Sharing Pizza in n Dimensions

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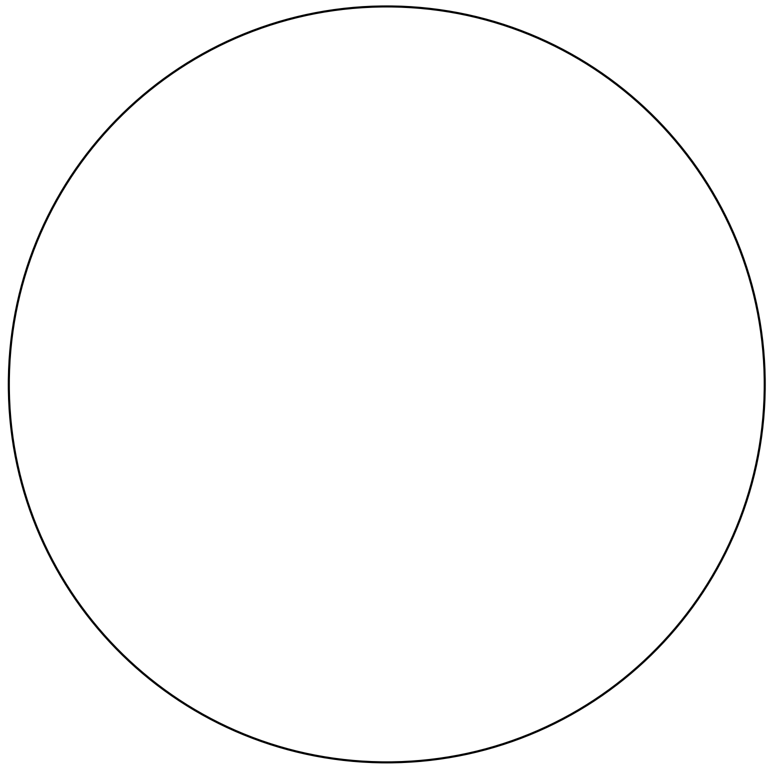
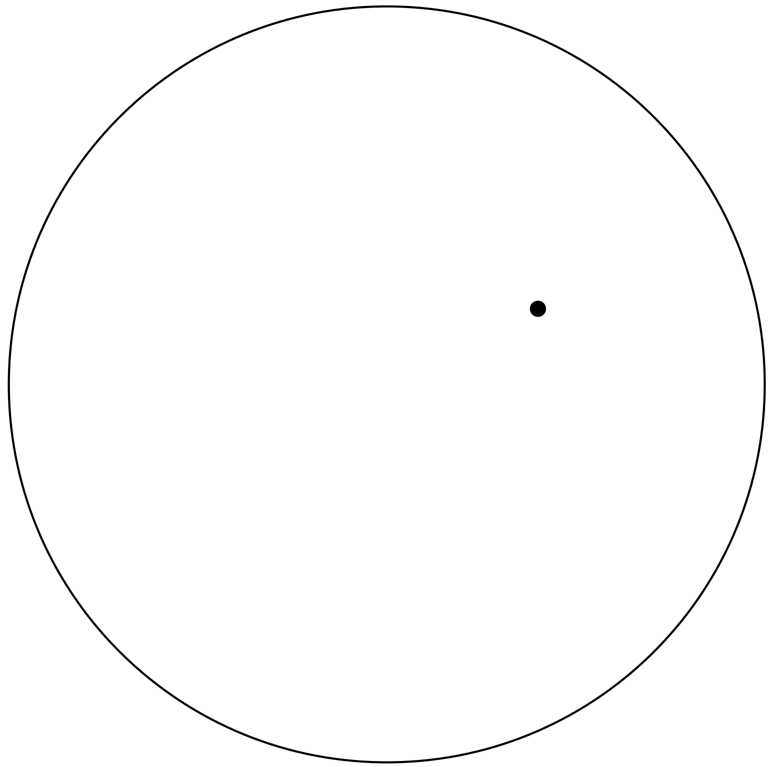
Pizza

