
& \qquad \qquad \qquad = h_0(n) + h_1(n)\varepsilon + h_2(n)\varepsilon^2 + \dots
\end{aligned}$$

$\Downarrow$ constant terms must agree

$$a_0(0, n)F_0(n) + a_1(0, n)F_0(n+1) + \dots + a_\delta(0, n)F_0(n+\delta) = h_0(n)$$

REC solver: Given the initial values $F_0(1), F_0(2), \dots, F_0(\delta)$,
decide if $F_0(n)$ can be written in terms of indefinite
 nested sums and products.

Ansatz (for power series)

$$\begin{aligned}
 & a_0(\varepsilon, n) \left[F_0(n) + F_1(n)\varepsilon + F_2(n)\varepsilon^2 + \dots \right] \\
 & + a_1(\varepsilon, n) \left[F_0(n+1) + F_1(n+1)\varepsilon + F_2(n+1)\varepsilon^2 + \dots \right] \\
 & + \\
 & \vdots \\
 & + a_\delta(\varepsilon, n) \left[F_0(n+\delta) + F_1(n+\delta)\varepsilon + F_2(n+\delta)\varepsilon^2 + \dots \right] \\
 & \qquad \qquad \qquad = h_0(n) + h_1(n)\varepsilon + h_2(n)\varepsilon^2 + \dots
 \end{aligned}$$

⇓ constant terms must agree

$$a_0(0, n)F_0(n) + a_1(0, n)F_0(n+1) + \dots + a_\delta(0, n)F_0(n+\delta) = h_0(n)$$

Ansatz (for power series)

$$\begin{aligned}
 & a_0(\varepsilon, n) \left[F_0(n) + F_1(n)\varepsilon + F_2(n)\varepsilon^2 + \dots \right] \\
 & + a_1(\varepsilon, n) \left[F_0(n+1) + F_1(n+1)\varepsilon + F_2(n+1)\varepsilon^2 + \dots \right] \\
 & + \\
 & \vdots \\
 & + a_\delta(\varepsilon, n) \left[F_0(n+\delta) + F_1(n+\delta)\varepsilon + F_2(n+\delta)\varepsilon^2 + \dots \right] \\
 & \qquad \qquad \qquad = h_0(n) + h_1(n)\varepsilon + h_2(n)\varepsilon^2 + \dots
 \end{aligned}$$

⇓ constant terms must agree

$$a_0(0, n)F_0(n) + a_1(0, n)F_0(n+1) + \dots + a_\delta(0, n)F_0(n+\delta) = h_0(n)$$

$$\begin{aligned} & a_0(\varepsilon, n) \left[F_1(n)\varepsilon + F_2(n)\varepsilon^2 + \dots \right] \\ & + a_1(\varepsilon, n) \left[F_1(n+1)\varepsilon + F_2(n+1)\varepsilon^2 + \dots \right] \\ & + \\ & \vdots \\ & + a_\delta(\varepsilon, n) \left[F_1(n+\delta)\varepsilon + F_2(n+\delta)\varepsilon^2 + \dots \right] \\ & \qquad \qquad \qquad = h'_0(n) + h'_1(n)\varepsilon + h'_2(n)\varepsilon^2 + \dots \end{aligned}$$

$$\begin{aligned}
& a_0(\varepsilon, n) \left[F_1(n)\varepsilon + F_2(n)\varepsilon^2 + \dots \right] \\
& + a_1(\varepsilon, n) \left[F_1(n+1)\varepsilon + F_2(n+1)\varepsilon^2 + \dots \right] \\
& + \\
& \vdots \\
& + a_\delta(\varepsilon, n) \left[F_1(n+\delta)\varepsilon + F_2(n+\delta)\varepsilon^2 + \dots \right] \\
& \qquad \qquad \qquad = \underbrace{h'_0(n) + h'_1(n)}_{=0} \varepsilon + h'_2(n)\varepsilon^2 + \dots
\end{aligned}$$

Divide by ε

$$\begin{aligned}
& a_0(\varepsilon, n) \left[F_1(n) + F_2(n)\varepsilon + \dots \right] \\
& + a_1(\varepsilon, n) \left[F_1(n+1) + F_2(n+1)\varepsilon + \dots \right] \\
& + \\
& \vdots \\
& + a_\delta(\varepsilon, n) \left[F_1(n+\delta) + F_2(n+\delta)\varepsilon + \dots \right] = h'_1(n) + h'_2(n)\varepsilon + \dots
\end{aligned}$$

Repeat to get $F_1(n), F_2(n), \dots$
--

Remark: Works the same for Laurent series.

Example: A master integral from Ladder and V -topologies

[arXiv:1509.08324]

$$F(\varepsilon, n) = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{\varepsilon/2} y^{\varepsilon/2} (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+n+1} \underbrace{(1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^n}_{f(\varepsilon, n, x, y, z)}$$

Ablinger's
MultiIntegrate.m

↓ (9 hours)

$$a_0(\varepsilon, n)F(\varepsilon, n) + a_1(\varepsilon, n)F(\varepsilon, n+1) + \cdots + a_5(\varepsilon, n)F(\varepsilon, n+5) = 0$$

Sigma.m

↓ (2 hours)

$$F(\varepsilon, n) = F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + \cdots + F_4(n)\varepsilon^4 + O(\varepsilon^5)$$

We get

$$F_{-3}(n) = \frac{8(-1)^n}{3(n+1)(n+2)} + \frac{8(2n+3)}{3(n+1)^2(n+2)}$$

We get

$$F_{-3}(n) = \frac{8(-1)^n}{3(n+1)(n+2)} + \frac{8(2n+3)}{3(n+1)^2(n+2)}$$

$$F_{-2}(n) = -\frac{4(-1)^n(3n^3+18n^2+31n+18)}{3(n+1)^3(n+2)^2} - \frac{4(6n^3+32n^2+51n+26)}{3(n+1)^3(n+2)^2}$$

We get

$$F_{-3}(n) = \frac{8(-1)^n}{3(n+1)(n+2)} + \frac{8(2n+3)}{3(n+1)^2(n+2)}$$

$$F_{-2}(n) = -\frac{4(-1)^n(3n^3+18n^2+31n+18)}{3(n+1)^3(n+2)^2} - \frac{4(6n^3+32n^2+51n+26)}{3(n+1)^3(n+2)^2}$$

$$\begin{aligned} F_{-1}(n) = & (-1)^n \left(\frac{2(9n^5 + 81n^4 + 295n^3 + 533n^2 + 500n + 204)}{3(n+1)^4(n+2)^3} + \frac{\zeta_2}{(n+1)(n+2)} \right) \\ & + \frac{2(18n^5 + 150n^4 + 490n^3 + 755n^2 + 536n + 132)}{3(n+1)^4(n+2)^3} + \frac{(2n+3)\zeta_2}{(n+1)^2(n+2)} \\ & + \left(-\frac{4}{(n+1)^2(n+2)} + \frac{4(-1)^n}{(n+1)(n+2)} \right) S_2(n) \\ & + \left(\frac{4(-1)^n}{3(n+1)(n+2)} - \frac{4(n+9)}{3(n+1)^2(n+2)} \right) S_{-2}(n) \end{aligned}$$

Find a recurrence and solve it for the integral/sum

$$F(\varepsilon, n) \equiv \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, n, x_1, x_2, \dots, x_7) dx_1 dx_2 \dots dx_7$$
$$\stackrel{?}{=} F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \dots$$

 ε -recurrence solver

multivariate
Almquist/Zeilberger
(Jakob Ablinger)

$$a_0(\varepsilon, n)F(\varepsilon, n) + \dots + a_d(\varepsilon, n)F(\varepsilon, n + d) = h(\varepsilon, n)$$

Find a recurrence and solve it for the integral/sum

$$F(\varepsilon, n) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, n, x_1, x_2, \dots, x_7) dx_1 dx_2 \dots dx_7$$

$$\stackrel{?}{=} F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \dots$$

 ε -recurrence solver

multivariate
Almquist/Zeilberger
(Jakob Ablinger)

$$\sum_{i_1} \cdots \sum_{i_7} f(\varepsilon, n, i_1, i_2, \dots, i_7)$$

MultiSum Package
(K. Wegschaider)

$$a_0(\varepsilon, n)F(\varepsilon, n) + \dots + a_d(\varepsilon, n)F(\varepsilon, n + d) = h(\varepsilon, n)$$

Find a recurrence and solve it for the integral/sum

$$F(\varepsilon, n) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, n, x_1, x_2, \dots, x_7) dx_1 dx_2 \dots dx_7$$

$$\stackrel{?}{=} F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \dots$$

 ε -recurrence solver

multivariate
Almquist/Zeilberger
(Jakob Ablinger)

$$\sum_{i_1} \cdots \sum_{i_7} f(\varepsilon, n, i_1, i_2, \dots, i_7)$$

MultiSum Package
(K. Wegschaider)

Holonomic/difference ring approach

$$a_0(\varepsilon, n)F(\varepsilon, n) + \dots + a_d(\varepsilon, n)F(\varepsilon, n + d) = h(\varepsilon, n)$$

Find a recurrence and solve it for the integral/sum

$$F(\varepsilon, n) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, n, x_1, x_2, \dots, x_7) dx_1 dx_2 \dots dx_7$$

$$\stackrel{?}{=} F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \dots$$

difference ring approach

 ε -recurrence solver

multivariate
Almquist/Zeilberger
(Jakob Ablinger)

$$\sum_{i_1} \cdots \sum_{i_7} f(\varepsilon, n, i_1, i_2, \dots, i_7)$$

MultiSum Package
(K. Wegschaider)

Holonomic/difference ring approach

$$a_0(\varepsilon, n)F(\varepsilon, n) + \dots + a_d(\varepsilon, n)F(\varepsilon, n + d) = h(\varepsilon, n)$$

symbolic
summation

recurrence
solving

symbolic
integration
RE finding

Definition: A **formal** power series $f(x) \in \mathbb{K}[[x]]$ is called **holonomic** if

there exist $b_0(x), \dots, b_\lambda(x) \in \mathbb{K}[x]$ (not all zero) with

$$b_0(x)f(x) + \dots + b_\lambda(x)D^\lambda f(x) = 0 \quad (\text{DE})$$

Definition: sequence $(F(n))_{n \in \mathbb{N}}$ is called **holonomic** if

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Feynman
integrals

symbolic
summation
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Feynman
integrals

A warm up example:

$$\begin{aligned} \text{GIVEN } F(\varepsilon, n) = & \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{e^{-e\gamma}}{\Gamma(\varepsilon + 1)} \times \\ & \times \left(\frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)} \right. \\ & \left. + \underbrace{\frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}}_{f(\varepsilon, n, k, j)} \right). \end{aligned}$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, **Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals**. 2006

A warm up example:

$$\begin{aligned} \text{GIVEN } F(\varepsilon, n) = & \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{e^{-e\gamma}}{\Gamma(\varepsilon + 1)} \times \\ & \times \left(\frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)} \right. \\ & \left. + \underbrace{\frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}}_{f(\varepsilon, n, k, j)} \right). \end{aligned}$$

FIND the first coefficients of the ε -expansion

$$F(\varepsilon, n) = F_0(n) + \varepsilon F_1(n) + \dots$$

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A warm up example:

$$\begin{aligned} \text{GIVEN } F(\varepsilon, n) = & \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{e^{-e\gamma}}{\Gamma(\varepsilon + 1)} \times \\ & \times \left(\frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)} \right. \\ & \left. + \underbrace{\frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}}_{f(\varepsilon, n, k, j)} \right). \end{aligned}$$

Step 1: Compute the first coefficients of the ε -expansion

$$f(\varepsilon, n, k, j) = f_0(n, k, j) + \varepsilon f_1(n, k, j) + \dots$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, **Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals**. 2006

A warm up example:

$$\begin{aligned} \text{GIVEN } F(\varepsilon, n) = & \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{e^{-e\gamma}}{\Gamma(\varepsilon + 1)} \times \\ & \times \left(\frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)} \right. \\ & \left. + \underbrace{\frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}}_{f(\varepsilon, n, k, j)} \right). \end{aligned}$$

Step 2: **Simplify** the sums in

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f(\varepsilon, n, k, j) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f_0(n, k, j) + \varepsilon \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f_1(n, k, j) + \dots$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, **Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals**. 2006

A warm-up example: simplify

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \underbrace{\frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)} \right)$$

where

$$S_1(n) = \sum_{i=1}^n \frac{1}{i} \quad (= H_n)$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \underbrace{\frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)} \right)$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \underbrace{\frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)} \right)$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

↑ summation package Sigma

$$g(j) = \frac{(j+k+1)(j+n+1)j!k!(j+k+n)!(S_1(j) - S_1(j+k) - S_1(j+n) + S_1(j+k+n))}{kn(j+k+1)!(j+n+1)!(k+n+1)!}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \underbrace{\frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)} \right)$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

Summing the telescoping equation over j from 0 to a gives

$$\sum_{j=0}^a f(j) = g(a+1) - g(0)$$

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$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \underbrace{\frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)} \right)$$

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Summing the telescoping equation over j from 0 to a gives

$$\sum_{j=0}^a f(j) = g(a+1) - g(0) \\ = \frac{(a+1)!(k-1)!(a+k+n+1)!(S_1(a) - S_1(a+k) - S_1(a+n) + S_1(a+k+n))}{n(a+k+1)!(a+n+1)!(k+n+1)!} \\ + \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)n!} + \frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}}_{a \rightarrow \infty}$$

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$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \underbrace{\frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)} \right)$$

$$\sum_{j=0}^{\infty} f(j) = \frac{1}{n!} \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}$$

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$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!} \right) \\ \underbrace{\hspace{10em}}_{f(j)}$$

$$\sum_{k=1}^{\infty} \sum_{j=0}^{\infty} f(j) = \frac{1}{n!} \sum_{k=1}^{\infty} \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}$$

Telescoping

GIVEN

$$A(n) := \sum_{k=1}^n \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(k)}.$$

FIND $g(k)$:

$$\boxed{g(k+1) - g(k)} = \boxed{f(k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

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no solution 😞

Zeilberger's creative telescoping paradigm

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$$A(n) := \sum_{k=1}^a \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n, k)}.$$

FIND $g(n, k)$

$$\boxed{g(n, k+1) - g(n, k)} = \boxed{f(n, k)}$$

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FIND $g(n, k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n, k+1) - g(n, k)} = \boxed{c_0(n)f(n, k) + c_1(n)f(n+1, k)}$$

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for all $0 \leq k \leq n$ and all $n \geq 0$.

Sigma computes: $c_0(n) = -n, c_1(n) = (n+2)$ and

$$g(n, k) = \frac{kS_1(k) + (-n-1)S_1(n) - kS_1(k+n) - 2}{(k+n+1)(n+1)^2}$$

Zeilberger's creative telescoping paradigm

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$$A(n) := \sum_{k=1}^a \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n, k)}.$$

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for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n, a+1) - g(n, 1)} = \boxed{\sum_{k=1}^a \left[c_0(n)f(n, k) + c_1(n)f(n+1, k) \right]}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$A(n) := \sum_{k=1}^a \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n, k)}.$$

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Zeilberger's creative telescoping paradigm

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Summing this equation over k from 1 to a gives:

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Zeilberger's creative telescoping paradigm

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for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 1 to a gives:

$$\begin{array}{ccc} \boxed{g(n, a+1) - g(n, 1)} & = & \boxed{c_0(n)A(n) + c_1(n)A(n+1)} \\ \parallel & & \parallel \\ \frac{(a+1)(S_1(a)+S_1(n)-S_1(a+n))}{(n+1)^2(a+n+2)} & & - nA(n) + (2+n)A(n+1) \\ + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)} & & \end{array}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n) + 1}{(n+1)^3}$$

recurrence finder


$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n) + 1}{(n+1)^3}$$

recurrence solver

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}$$

$\in$

$$\left\{ c \times \frac{1}{n(n+1)} + \frac{S_1(n)^2 + S_2(n)}{2n(n+1)} \mid c \in \mathbb{R} \right\}$$

where

$$S_1(n) = \sum_{i=1}^n \frac{1}{i}$$

$$S_2(n) = \sum_{i=1}^n \frac{1}{i^2}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n) + 1}{(n+1)^3}$$

Summation package Sigma

(based on difference field/ring algorithms/theory

see, e.g., Abramov, Karr 1981, Bronstein 2000, Schneider 2001/2004/2005a-c/2007/2008/2010a-c)

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}$$

=

$$0 \times \frac{1}{n(n+1)} + \frac{S_1(n)^2 + S_2(n)}{2n(n+1)}$$

where

$$S_1(n) = \sum_{i=1}^n \frac{1}{i}$$

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A warm-up example: simplify

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!} \right) \\ \underbrace{\hspace{15em}}_{f(j)}$$

$$\sum_{k=1}^{\infty} \sum_{j=0}^{\infty} f(j) = \frac{1}{n!} \sum_{k=1}^{\infty} \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)} \\ = \frac{1}{n!} \frac{S_1(n)^2 + S_2(n)}{2n(n+1)}$$

where

$$S_1(n) = \sum_{i=1}^n \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^n \frac{1}{i^2}$$

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$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!} \right) \\ \underbrace{\hspace{10em}}_{f(n,k,j)}$$

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f(n,k,j) = \frac{S_1(n)^2 + 3S_2(n)}{2n(n+1)!}$$

where

$$S_1(n) = \sum_{i=1}^n \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^n \frac{1}{i^2}$$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a **definite** sum

$$A(n) = \sum_{k=0}^n f(n, k);$$

$f(n, k)$: indefinite nested product-sum in k ;
 n : extra parameter

FIND a **recurrence** for $A(n)$

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2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)A(n) + \dots + a_d(n)A(n+d) = h(n);$$

FIND **all solutions** expressible by indefinite nested products/sums

(Abramov/Bronstein/Petkovšek/CS, JSC 2021)

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FIND **all solutions** expressible by indefinite nested products/sums
 (Abramov/Bronstein/Petkovšek/CS, JSC 2021)

3. Find a “closed form”

$A(n)$ =combined solutions in terms of **indefinite nested** sums.

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

Simple sum

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

||

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \boxed{\sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

||

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \left[\sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!} \right]$$

||

$$\left(\binom{j+1}{r} \left(\frac{(-1)^r (-j+n-2)! r!}{(n+1)(-j+n+r-1)(-j+n+r)!} + \frac{(-1)^{n+r} (j+1)! (-j+n-2)! (-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)! (-j-1)_r (2-n)_j} \right) \right)$$

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

||

$$\sum_{j=0}^{n-2} \left[\sum_{r=0}^{j+1} \binom{j+1}{r} \left(\frac{(-1)^r (-j+n-2)! r!}{(n+1)(-j+n+r-1)(-j+n+r)!} + \frac{(-1)^{n+r} (j+1)! (-j+n-2)! (-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)! (-j-1)_r (2-n)_j} \right) \right]$$

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

||

$$\sum_{j=0}^{n-2} \left[\sum_{r=0}^{j+1} \binom{j+1}{r} \left(\frac{(-1)^r (-j+n-2)! r!}{(n+1)(-j+n+r-1)(-j+n+r)!} + \frac{(-1)^{n+r} (j+1)! (-j+n-2)! (-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)! (-j-1)_r (2-n)_j} \right) \right]$$

||

$$\left(\frac{n^2 - n + 1}{(n-1)^2 n^2 (n+1) (2-n)_j} + \frac{\sum_{i=1}^j \frac{(2-n)_i}{(-i+n-1)^2 (i+1)!}}{(n+1) (2-n)_j} + \frac{(-1)^{j+n} (-j-2) (-j+n-2)!}{(j-n+1) (n+1)^2 n!} \right) (j+1)! - \frac{1}{(n+1)^2 (-j+n-1)}$$