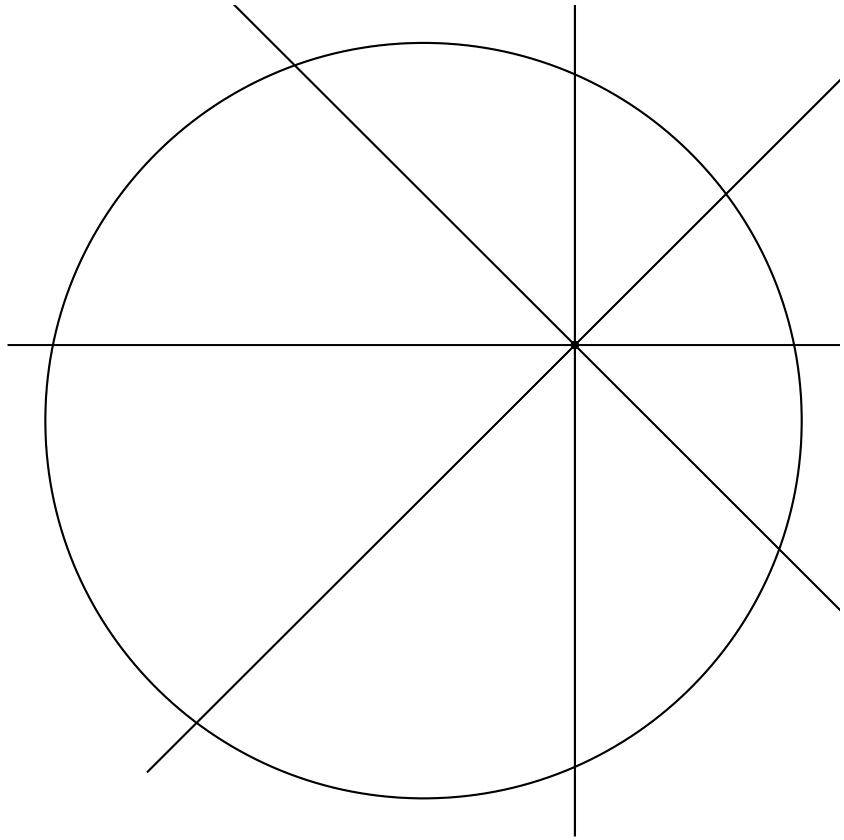
Pizza



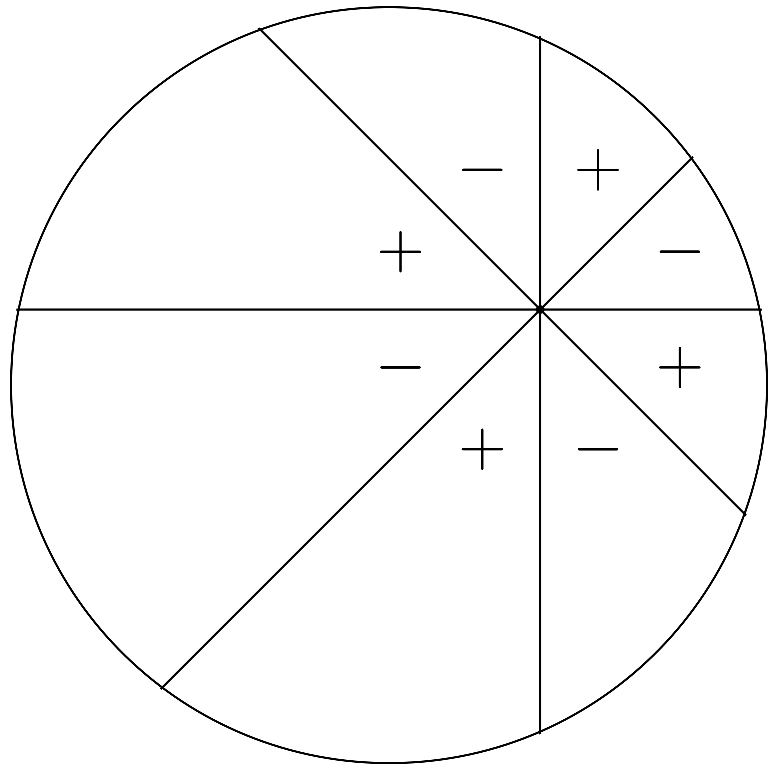
Pick any point



Cut with four equidistributed lines



Pizza Theorem: Alternating sum of the areas is equal to 0.



History

1967 Upton stated the problem in *Mathematics Magazine*

1968 Goldberg solved the problem and stated the result for $2k$ equidistributed lines, $k \geq 2$

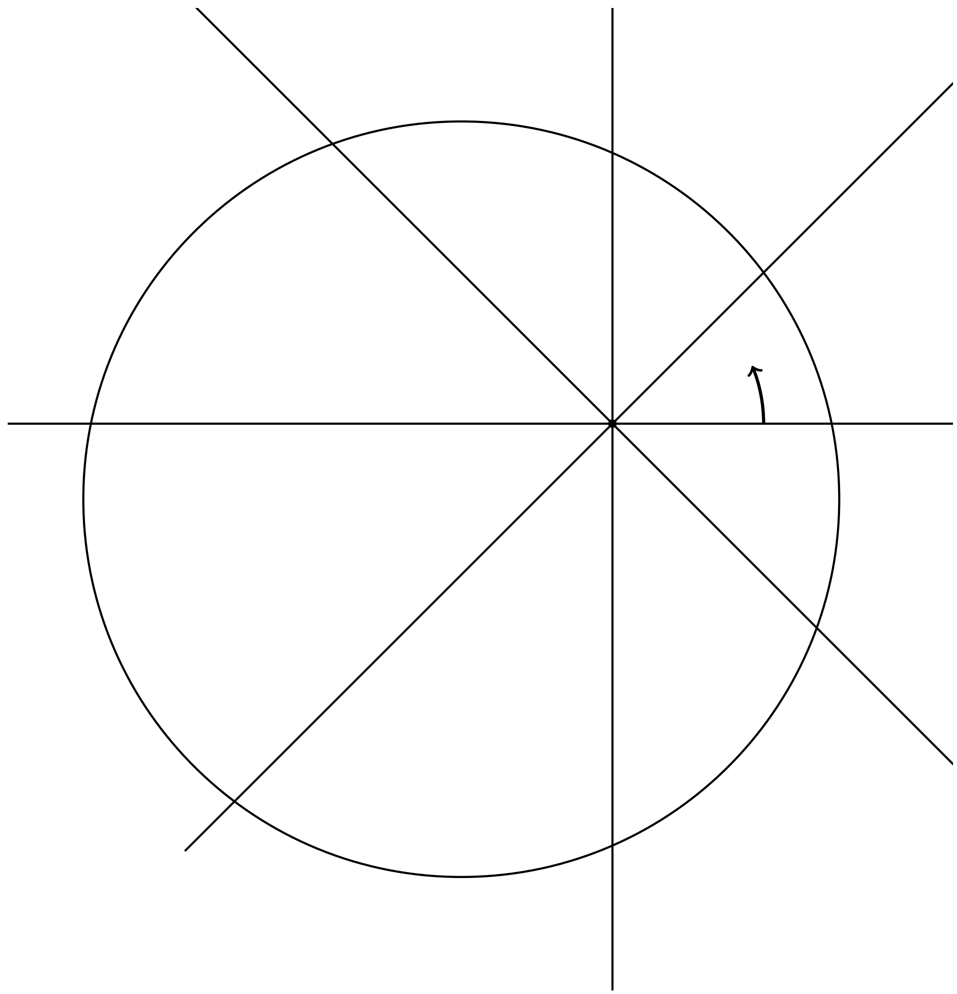
1994 Carter and Wagon gave a dissection proof