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

||

$$\sum_{j=0}^{n-2} \left(\left(\frac{n^2 - n + 1}{(n-1)^2 n^2 (n+1)(2-n)_j} + \frac{\sum_{i=1}^j \frac{(2-n)_i}{(-i+n-1)^2 (i+1)!}}{(n+1)(2-n)_j} + \frac{(-1)^{j+n} (-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2 n!} \right) (j+1)! - \frac{1}{(n+1)^2 (-j+n-1)} \right)$$

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

||

$$\sum_{j=0}^{n-2} \left(\left(\frac{n^2 - n + 1}{(n-1)^2 n^2 (n+1)(2-n)_j} + \frac{\sum_{i=1}^j \frac{(2-n)_i}{(-i+n-1)^2 (i+1)!}}{(n+1)(2-n)_j} + \frac{(-1)^{j+n} (-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2 n!} \right) (j+1)! - \frac{1}{(n+1)^2 (-j+n-1)} \right)$$

||

$$\frac{-n^2 - n - 1}{n^2(n+1)^3} + \frac{(-1)^n (n^2 + n + 1)}{n^2(n+1)^3} - \frac{2S_{-2}(n)}{n+1} + \frac{S_1(n)}{(n+1)^2} + \frac{S_2(n)}{-n-1}$$

Note: $S_a(n) = \sum_{i=1}^N \frac{\text{sign}(a)^i}{i^{|a|}}$, $a \in \mathbb{Z} \setminus \{0\}$.

In[1]:= << **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= << **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

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In[3]:= << **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

$$\text{In[4]:= mySum} = \sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!};$$

In[5]:= **EvaluateMultiSum**[mySum, {}, {n}, {1}]

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Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= << **HarmonicSums.m**

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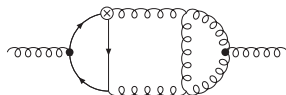
EvaluateMultiSums by Carsten Schneider © RISC-Linz

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In[5]:= **EvaluateMultiSum**[mySum, {}, {n}, {1}]

$$\text{Out[5]=} \frac{-n^2 - n - 1}{n^2(n+1)^3} + \frac{(-1)^n (n^2 + n + 1)}{n^2(n+1)^3} - \frac{2S[-2, n]}{n+1} + \frac{S[1, n]}{(n+1)^2} + \frac{S[2, n]}{-n-1}$$

Feynman integral



a 3-loop massive ladder
diagram [arXiv:1509.08324]

$$\sum_{j=0}^{n-3} \sum_{k=0}^j \binom{n-1}{j+2} \binom{j+1}{k+1}$$

$$||$$

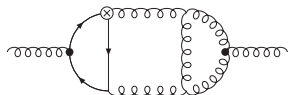
$$\times \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \theta(1-x_5-x_6)(1-x_2)(1-x_4)x_2^{-\varepsilon}$$

$$(1-x_2)^{-\varepsilon} x_4^{\varepsilon/2-1} (1-x_4)^{\varepsilon/2-1} x_5^{\varepsilon-1} x_6^{-\varepsilon/2}$$

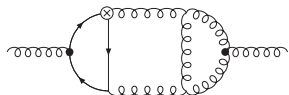
$$\left[\begin{aligned} & [-x_3(1-x_4) - x_4(1-x_5-x_6+x_5x_1+x_6x_3)]^k \\ & + [x_3(1-x_4) - (1-x_4)(1-x_5-x_6+x_5x_1+x_6x_3)]^k \end{aligned} \right]$$

$$\times (1-x_5-x_6+x_5x_1+x_6x_3)^{j-k} (1-x_2)^{n-3-j}$$

$$\times [x_1 - (1-x_5-x_6) - x_5x_1 - x_6x_3]^{n-3-j} dx_1 dx_2 dx_3 dx_4 dx_5 dx_6$$



$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}$$



$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}$$

||

Simplify

$$\sum_{j=0}^{n-3} \sum_{k=0}^j \sum_{l=0}^k \sum_{q=0}^{-j+n-3} \sum_{s=1}^{-l+n-q-3} \sum_{r=0}^{-l+n-q-s-3} (-1)^{-j+k-l+n-q-3} \times$$

$$\times \frac{\binom{j+1}{k+1} \binom{k}{l} \binom{n-1}{j+2} \binom{-j+n-3}{q} \binom{-l+n-q-3}{s} \binom{-l+n-q-s-3}{r} r! (-l+n-q-r-s-3)! (s-1)!}{(-l+n-q-2)! (-j+n-1) (n-q-r-s-2) (q+s+1)}$$

$$\left[4S_1(-j+n-1) - 4S_1(-j+n-2) - 2S_1(k) \right.$$

$$\left. - (S_1(-l+n-q-2) + S_1(-l+n-q-r-s-3) - 2S_1(r+s)) \right.$$

$$\left. + 2S_1(s-1) - 2S_1(r+s) \right] + \mathbf{3 \text{ further 6-fold sums}}$$

$$\boxed{F_0(n)} =$$

$$\begin{aligned}
& \frac{7}{12}S_1(n)^4 + \frac{(17n+5)S_1(n)^3}{3n(n+1)} + \left(\frac{35n^2-2n-5}{2n^2(n+1)^2} + \frac{13S_2(n)}{2} + \frac{5(-1)^n}{2n^2} \right) S_1(n)^2 \\
& + \left(-\frac{4(13n+5)}{n^2(n+1)^2} + \left(\frac{4(-1)^n(2n+1)}{n(n+1)} - \frac{13}{n} \right) S_2(n) + \left(\frac{29}{3} - (-1)^n \right) S_3(n) \right. \\
& + (2 + 2(-1)^n)S_{2,1}(n) - 28S_{-2,1}(n) + \frac{20(-1)^n}{n^2(n+1)} S_1(n) + \left(\frac{3}{4} + (-1)^n \right) S_2(n)^2 \\
& - 2(-1)^n S_{-2}(n)^2 + S_{-3}(n) \left(\frac{2(3n-5)}{n(n+1)} + (26 + 4(-1)^n)S_1(n) + \frac{4(-1)^n}{n+1} \right) \\
& + \left(\frac{(-1)^n(5-3n)}{2n^2(n+1)} - \frac{5}{2n^2} \right) S_2(n) + S_{-2}(n) (10S_1(n)^2 + \left(\frac{8(-1)^n(2n+1)}{n(n+1)} \right. \\
& + \frac{4(3n-1)}{n(n+1)} \left. \right) S_1(n) + \frac{8(-1)^n(3n+1)}{n(n+1)^2} + (-22 + 6(-1)^n)S_2(n) - \frac{16}{n(n+1)} \\
& + \left(\frac{(-1)^n(9n+5)}{n(n+1)} - \frac{29}{3n} \right) S_3(n) + \left(\frac{19}{2} - 2(-1)^n \right) S_4(n) + (-6 + 5(-1)^n)S_{-4}(n) \\
& + \left(-\frac{2(-1)^n(9n+5)}{n(n+1)} - \frac{2}{n} \right) S_{2,1}(n) + (20 + 2(-1)^n)S_{2,-2}(n) + (-17 + 13(-1)^n)S_{3,1}(n) \\
& - \frac{8(-1)^n(2n+1) + 4(9n+1)}{n(n+1)} S_{-2,1}(n) - (24 + 4(-1)^n)S_{-3,1}(n) + (3 - 5(-1)^n)S_{2,1,1}(n) \\
& + 32S_{-2,1,1}(n) + \left(\frac{3}{2}S_1(n)^2 - \frac{3S_1(n)}{n} + \frac{3}{2}(-1)^n S_{-2}(n) \right) \zeta(2)
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& + \left(-\frac{4(1-(-1)^n)}{n^2} \right) S_1(n) = \sum_{i=1}^n \frac{1}{i} \left(\frac{2n+1}{n+1} - \frac{13}{n} \right) S_2(n) + \left(\frac{29}{3} - (-1)^n \right) S_3(n) \\
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& + (2+2(-1)^n) S_{-2,1}(n) + \frac{20(-1)^n}{n^2(n+1)} S_2(n) = \sum_{i=1}^n \frac{1}{i^2} S_2(n)^2 \\
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& + \left(\frac{(-1)^n(5-3n)}{2n^2(n+1)} - \frac{5}{2n^2} \right) S_2(n) + S_{-2}(n) (10S_1(n)^2 + \left(\frac{8(-1)^n(2n+1)}{n(n+1)} \right. \\
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& + \left(\frac{(-1)^n(5-2n)}{2n^2(n+1)} + \frac{4(3n-1)}{n(n+1)} \right) S_2(n) - \frac{16}{n(n+1)} \\
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\end{aligned}$$

Sigma.m is based on difference ring/field theory

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symbolic
summation
RE finding

recurrence
solving

symbolic
integration
RE finding

Definition: A **formal** power series $f(x) \in \mathbb{K}[[x]]$ is called **holonomic** if

there exist $b_0(x), \dots, b_\lambda(x) \in \mathbb{K}[x]$ (not all zero) with

$$b_0(x)f(x) + \dots + b_\lambda(x)D^\lambda f(x) = 0 \quad (\text{DE})$$

Definition: sequence $(F(n))_{n \in \mathbb{N}}$ is called **holonomic** if

there exist $a_0(x), \dots, a_\delta(x) \in \mathbb{K}[x]$ (not all zero) with

$$a_0(n)F(n) + \dots + a_\delta(n)F(n + \delta) = 0 \quad (\text{RE})$$

Feynman
integrals

symbolic
summation

RE&DE finding

recurrence
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Motivation: solving recurrences and D-equations

[coming, e.g., from IBP methods]

Given invert. $A(x) \in \mathbb{K}(x)^{\lambda \times \lambda}$ and $\hat{R}_1(x), \dots, \hat{R}_\lambda(x)$ (in terms of special functions)

Determine $\hat{I}_1(x), \dots, \hat{I}_\lambda(x)$ (for given initial values) s.t.

$$D_x \begin{pmatrix} \hat{I}_1(x) \\ \dots \\ \hat{I}_\lambda(x) \end{pmatrix} = A(x) \begin{pmatrix} \hat{I}_1(x) \\ \dots \\ \hat{I}_\lambda(x) \end{pmatrix} + \begin{pmatrix} \hat{R}_1(x) \\ \dots \\ \hat{R}_\lambda(x) \end{pmatrix}$$

given

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↓
uncoupling algorithms
(Zürcher, Abramov/Zima, Gauss,...)