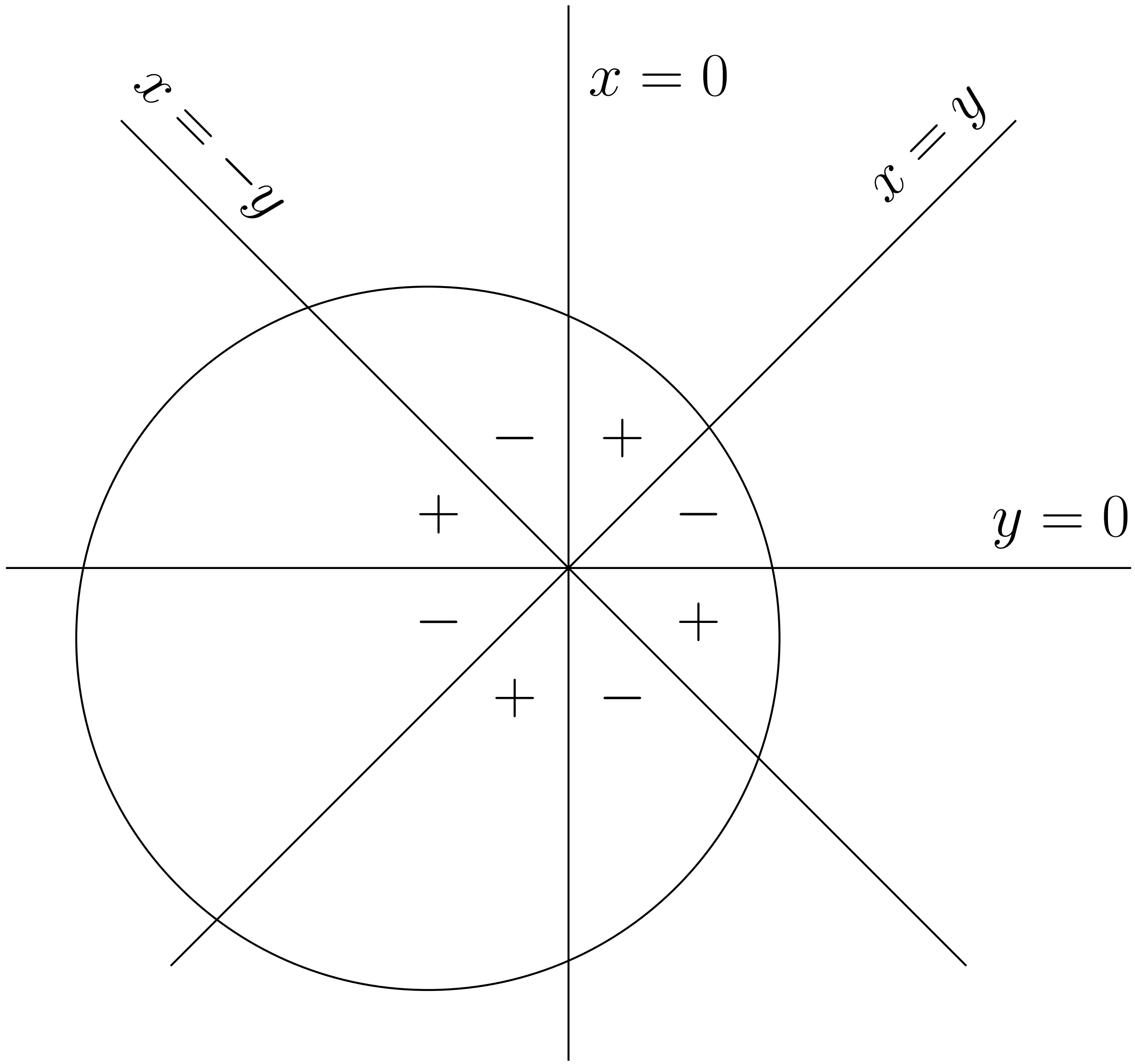
1999 Hirschhorn, Hirschhorn, Hirschhorn, Hirschhorn and Hirschhorn considered p people sharing pizza

2009 Mabry and Deiermann showed the result fails for an odd number of equidistributed lines

2012 Frederickson provided dissection proofs

Classical proof





$x = 0$
$x = -y$
$x = y$
$y = 0$
$-$
$+$
$+$
$-$
$-$
$+$
$+$
$-$

Higher dimensions?