1. $\hat{I}_1(x)$ is a solution of

$$b_0(x)\hat{I}_1(x) + b_1(x)D_x\hat{I}_1(x) + \dots + b_\lambda(x)D_x^\lambda\hat{I}_1(x) = \hat{r}(x)$$

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2. For $i = 2, \dots, r$ we get

$$\hat{I}_i(x) = \text{LinCom}(\hat{I}_1(x), \dots, D_x^{\lambda-1}\hat{I}_1(x)) + \text{LinCom}(\dots, D^i\hat{R}_i(x), \dots)$$

Given invert. $A(x) \in \mathbb{K}(x)^{\lambda \times \lambda}$ and $\hat{R}_1(x), \dots, \hat{R}_\lambda(x)$ (in terms of special functions)

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uncoupling algorithms
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DE-solver

I. A differential equation solver (HarmonicSums.m)

GIVEN a linear differential equation $b_0(x), \dots, b_\lambda(x) \in \mathbb{K}[x]$

$$b_0(x)f(x) + \dots + b_\lambda(x)D^\lambda f(x) = 0;$$

together with initial values $f(0), \dots, D^{\lambda-1}f(x)|_{x=0} \in \mathbb{K}$

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DECIDE constructively if $f(x)$ can be expressed in terms of **iterated integrals** defined over **hyperexponential functions**.

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Special cases of **iterated integrals** over hyperexponential functions:

$$H_{1,-1}(x) = \int_0^x \frac{1}{1-\tau_1} \int_0^{\tau_1} \frac{1}{1+\tau_2} d\tau_2 d\tau_1 \quad (\text{harmonic polylogarithm})$$

E. Remiddi, E. and J.A.M. Vermaseren, Int. J. Mod. Phys. **A15** (2000) [arXiv:hep-ph/9905237]

I. A differential equation solver (HarmonicSums.m)

GIVEN a linear differential equation $b_0(x), \dots, b_\lambda(x) \in \mathbb{K}[x]$

$$b_0(x)f(x) + \dots + b_\lambda(x)D^\lambda f(x) = 0;$$

together with initial values $f(0), \dots, D^{\lambda-1}f(x)|_{x=0} \in \mathbb{K}$

DECIDE constructively if $f(x)$ can be expressed in terms of **iterated integrals** defined over **hyperexponential functions**.

Special cases of **iterated integrals** over hyperexponential functions:

$$H_{2,-2}(x) = \int_0^x \frac{1}{2 - \tau_1} \int_0^{\tau_1} \frac{1}{2 + \tau_2} d\tau_2 d\tau_1 \quad (\text{generalized polylogarithms})$$

S. Moch, P. Uwer and S. Weinzierl, J. Math. Phys. **43** (2002) 3363 [hep-ph/0110083];

J. Ablinger, J. Blümlein and CS, J. Math. Phys. **54** (2013) 082301 [arXiv:1302.0378].

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Special cases of iterated integrals over hyperexponential functions:

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J. Ablinger, J. Blümlein and CS, J. Math. Phys. **52** (2011) 102301 [arXiv:1105.6063].

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$$\int_0^x \frac{1}{\sqrt{1+\tau_1}} \int_0^{\tau_1} \frac{1}{1+\tau_2} d\tau_2 d\tau_1 \quad (\text{radical integrals})$$

J. Ablinger, J. Blümlein, C. G. Raab and CS, J. Math. Phys. **55** (2014) 112301 [arXiv:1407.1822].

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Special cases of iterated integrals over hyperexponential functions:

$$\int_0^x \frac{1}{1 - \tau_1 + \eta\tau_1} \int_0^{\tau_1} \sqrt{1 - \tau_2} \sqrt{1 - \tau_2 + \eta\tau_2} d\tau_2 d\tau_1 \quad \text{(generalized radical integrals)}$$

J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, CS, K. Schönwald. Nucl.Phys.B 932. 2018. [arXiv:1804.02226].

J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, M. Saragnese, CS, K. Schönwald. Nucl.Phys.B 955. 2020. [arXiv:2004.08916]

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A more general example:

$$\int_0^x e^{\int_1^{\tau_1} \frac{1}{1+y+y^2} dy} \int_0^{\tau_1} \frac{1}{1+\tau_2} d\tau_2 d\tau_1$$

HarmonicSums can also deal with Liouvillian solutions (i.e., it contains Kovacic's algorithm):

$$(11 + 20x)f'(x) + (1 + x)(35 + 134x)f''(x) \\ + 3(1 + x)^2(4 + 37x)f^{(3)}(x) + 18x(1 + x)^3f^{(4)}(x) = 0$$

↓

$$\left\{ c_1 + c_2 \int_0^x \frac{1}{1 + \tau_1} d\tau_1 + c_3 \int_0^x \frac{1}{1 + \tau_1} \int_0^{\tau_1} \frac{\sqrt[3]{1 + \sqrt{1 + \tau_2}}}{1 + \tau_2} d\tau_2 d\tau_1 \right. \\ \left. + c_4 \int_0^x \frac{1}{1 + \tau_1} \int_0^{\tau_1} \frac{\sqrt[3]{1 - \sqrt{1 + \tau_2}}}{1 + \tau_2} d\tau_2 d\tau_1 \mid c_1, c_2, c_3, c_4 \in \mathbb{K} \right\}$$

Definition: A **formal** power series $f(x) \in \mathbb{K}[[x]]$ is called **holonomic** if

there exist $b_0(x), \dots, b_\lambda(x) \in \mathbb{K}[x]$ (not all zero) with

$$b_0(x)f(x) + \dots + b_\lambda(x)D^\lambda f(x) = 0 \quad (\text{DE})$$

Definition: sequence $(F(n))_{n \geq 0} \in \mathbb{K}^{\mathbb{N}}$ is called **holonomic** if

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$$a_0(n)F(n) + \dots + a_\delta(n)F(n + \delta) = 0 \quad (\text{RE})$$

Connection: $DE \longleftrightarrow REC$

Let

$$f(x) = \sum_{n=0}^{\infty} F(n)x^n$$

be a (formal) power series. Then:

there exist $b_0(x), \dots, b_\lambda(x) \in \mathbb{K}[x]$ (not all zero) with

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$\Updownarrow$ algorithmic

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Given invert. $A(x) \in \mathbb{K}(x)^{\lambda \times \lambda}$ and $\hat{R}_1(x), \dots, \hat{R}_\lambda(x)$ (in terms of special functions)

Determine $\hat{I}_1(x), \dots, \hat{I}_\lambda(x)$ (for given initial values) s.t.

$$D_x \begin{pmatrix} \hat{I}_1(x) \\ \dots \\ \hat{I}_\lambda(x) \end{pmatrix} = A(x) \begin{pmatrix} \hat{I}_1(x) \\ \dots \\ \hat{I}_\lambda(x) \end{pmatrix} + \begin{pmatrix} \hat{R}_1(x) \\ \dots \\ \hat{R}_\lambda(x) \end{pmatrix}$$

uncoupling algorithms
(Zürcher, Abramov/Zima, Gauss,...)

1. $\hat{I}_1(x)$ is a solution of

$$b_0(x)\hat{I}_1(x) + b_1(x)D_x\hat{I}_1(x) + \dots + b_\lambda(x)D_x^\lambda\hat{I}_1(x) = \hat{r}(x)$$

DE-solver

Given invert. $A(x) \in \mathbb{K}(x)^{\lambda \times \lambda}$ and $\hat{R}_1(x), \dots, \hat{R}_\lambda(x)$ (in terms of special functions)
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DE-solver

REC-solver

Example 1: Find a power series solution

$$f(x) = \sum_{n=0}^{\infty} F(n)x^n$$

for

$$\begin{aligned} & - (x^4 - 64x^3) f^{(4)}(x) - 2 (5x^3 - 144x^2) f^{(3)}(x) \\ & - (25x^2 - 208x) f''(x) - (15x - 8) f'(x) - f(x) = 0 \end{aligned}$$

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$$F(n) = \frac{1}{\binom{2n}{n}^3} = \frac{(1)_n (1)_n (1)_n (1)_n}{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n n!} \frac{1}{64^n}$$

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$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{\binom{2n}{n}^3} = {}_4F_3 \left[\begin{matrix} 1, 1, 1, 1 \\ \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \end{matrix}; \frac{x}{64} \right]$$

for

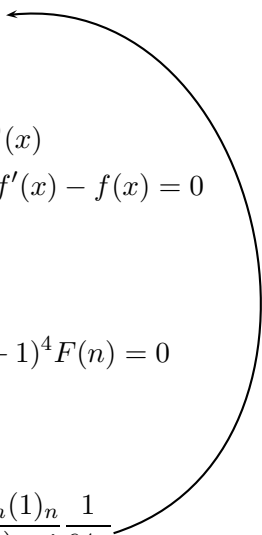
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for further transformations
see [arXiv:1706.01299]

for

$$\begin{aligned} & - (x^4 - 64x^3) f^{(4)}(x) - 2(5x^3 - 144x^2) f^{(3)}(x) \\ & - (25x^2 - 208x) f''(x) - (15x - 8) f'(x) - f(x) = 0 \end{aligned}$$