V real vector space of dimension n with inner product $(\cdot, \cdot)$

Index set E finite set of unit vectors such that $E \cap (-E) = \emptyset$

Hyperplane $H_e = \{v \in V : (v, e) = 0\}$

Hyperplane arrangement $\mathcal{H} = \{H_e\}_{e \in E}$

A chamber T is a connected component of $V - \bigcup_{e \in E} H_e$

$\mathscr{T}$ set of all chambers

Pick T_0 base chamber

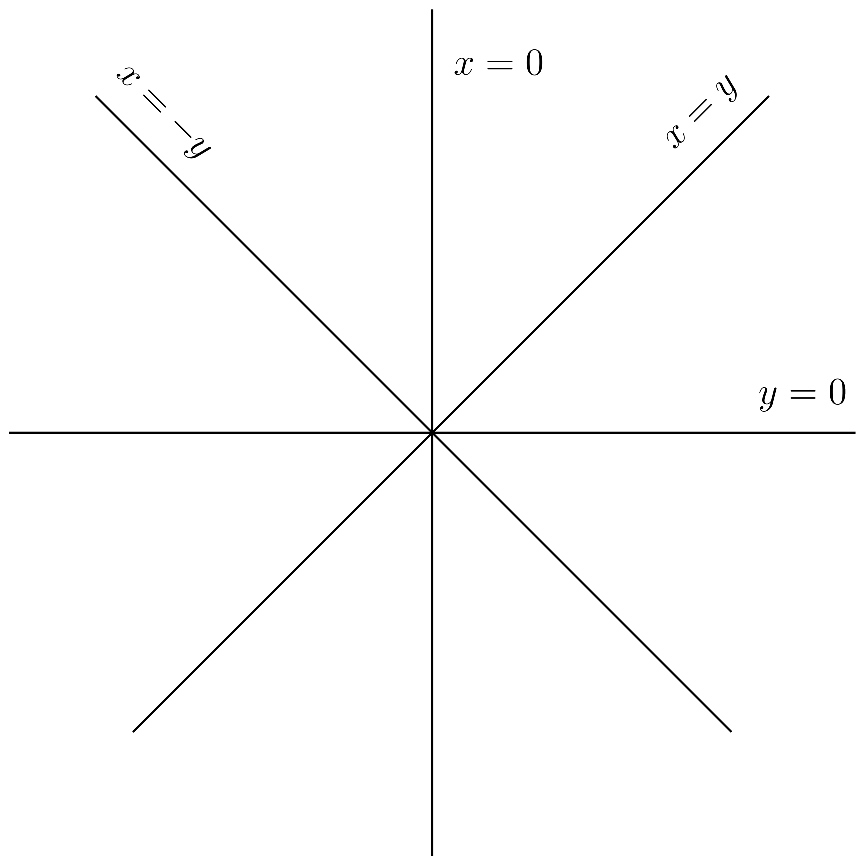
Sign $(-1)^T = (-1)^k$ where k is the number of hyperplanes separating T from T_0

Pizza quantity

$$P(\mathcal{H}, K) = \sum_{T \in \mathscr{T}} (-1)^{T} \operatorname{Vol}(K \cap T)$$

$\mathcal{H}$ is a *Coxeter arrangement* if

– the group W generated by the orthogonal reflections in the hyperplanes of $\mathcal{H}$ is finite and
– the arrangement is closed under all such reflections



$\mathcal{H}_i$ arrangement in V_i

$\mathcal{H}_1 \times \mathcal{H}_2$ arrangement in $V_1 \times V_2$ with hyperplanes

$$\{H \times V_2 : H \in \mathcal{H}_1\} \cup \{V_1 \times H : H \in \mathcal{H}_2\}$$

$\mathcal{H}_1$ and $\mathcal{H}_2$ Coxeter $\Longrightarrow \mathcal{H}_1 \times \mathcal{H}_2$ Coxeter