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Example 2: Find a power series solution

$$f(x) = \sum_{n=0}^{\infty} F(n)x^n$$

for

$$\begin{aligned} & (x^6 - 32x^5 + 256x^4) f^{(6)}(x) + (23x^5 - 528x^4 + 2560x^3) f^{(5)}(x) \\ & + (171x^4 - 2552x^3 + 6272x^2) f^{(4)}(x) + 2(245x^3 - 2002x^2 + 1728x) f^{(3)}(x) \\ & + 2(253x^2 - 786x + 72) f''(x) + 4(35x - 12) f'(x) + 4f(x) = 0 \end{aligned}$$

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↓ Sigma.m

$$F(n) = \frac{1}{\binom{2n}{n}^2} \left(c_1 + c_2 S_1(n) \right) = \frac{(1)_n (1)_n (1)_n}{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n n!} \frac{1}{16^n} \left(c_1 + c_2 S_1(n) \right)$$

Example 2: Find a power series solution

$$f(x) = c_1 \cdot {}_3F_2 \left[\begin{matrix} 1, 1, 1 \\ \frac{1}{2}, \frac{1}{2} \end{matrix}; \frac{x}{16} \right] + c_2 \sum_{n=0}^{\infty} \frac{S_1(n)}{\binom{2n}{n}^2} x^n$$

for

$$\begin{aligned} & (x^6 - 32x^5 + 256x^4) f^{(6)}(x) + (23x^5 - 528x^4 + 2560x^3) f^{(5)}(x) \\ & + (171x^4 - 2552x^3 + 6272x^2) f^{(4)}(x) + 2(245x^3 - 2002x^2 + 1728x) f^{(3)}(x) \\ & + 2(253x^2 - 786x + 72) f''(x) + 4(35x - 12) f'(x) + 4f(x) = 0 \end{aligned}$$

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symbolic
summation

RE&DE finding

recurrence
solving

DE
solving

symbolic
integration

RE&DE finding

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Feynman
integrals

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symbolic
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coupled systems

$$\text{for } f(x) = \sum_{n=0}^{\infty} P(n)x^n$$

SolveCoupledSystem.m



large no. of moments,
say $P(0), \dots, P(10000)$

coupled systems

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numerics

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guessing (ore_algebra in Sage)

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RE solving

indefinite nested sums
over hypergeo. products

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guessing (ore_algebra in Sage)



recurrence

numerics

asymptotics

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over hypergeo. products

indefinite nested sums
over ${}_pF_q$ s (e.g., elliptic fus)
[iterative-noniterative sums]

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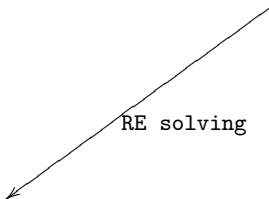
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guessing (ore_algebra in Sage)

recurrence

RE solving



indefinite nested sums
over hypergeo. products

Example (J. Blümlein, P. Marquard, CS, K. Schönwald. Nucl. Phys. B 971, pp. 1-44. 2021)

In[6]:= << **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[7]:= **initial = << iFile16**

```
In[6]:= << Sigma.m
```

```
In[7]:= initial == << iFile16
```

```
Out[7]= {37, 34577/1296, 7598833/151875, 13675395569/230496000,  
475840076183/7501410000, 1432950323678333/21965628762000,  
21648380901382517/328583783127600,  
52869784323778576751/802218994536960000,  
49422862094045523994231/753773992230616156800,  
33131879832907935920726113/509557943985299969760000,  
5209274721836755168448777/80949984111854180459136,  
56143711997344769021041145213/882589266383586456384353664,  
453500433353845628194790025124807/7217228048879468556886950000000,  
14061543374120479886110159898869387/226643167590350326435656036000000,  
715586522666491903324905785178619936571168370307700222807811495895030000000,  
16286729046359273892841271257418854056836413/269396588055480390401343344736943104000000,  
1428729642632302467951426905844691837805299/23940759575034122827861315961573673600000,  
498938690219595294505102809199154550783080767/8468883667852979813171262304054002720000000,  
55522222121221222222127375373737122217122222122222/2512212222221212225522222222221777171772222222
```

In[8]:= **rec** == << **rFile16**

Out[8]= $(n+1)^4(n+2)^2(2n+3)(2n+5)(2n+7)(2n+9)(2n+11) \left(309237645312n^{32} + 38256884318208n^{31} + \right.$
 $2282100271087616n^{30} + 87428170197762048n^{29} + 2417273990256001024n^{28} + 51388547929265405952n^{27} +$
 $873862324676687036416n^{26} + 12209268055143308328960n^{25} + 142860861222820240162816n^{24} +$
 $1419883954103469621510144n^{23} + 12115561235109256405319680n^{22} + 89479384946084038000803840n^{21} +$
 $575561340618928527623274496n^{20} + 3239547818363227419971647488n^{19} + 16009805333085271423330779136n^{18} +$
 $69631814641718655426881659392n^{17} + 266892117418348771052573667328n^{16} +$
 $901901113782416884441719270144n^{15} + 2685821385767154471801366647296n^{14} +$
 $7038702625583766161604414471744n^{13} + 16195069575749412648646633248128n^{12} +$
 $32602540883321212533013752639288n^{11} + 57154680141624618025310553466704n^{10} +$
 $86710462147941775492301231896818n^9 + 112917328975807075881545543668548n^8 +$
 $124873767581470867343743078943772n^7 + 115624836314544572769501784072647n^6 +$
 $87938536330971046886456627610048n^5 + 53481897815980319933589323279298n^4 +$
 $25000430622737750756669804052204n^3 + 8430930497463933665464836129855n^2 +$
 $1825177817831282261293155379650n + 190428196025667395685609855000 \Big) (2n+1)^4 P[n]$

$$\begin{aligned}
& -(n+2)^3(2n+3)^3(2n+7)(2n+9)(2n+11) \Big(12369505812480n^{38} + 1613151061671936n^{37} + \\
& 101748284195864576n^{36} + 4135139115563745280n^{35} + 121713599527855849472n^{34} + \\
& 2765050919624810430464n^{33} + 50453046277771391664128n^{32} + 759760507477065230974976n^{31} + \\
& 9628262076527899425374208n^{30} + 104191253579306374131613696n^{29} + 973595596739520084325171200n^{28} + \\
& 7924537790312611436520013824n^{27} + 56571687381518195331462463488n^{26} + \\
& 356133102136059681954436399104n^{25} + 1985507231916669869451824553984n^{24} + \\
& 9836060321685410187563260035072n^{23} + 43406506634905372676489415905280n^{22} + \\
& 170945808151999530921656848106496n^{21} + 601507760131008511164113355409920n^{20} + \\
& 1892149418896523531194676203153920n^{19} + 532117380629233448534132495165440n^{18} + \\
& 13370912745727662541153592039812160n^{17} + 29987002021632029091547005084057760n^{16} + \\
& 59921270253255984811455083696758912n^{15} + 106434458966741189159011567116493072n^{14} + \\
& 167533688453539238956436945725341004n^{13} + 232781742346547554435545097479210510n^{12} + \\
& 284125621128876904663642986868770746n^{11} + 302806836393712159148051277734975424n^{10} + \\
& 279679164311116651162116055961513301n^9 + 221781415386984655607595031093415136n^8 + \\
& 149214365004640710156345950062395186n^7 + 83882523964213110328265187672574356n^6 + \\
& 38609679702395410742361774562392789n^5 + 14149471988638475521561721269939086n^4 + \\
& 3963748138857399502678254252169734n^3 + 795659668131014454843348852372480n^2 + \\
& 101701393436276172443717692853400n + 6204709909986751913151675960000) P[n+1]
\end{aligned}$$

$$\begin{aligned}
& +2(n+3)^2(2n+5)^3(2n+9)(2n+11) \left(24739011624960n^{40} + 3317836466356224n^{39} + 215508170284466176n^{38} + 9032884062187945984n^{37} + \right. \\
& 274636134389959884800n^{36} + 6455501959255126179840n^{35} + 122094572934385260036096n^{34} + 1909387225793663151898624n^{33} + \\
& 25180108291969215434326016n^{32} + 284171960705270647479074816n^{31} + 2775794400720227034854326272n^{30} + \\
& 23677622163992853854566219776n^{29} + 177624312783583749157935120384n^{28} + 1178515602115604757944201871360n^{27} + \\
& 6947091965313419323781358354432n^{26} + 36515023100308314818702129258496n^{25} + 171621148571344894953594594017280n^{24} + \\
& 722837793013976317556258102507520n^{23} + 2732534027077907914497042720534528n^{22} + 9281028665970648470895368668485120n^{21} + \\
& 28337819215557708948254385336117248n^{20} + 77786125749274632150536464583130752n^{19} + 191877161455672780973502244537632256n^{18} + \\
& 424953221702140663089937921965135648n^{17} + 843818276409975584824720931649555264n^{16} + \\
& 1499359936674956711935311062995422344n^{15} + 2378007025570977662661938772843220240n^{14} + \\
& 3355671771434535852147325502571953770n^{13} + 4196375762867184563407432891655585484n^{12} + \\
& 4627675779563752366067861596232781096n^{11} + 4473175960511956000526499430851993603n^{10} + \\
& 3761696365025837909581516781307249585n^9 + 2726553473467254373993685951699145492n^8 + \\
& 1683383212304999468664293798012773485n^7 + 871926653651504419744271839781064837n^6 + \\
& 371307437598003570058538796122994147n^5 + 126427972742886389602285855482966072n^4 + 33048762330145623969058704448697313n^3 + \\
& 6217924746857741077419160100404560n^2 + 748298077423337427195946099994100n + 43181089548034246077698611794000) P[n+2]
\end{aligned}$$