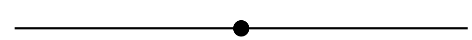
Type A_n

$$V = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1 + x_2 + \cdots + x_{n+1} = 0\}$$

$$\mathcal{H} = \{x_i = x_j : 1 \leq i < j \leq n+1\}$$

Symmetries of the n-dimensional simplex

A_1



$$\begin{aligned} A_1^n &= A_1 \times A_1 \times \cdots \times A_1 \\ &= \{x_i = 0 : 1 \leq i \leq n\} \end{aligned}$$

Type B_n (and type C_n) $\quad n \geq 2$

$$V = \mathbb{R}^n$$

$$\mathcal{H} = \{x_i = 0 : 1 \leq i \leq n\} \cup \{x_i = \pm x_j : 1 \leq i < j \leq n\}$$

Symmetries of the n-dimensional cube and crosspolytope

Type D_n $\quad n \geq 4$

$$V = \mathbb{R}^n$$

$$\mathcal{H} = \{x_i = \pm x_j : 1 \leq i < j \leq n\}$$

$D_2 = A_1^2 \qquad D_3 = A_3$

Type E_6, E_7 and E_8

Type F_4

$$V = \mathbb{R}^4$$

$$\begin{aligned}\mathcal{H} = &\{x_i = 0 : 1 \leq i \leq 4\} \\ &\cup \{x_i = \pm x_j : 1 \leq i < j \leq 4\} \\ &\cup \{x_1 \pm x_2 \pm x_3 \pm x_4 = 0\}\end{aligned}$$

F_4 = symmetries of the 24-cell

Type G_2

$G_2 = I_2(6)$

Type H_3 and H_4

$H_3 =$ symmetries of the dodecahedron and the icosahedron

$H_4 =$ symmetries of the 120-cell and 600-cell

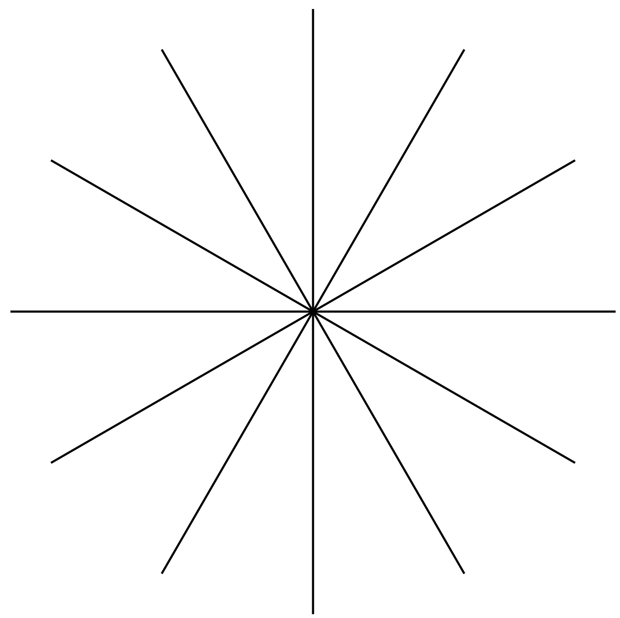
Do not arise from root systems

Type $I_2(k)$ $\qquad k \geq 2$

$I_2(k)$ = symmetries of the k-gon

$I_2(k)$ consists of k lines

$I_2(2) = A_1^2 \qquad I_2(3) = A_2 \qquad I_2(4) = B_2$



$$\mathbb{B}(a,R) = \{x \in V : \|x - a\| \leq R\}.$$

Theorem [Goldberg] Let $\mathcal{H}$ be the dihedral arrangement $I_2(2k)$ in $\mathbb{R}^2$ for $k \geq 2$. For every $a \in \mathbb{R}^2$ such that $0 \in \mathbb{B}(a,R)$, the pizza quantity for the ball $\mathbb{B}(a,R)$ vanishes:

$$P(\mathcal{H}, \mathbb{B}(a,R)) = 0.$$

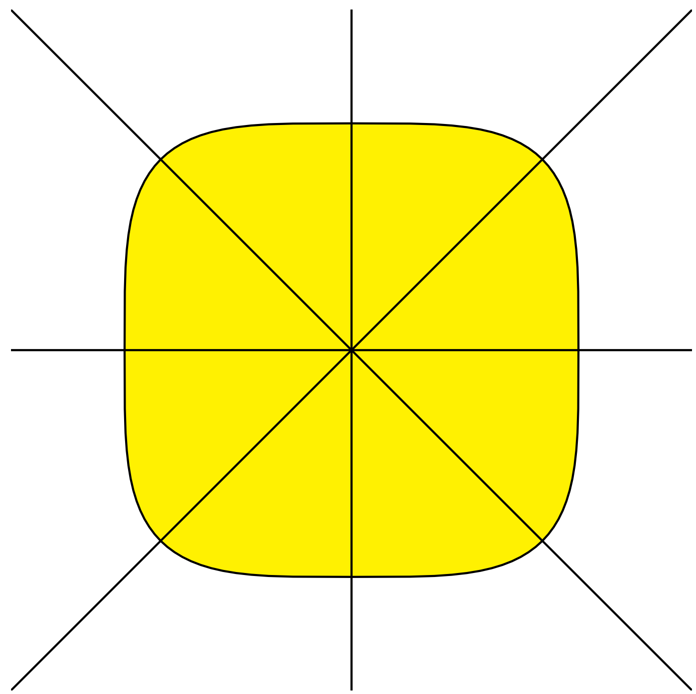
Theorem [Frederickson] Let $\mathcal{H}$ be of type $A_1 \times I_2(2k)$ in $\mathbb{R}^3$ for $k \geq 2$. For every $a \in \mathbb{R}^3$ such that $0 \in \mathbb{B}(a,R)$, the pizza quantity for the ball $\mathbb{B}(a,R)$ vanishes:

$$P(\mathcal{H}, \mathbb{B}(a,R)) = 0.$$

A set $K \subseteq V$ is stable under the group W if

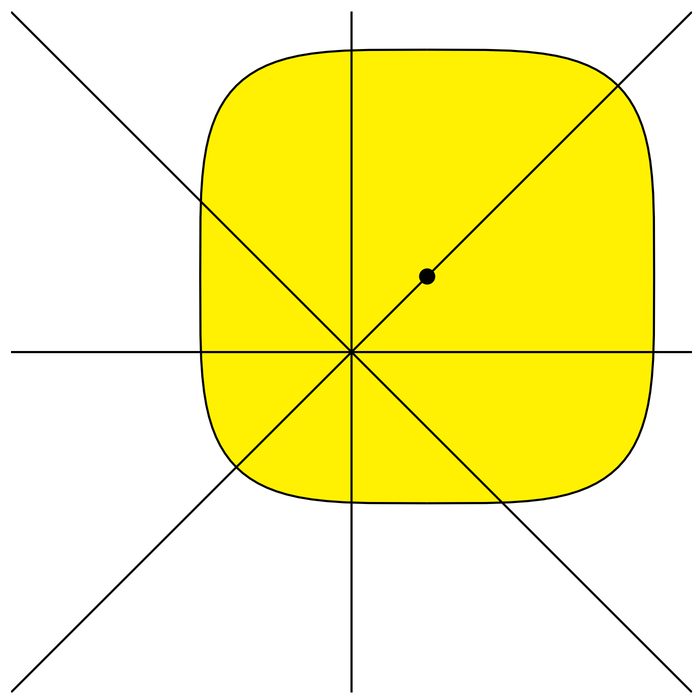
$$w(K) = K$$

for all $w \in W$



Lemma. $\mathcal{H}$ Coxeter arrangement with group W. If K is stable under W and $a \in H \in \mathcal{H}$ then

$$P(\mathcal{H}, K + a) = 0$$



Theorem. Let $\mathcal{H}$ be a Coxeter arrangement on V such that the map $-\mathrm{id}_V$ belongs to the Coxeter group W and where we furthermore assume that $\mathcal{H}$ is not of type A_1^n. Then the pizza quantity of $\mathcal{H}$ vanishes, that is, for a convex set K containing the origin that is the translate of a set stable under reflections in the hyperplanes of the arrangement $\mathcal{H}$, the identity $P(\mathcal{H}, K) = 0$ holds.