$$\begin{aligned}
& -2(n+4)^2(2n+5)(2n+7)^3(2n+11) \left(24739011624960n^{40} + 3322784268681216n^{39} + 216160919414112256n^{38} + 9074528155284275200n^{37} + \right. \\
& 276348048819456311296n^{36} + 6506479077331107315712n^{35} + 123266585640616142569472n^{34} + 1931040885785102661976064n^{33} + \\
& 25510503383281445462081536n^{32} + 288418124175428279391485952n^{31} + 2822442799033603081019326464n^{30} + \\
& 24120717233320712351821332480n^{29} + 181295944719289040999116701696n^{28} + 1205246297785423925076555694080n^{27} + \\
& 7119049557560114436136213413888n^{26} + 37496933571993839665392189775872n^{25} + 176616172467048982234270428880896n^{24} + \\
& 745539218875020737621728364206080n^{23} + 2824909633156578132652259733712896n^{22} + 9618101958268071244680677589035520n^{21} + \\
& 29441860528446423517613263360742912n^{20} + 81033563306363873505877563416477312n^{19} + 200454769103641040142838133702338304n^{18} + \\
& 445286624972461749049425309485328992n^{17} + 887028447418790661018847407251573152n^{16} + \\
& 1581538101499869694224895701784875304n^{15} + 2517550244392724509968791166585362672n^{14} + \\
& 3566593026520465155504695877897282630n^{13} + 4479066125207404898722179511912639638n^{12} + \\
& 4962006990874351800791769650243464872n^{11} + 4819992643914265990647887896664485209n^{10} + \\
& 407489538669418224094153822230233221n^9 + 2970477229398746689186622534784613554n^8 + \\
& 1845274131994015990683957902602775337n^7 + 962091291302144537393228847830431614n^6 + \\
& 412595107814836563208757757032740146n^5 + 141540723940232563767779647013785485n^4 + 37292931812630561528276365992452010n^3 + \\
& 7074865777225416725452872895397100n^2 + 858794112392644074221312049837000n + 49997386738260112603615104780000) P[n+3]
\end{aligned}$$

$$\begin{aligned}
& + (n+5)^3(2n+5)(2n+7)(2n+9)^4 \left(12369505812480n^{38} + 1546355730284544n^{37} + 93441851805138944n^{36} + \right. \\
& 3636063211393908736n^{35} + 102413434086873890816n^{34} + 2225107112182077718528n^{33} + \\
& 38808234188348931964928n^{32} + 558299807912629375074304n^{31} + 6755648626273815474733056n^{30} + \\
& 69769132238801205785001984n^{29} + 621900006220029229458259968n^{28} + 4826558182244413850688946176n^{27} + \\
& 32840774268722977511855751168n^{26} + 196981883700048989849717882880n^{25} + \\
& 1046061529031136798450810839040n^{24} + 4934888224954929426023144030208n^{23} + \\
& 20735286278224836075286873214976n^{22} + 77745549200390911029444008457216n^{21} + \\
& 260448286122609254214904458392064n^{20} + 780087654447729149285799146869248n^{19} + \\
& 2089276462852113795051294249728512n^{18} + 5001455921015163002705347586646080n^{17} + \\
& 10691068512696184477385875851523744n^{16} + 20374769440121072185247660725156544n^{15} + \\
& 34542976501702600883669655947085712n^{14} + 51947527795197316142253213880200764n^{13} + \\
& 69039779136078090572935768218052854n^{12} + 80712286124402599779679594199103258n^{11} + \\
& 82519759833385882007812859351392458n^{10} + 73248127158607338722648198918322201n^9 + \\
& 55935262205790259307904762197107653n^8 + 36322355479155199114489624391144238n^7 + \\
& 19756597118002557191991191826327042n^6 + 8822212911433711339358062994077203n^5 + \\
& 3145597282374650512689680780380605n^4 + 859907105684964990690798899478888n^3 + \\
& 168963309995629650025632011492580n^2 + 21205680751316222158938757272000n + \\
& 1274120732351744651125603886400) P[n+4]
\end{aligned}$$

$$\begin{aligned}
& -(n+5)^2(n+6)^4(2n+5)(2n+7)(2n+9)^3(2n+11)^4 \Big(309237645312n^{32} + 28361279668224n^{31} + \\
& 1249518729297920n^{30} + 35220794552352768n^{29} + 713726163159089152n^{28} + 11076866026783113216n^{27} + \\
& 136959486138712588288n^{26} + 1385658801437173350400n^{25} + 11691772665924577918976n^{24} + \\
& 83438339505976242995200n^{23} + 508989054278115477684224n^{22} + 2675508113418826174332928n^{21} + \\
& 12193213796145039633072128n^{20} + 48399020537651722726242304n^{19} + 167881257973769248139515904n^{18} + \\
& 510012482113388176546187776n^{17} + 1358662126092561923541267968n^{16} + 3174925021159974655053814528n^{15} + \\
& 6504205668151125355938798848n^{14} + 11663792381020901870157176128n^{13} + \\
& 18263581057905911985340656960n^{12} + 24881010123632244515458585528n^{11} + \\
& 29346856353503020415409305704n^{10} + 29775859546803351930591002266n^9 + 25770328899499991754425455738n^8 + \\
& 18817114309842270306167785140n^7 + 11424980760825630752861027739n^6 + 5656051955667821083952617134n^5 + \\
& 2221448212382554437709999491n^4 + 664859653803075491350122060n^3 + 142190920852333874895041748n^2 + \\
& 19313175036907229252501700n + 1248723341516324359641600 \Big) P[n+5] = 0
\end{aligned}$$

```
In[9]:= recSol = SolveRecurrence[rec, P[n]]
```

In[9]:= **recSol = SolveRecurrence[rec, P[n]]**

$$\begin{aligned} \text{Out[9]} = & \left\{ \left\{ 0, \frac{(3+2n)(3+4n)}{(1+n)^2(1+2n)^2} \right\} \right. \\ & \left\{ 0, -\frac{(3+2n)(-8-9n+2n^2)}{(1+n)^2(1+2n)^2} \right\} \\ & \left\{ 0, -\frac{(3+2n)(-5+8n^2)}{2(1+n)^2(1+2n)^2} + \frac{(3+2n) \sum_{i=1}^n \frac{1}{i}}{(1+n)(1+2n)} + \frac{2(3+2n) \sum_{i=1}^n \frac{1}{-1+2i}}{(1+n)(1+2n)} \right\} \\ & \left\{ 0, \frac{(3+2n)(-513-2184n-2416n^2+768n^4)}{2(1+n)^3(1+2n)^3} + \frac{14(3+2n) \sum_{i=1}^n \frac{1}{i^2}}{(1+n)(1+2n)} + \left(-\frac{2(3+2n)(3+44n+48n^2)}{(1+n)^2(1+2n)^2} + \frac{48(3+2n) \sum_{i=1}^n \frac{1}{-1+2i}}{(1+n)(1+2n)} \right) \sum_{i=1}^n \frac{1}{i} + \right. \\ & \left. \frac{12(3+2n) \left(\sum_{i=1}^n \frac{1}{i} \right)^2}{(1+n)(1+2n)} + \frac{56(3+2n) \sum_{i=1}^n \frac{1}{(-1+2i)^2}}{(1+n)(1+2n)} - \right. \\ & \left. \frac{4(3+2n)(3+44n+48n^2) \sum_{i=1}^n \frac{1}{-1+2i}}{(1+n)^2(1+2n)^2} + \frac{48(3+2n) \left(\sum_{i=1}^n \frac{1}{-1+2i} \right)^2}{(1+n)(1+2n)} \right\} \end{aligned}$$

$$\begin{aligned}
& \{0, \frac{1}{16(1+n)^4(1+2n)^4} (72519 + 572343n + 1814716n^2 + 2918100n^3 + 2442240n^4 + 912896n^5 + 24576n^6 - \\
& 49152n^7) + \frac{16(3+2n) \sum_{i=1}^n \frac{1}{i^3}}{3(1+n)(1+2n)} + (-\frac{(3+2n)(29+307n+322n^2)}{4(1+n)^2(1+2n)^2} + \frac{44(3+2n) \sum_{i=1}^n \frac{1}{-1+2i}}{(1+n)(1+2n)}) \sum_{i=1}^n \frac{1}{i^2} + \\
& (\frac{(3+2n)(91+259n+974n^2+1784n^3+1024n^4)}{4(1+n)^3(1+2n)^3} + \frac{22(3+2n) \sum_{i=1}^n \frac{1}{i^2}}{(1+n)(1+2n)} + \frac{24(3+2n) \sum_{i=1}^n \frac{1}{(-1+2i)^2}}{(1+n)(1+2n)} - \\
& \frac{4(3+2n)(-13-4n+16n^2) \sum_{i=1}^n \frac{1}{-1+2i}}{(1+n)^2(1+2n)^2} + \frac{16(3+2n)(\sum_{i=1}^n \frac{1}{-1+2i})^2}{(1+n)(1+2n)}) \sum_{i=1}^n \frac{1}{i} + (- \\
& \frac{(3+2n)(19+92n+80n^2)}{(1+n)^2(1+2n)^2} + \frac{40(3+2n) \sum_{i=1}^n \frac{1}{-1+2i}}{(1+n)(1+2n)}) (\sum_{i=1}^n \frac{1}{i})^2 + \frac{20(3+2n)(\sum_{i=1}^n \frac{1}{i})^3}{3(1+n)(1+2n)} + \\
& \frac{64(3+2n) \sum_{i=1}^n \frac{1}{(-1+2i)^3}}{3(1+n)(1+2n)} - \frac{3(3+2n)(63+209n+150n^2) \sum_{i=1}^n \frac{1}{(-1+2i)^2}}{(1+n)^2(1+2n)^2} + \\
& (\frac{(3+2n)(347+1795n+4302n^2+4856n^3+2048n^4)}{2(1+n)^3(1+2n)^3} + \frac{48(3+2n) \sum_{i=1}^n \frac{1}{(-1+2i)^2}}{(1+n)(1+2n)}) \sum_{i=1}^n \frac{1}{-1+2i} - \\
& \frac{4(3+2n)(19+92n+80n^2)(\sum_{i=1}^n \frac{1}{-1+2i})^2}{(1+n)^2(1+2n)^2} + \frac{32(3+2n)(\sum_{i=1}^n \frac{1}{-1+2i})^3}{3(1+n)(1+2n)} - \\
& \frac{8(3+2n) \sum_{i=1}^n \frac{(\sum_{j=1}^i \frac{1}{j})^2}{i}}{(1+n)(1+2n)} - \frac{16(3+2n) \sum_{i=1}^n \frac{(\sum_{j=1}^i \frac{1}{j})^2}{-1+2i}}{(1+n)(1+2n)} \\
& - \frac{32(3+2n) \sum_{i=1}^n \frac{(\sum_{j=1}^i \frac{1}{j}) \sum_{j=1}^i \frac{1}{-1+2j}}{i}}{(1+n)(1+2n)} + \frac{64(3+2n) \sum_{i=1}^n \frac{(\sum_{j=1}^i \frac{1}{j}) \sum_{j=1}^i \frac{1}{-1+2j}}{-1+2i}}{(1+n)(1+2n)} + \\
& \frac{32(3+2n) \sum_{i=1}^n \frac{(\sum_{j=1}^i \frac{1}{-1+2j})^2}{i}}{(1+n)(1+2n)} + \frac{64(3+2n) \sum_{i=1}^n \frac{(\sum_{j=1}^i \frac{1}{-1+2j})^2}{-1+2i}}{(1+n)(1+2n)} \}, \{1, 0\} \}
\end{aligned}$$

```
In[10]:= sol = FindLinearCombination[recSol, {0, initial}, n, 7, MinInitialValue → 1]
```

In[10]:= sol = FindLinearCombination[recSol, {0, initial}, n, 7, MinInitialValue → 1]

$$\begin{aligned} \text{Out}[10]= & \frac{1}{3(1+n)^4(1+2n)^4} (111 + 1920n + 11765n^2 + 32545n^3 + 46476n^4 + 35376n^5 + 13440n^6 + 1968n^7) + \frac{32(3+2n) \sum_{i=1}^n \frac{1}{i^3}}{9(1+n)(1+2n)} - \\ & \frac{(3+2n)(-3+101n+126n^2) \sum_{i=1}^n \frac{1}{i^2}}{(3+2n)(115+921n+1967n^2+1524n^3+340n^4) \sum_{i=1}^n \frac{1}{i}} + \\ & \frac{3(1+n)^2(1+2n)^2}{44(3+2n)(\sum_{i=1}^n \frac{1}{i^2}) \sum_{i=1}^n \frac{1}{i}} - \frac{3(1+n)^3(1+2n)^3}{(3+2n)(23+139n+130n^2)(\sum_{i=1}^n \frac{1}{i})^2} + \frac{40(3+2n)(\sum_{i=1}^n \frac{1}{i})^3}{4(3+2n)(77+261n+190n^2) \sum_{i=1}^n \frac{1}{(-1+2i)^2}} + \\ & \frac{3(1+n)(1+2n)}{128(3+2n) \sum_{i=1}^n \frac{1}{(-1+2i)^3}} - \frac{3(1+n)^2(1+2n)^2}{4(3+2n)(77+261n+190n^2) \sum_{i=1}^n \frac{1}{(-1+2i)^2}} + \frac{9(1+n)(1+2n)}{16(3+2n)(\sum_{i=1}^n \frac{1}{i}) \sum_{i=1}^n \frac{1}{(-1+2i)^2}} + \\ & \frac{9(1+n)(1+2n)}{2(3+2n)(13-153n-303n^2+12n^3+172n^4) \sum_{i=1}^n \frac{1}{-1+2i}} + \frac{88(3+2n)(\sum_{i=1}^n \frac{1}{i^2}) \sum_{i=1}^n \frac{1}{-1+2i}}{(1+n)(1+2n)} - \\ & \frac{3(1+n)^3(1+2n)^3}{4(3+2n)(-41-53n+2n^2)(\sum_{i=1}^n \frac{1}{i}) \sum_{i=1}^n \frac{1}{-1+2i}} + \frac{80(3+2n)(\sum_{i=1}^n \frac{1}{i})^2 \sum_{i=1}^n \frac{1}{-1+2i}}{3(1+n)^2(1+2n)^2} + \\ & \frac{32(3+2n)(\sum_{i=1}^n \frac{1}{(-1+2i)^2}) \sum_{i=1}^n \frac{1}{-1+2i}}{4(3+2n)(23+139n+130n^2)(\sum_{i=1}^n \frac{1}{-1+2i})^2} + \\ & \frac{32(3+2n)(\sum_{i=1}^n \frac{1}{i})(\sum_{i=1}^n \frac{1}{-1+2i})^2}{64(3+2n)(\sum_{i=1}^n \frac{1}{-1+2i})^3} - \frac{16(3+2n) \sum_{i=1}^n \frac{(\sum_{j=1}^i \frac{1}{j})^2}{i}}{3(1+n)(1+2n)} - \\ & \frac{32(3+2n) \sum_{i=1}^n \frac{(\sum_{j=1}^i \frac{1}{j})^2}{-1+2i}}{64(3+2n) \sum_{i=1}^n \frac{(\sum_{j=1}^i \frac{1}{j}) \sum_{j=1}^i \frac{1}{-1+2j}}{i}} + \\ & \frac{32(3+2n) \sum_{i=1}^n \frac{(\sum_{j=1}^i \frac{1}{j}) \sum_{j=1}^i \frac{1}{-1+2j}}{-1+2i}}{64(3+2n) \sum_{i=1}^n \frac{(\sum_{j=1}^i \frac{1}{-1+2j})^2}{i}} + \\ & \frac{128(3+2n) \sum_{i=1}^n \frac{(\sum_{j=1}^i \frac{1}{j}) \sum_{j=1}^i \frac{1}{-1+2j}}{-1+2i}}{64(3+2n) \sum_{i=1}^n \frac{(\sum_{j=1}^i \frac{1}{-1+2j})^2}{i}} + \\ & \frac{128(3+2n) \sum_{i=1}^n \frac{(\sum_{j=1}^i \frac{1}{-1+2j})^2}{-1+2i}}{3(1+n)(1+2n)} \end{aligned}$$

```
In[11]:= << HarmonicSums.m
```

```
HarmonicSums by Jakob Ablinger © RISC-Linz
```

```
In[12]:= sol = TransformToSSums[sol];
```

```
In[13]:= sol = ReduceToBasis[MultipleSumLimit[sol,  
n, 2]//ToStandardForm, n]//CollectProdSum;
```

In[11]:= << HarmonicSums.m

HarmonicSums by Jakob Ablinger © RISC-Linz

In[12]:= sol = TransformToSSums[sol];

In[13]:= sol = ReduceToBasis[MultipleSumLimit[sol,
n, 2]//ToStandardForm, n]//CollectProdSum;

$$\begin{aligned} \text{Out[13]} = & \frac{1}{3(1+n)^4(1+2n)^4} (111 + 1920n + 11765n^2 + 32545n^3 + 46476n^4 + 35376n^5 + 13440n^6 + \\ & 1968n^7) + \frac{64(3+2n)^2 S[1, n]}{3(1+n)(1+2n)^2} + \frac{64(3+2n)(2+3n) S[1, n]^2}{3(1+n)(1+2n)^2} + (- \\ & \frac{2(3+2n)(147 + 985n + 1871n^2 + 1268n^3 + 212n^4)}{3(1+n)^3(1+2n)^3} + \frac{224(3+2n) S[2, 2n]}{3(1+n)(1+2n)} + \\ & \frac{128(3+2n) S[-2, 2n]}{3(1+n)(1+2n)}) S[1, 2n] - \frac{4(3+2n)(23 + 123n + 114n^2) S[1, 2n]^2}{3(1+n)^2(1+2n)^2} + \\ & \frac{64(3+2n) S[1, 2n]^3}{3(1+n)(1+2n)} + \frac{64(3+2n) S[2, n]}{3(1+n)(1+2n)} - \frac{4(3+2n)(53 + 229n + 190n^2) S[2, 2n]}{3(1+n)^2(1+2n)^2} + \\ & \frac{64(3+2n) S[3, 2n]}{3(1+n)(1+2n)} + (- \frac{64(3+2n)^2}{3(1+n)(1+2n)^2} - \frac{128(3+2n)(2+3n) S[1, 2n]}{3(1+n)(1+2n)^2}) S[-1, 2n] - \\ & \frac{64(3+2n)(2+3n) S[-1, 2n]^2}{3(1+n)(1+2n)^2} - \frac{32(3+2n)(1+8n+8n^2) S[-2, 2n]}{3(1+n)^2(1+2n)^2} + \\ & \frac{64(3+2n) S[-3, 2n]}{3(1+n)(1+2n)} - \frac{128(3+2n) S[-2, 1, 2n]}{3(1+n)(1+2n)} \end{aligned}$$

```
In[11]:= << HarmonicSums.m
```

```
HarmonicSums by Jakob Ablinger © RISC-Linz
```

```
In[12]:= sol = TransformToSSums[sol];
```

```
In[13]:= sol = ReduceToBasis[MultipleSumLimit[sol,  
n, 2]//ToStandardForm, n]//CollectProdSum;
```

```
In[14]:= SExpansion[sol, n, 2]
```

```
Out[14]= ln2^2 ( (64LG[n])/n + 160/(3n^2) - 44/n ) +  
ln2 ( ( (320/(3n^2) - 88/n) LG[n] + 64LG[n]^2/n - 430/(3n^2) + 160ζ2/3n - 14/n ) +  
ζ2 ( (160LG[n]/3n + 40/n^2 - 84/n) + (160/(3n^2) - 44/n) LG[n]^2 + ( -430/(3n^2) - 14/n ) LG[n] + 64LG[n]^3/3n +  
64ln2^3/3n + 145/(2n^2) + 32ζ3/n + 41/n
```

```
In[11]:= << HarmonicSums.m
```

HarmonicSums by Jakob Ablinger © RISC-Linz

```
In[12]:= sol = TransformToSSums[sol];
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n, 2]//ToStandardForm, n]//CollectProdSum;
```

```
In[14]:= SExpansion[sol, n, 2]
```

$$\begin{aligned} \text{Out[14]} = & \ln^2 \left(\frac{64 \text{LG}[n]}{n} + \frac{160}{3n^2} - \frac{44}{n} \right) + \\ & \ln 2 \left(\left(\frac{320}{3n^2} - \frac{88}{n} \right) \text{LG}[n] + \frac{64 \text{LG}[n]^2}{n} - \frac{430}{3n^2} + \frac{160 \zeta_2}{3n} - \frac{14}{n} \right) + \\ & \zeta_2 \left(\frac{160 \text{LG}[n]}{3n} + \frac{40}{n^2} - \frac{84}{n} \right) + \left(\frac{160}{3n^2} - \frac{44}{n} \right) \text{LG}[n]^2 + \left(-\frac{430}{3n^2} - \frac{14}{n} \right) \text{LG}[n] + \frac{64 \text{LG}[n]^3}{3n} + \\ & \frac{64 \ln^2 3}{3n} + \frac{145}{2n^2} + \frac{32 \zeta_3}{n} + \frac{41}{n} \end{aligned}$$

Special function algorithms

► HarmonicSums package

Ablinger, Blümlein, CS, J. Math. Phys. 54, 2013, arXiv:1302.0378 [math-ph]

Ablinger, Blümlein, CS, J. Math. Phys. 52, 2011, arXiv:1302.0378 [math-ph]

Ablinger, Blümlein, CS, ACAT 2013, arXiv:1310.5645 [math-ph]

Ablinger, Blümlein, Raab, CS, J. Math. Phys. 55, 2014, arXiv:1407.1822 [hep-th]

► RICA package

Blümlein, Fadeev, CS. ACM Communications in Computer Algebra 57(2), pp. 31-34. 2023.

coupled systems

$$\text{for } f(x) = \sum_{n=0}^{\infty} P(n)x^n$$

SolveCoupledSystem.m



large no. of moments,
say $P(0), \dots, P(10000)$

guessing (ore_algebra in Sage)



recurrence

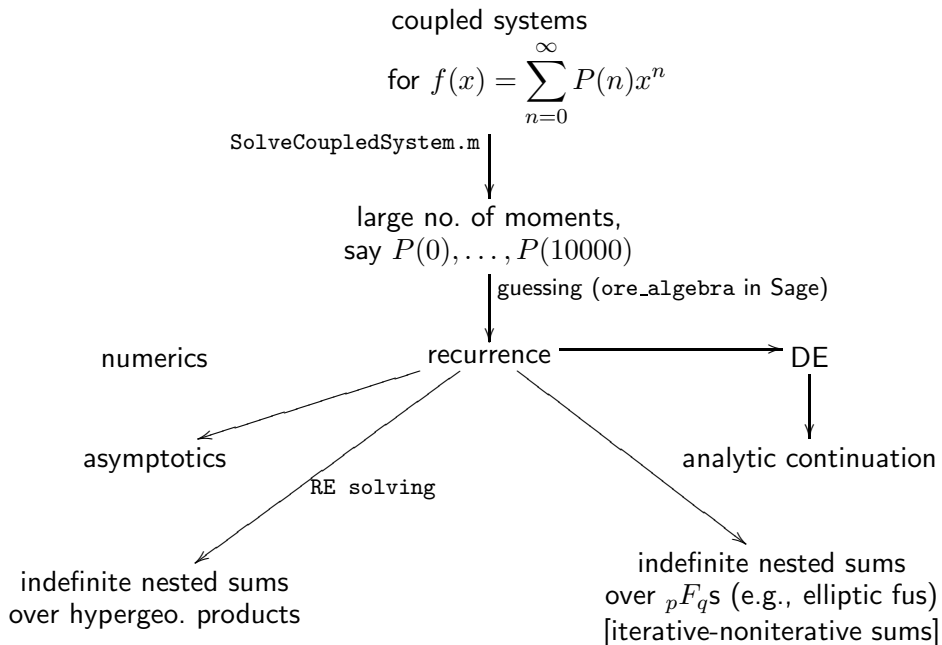
numerics

asymptotics

RE solving

indefinite nested sums
over hypergeo. products

indefinite nested sums
over ${}_pF_q$ s (e.g., elliptic fus)
[iterative-noniterative sums]



The easy (quarkonic) case of our Form Factor project

Evaluate beyond 0

convergency
radius

$$r = 1$$

0

1

$$\sum_{n=0}^{\infty} f_n (1-x)^n$$

given $f_n \in \mathbb{Q}$
by a guessed rec
of order 55

The easy (quarkonic) case of our Form Factor project

Matching evaluations at a common point x



$$\sum_{j=0}^6 \log(x)^j \sum_{n=-2}^{\infty} g_{j,n} x^n$$

find $g_{j,n} \in \mathbb{R}$

$$\sum_{n=0}^{\infty} f_n (1-x)^n$$

given $f_n \in \mathbb{Q}$
by a guessed rec
of order 55

The easy (quarkonic) case of our Form Factor project

Matching evaluations at a common point x

$$r = 0.078$$

convergency
radius

$$r = 1$$

0

x

1

$$\sum_{j=0}^6 \log(x)^j \sum_{n=-2}^{\infty} g_{j,n} x^n$$

$$\sum_{n=0}^{\infty} f_n (1-x)^n$$

find $g_{j,n} \in \mathbb{R}$

given $f_n \in \mathbb{Q}$
by a guessed rec
of order 55

The easy (quarkonic) case of our Form Factor project

Matching evaluations at a common point x

$$r = 0.078$$

convergency
radius

$$r = 1$$

$$0$$

$$x < 0.078$$

$$1$$

$$\sum_{j=0}^6 \log(x)^j \sum_{n=-2}^{\infty} g_{j,n} x^n$$

$$\sum_{n=0}^{\infty} f_n (1-x)^n$$

find $g_{j,n} \in \mathbb{R}$

given $f_n \in \mathbb{Q}$
by a guessed rec
of order 55

The easy (quarkonic) case of our Form Factor project

Matching evaluations at a common point x

$$r = 0.078$$

convergency
radius

$$r = 1$$

$$0$$

$$x < 0.078$$

$$1$$

$$\sum_{j=0}^6 \log(x)^j \sum_{n=-2}^{\infty} g_{j,n} x^n$$

$$\sum_{n=0}^{500000} f_n (1-x)^n$$

find $g_{j,n} \in \mathbb{R}$

given $f_n \in \mathbb{Q}$
by a guessed rec
of order 55

The easy (quarkonic) case of our Form Factor project

Matching evaluations at a common point x

$$r = 0.078$$

convergency
radius

$$r = 1$$

$$0$$

$$x < 0.078$$

$$1$$

$$\sum_{j=0}^6 \log(x)^j \sum_{n=-2}^{10000} g_{j,n} x^n$$

$$\sum_{n=0}^{500000} f_n (1-x)^n$$

find $g_{j,n} \in \mathbb{R}$

given $f_n \in \mathbb{Q}$
by a guessed rec
of order 55

The easy (quarkonic) case of our Form Factor project

Matching evaluations at a common point x

$$r = 0.078$$

convergency
radius

$$r = 1$$

$$0$$

$$x < 0.078$$

$$1$$

$$\sum_{j=0}^6 \log(x)^j \sum_{n=-2}^{10000} g_{j,n} x^n$$

$$\text{DE (order 48, deg 2800)} \quad \sum_{n=0}^{500000} f_n (1-x)^n$$

find $g_{j,n} \in \mathbb{R}$

given $f_n \in \mathbb{Q}$
by a guessed rec
of order 55

The easy (quarkonic) case of our Form Factor project

Matching evaluations at a common point x

$$r = 0.078$$

convergency
radius

$$r = 1$$

$$0$$

$$x = 0.005$$

$$1$$

$$\sum_{j=0}^6 \log(x)^j \sum_{n=-2}^{10000} g_{j,n} x^n$$

$$\text{DE (order 48, deg 2800)} \quad \sum_{n=0}^{500000} f_n (1-x)^n$$

find $g_{j,n} \in \mathbb{R}$
1400 digits precision

given $f_n \in \mathbb{Q}$
by a guessed rec
of order 55

The easy (quarkonic) case of our Form Factor project

Matching evaluations at a common point x

$$r = 0.078$$

convergency
radius

$$r = 1$$

$$0$$

$$x = 0.005$$

$$1$$

$$\sum_{j=0}^6 \log(x)^j \sum_{n=-2}^{10000} g_{j,n} x^n$$

$$\text{DE (order 48, deg 2800)} \quad \sum_{n=0}^{500000} f_n (1-x)^n$$

find $g_{j,n} \in \mathbb{R}$
1400 digits precision

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PSLQ

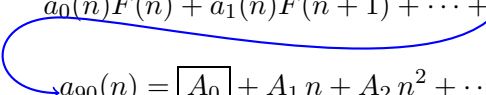
$$g_{j,n} \in \mathbb{Q}(\pi, \zeta_3, \dots)$$

World record recurrence:

$$a_0(n)F(n) + a_1(n)F(n+1) + \cdots + \boxed{a_{90}(n)}F(n+90) = 0$$

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$$a_0(n)F(n) + a_1(n)F(n+1) + \cdots + \boxed{a_{90}(n)}F(n+90) = 0$$


$$a_{90}(n) = \boxed{A_0} + A_1 n + A_2 n^2 + \cdots + A_{3900} n^{3900} \in \mathbb{Z}[n]$$

~650K A4-pages

[illegible]

(8084 decimal digits

Conclusion

1. Various **computer algebra** tools have been developed at RISC:
 - ▶ multi-summation and integration packages
 - ▶ up-to-date solvers for linear recurrences and DEs
(within `Sigma.m` and `HarmonicSums.m`)
 - ▶ a prototype method to solve partial linear DE/RE equations in QCD

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5. Results are contained in about 100 articles produced jointly within the RISC–DESY cooperation