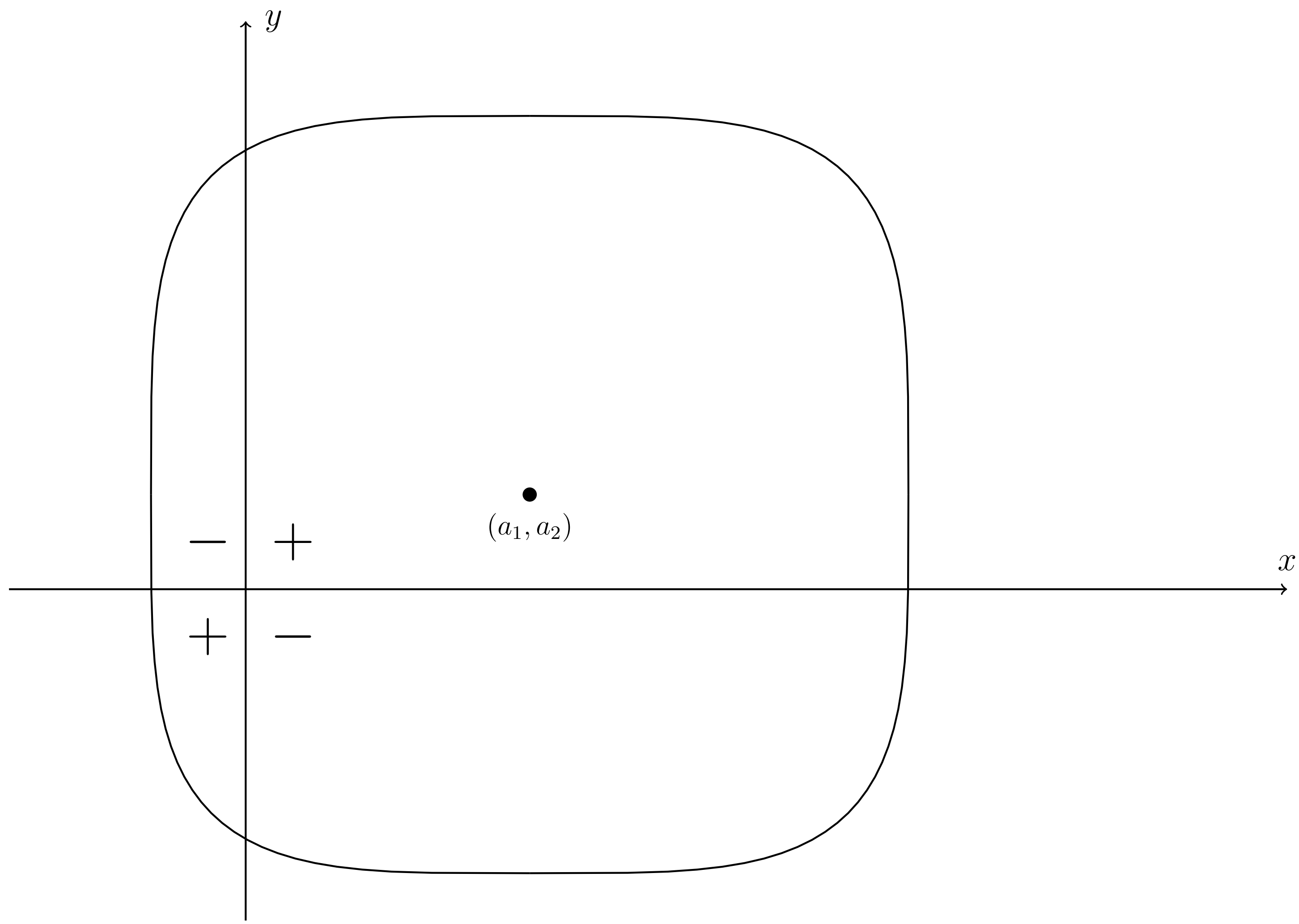
$$-\operatorname{id}_V \in W$$

$$\iff$$

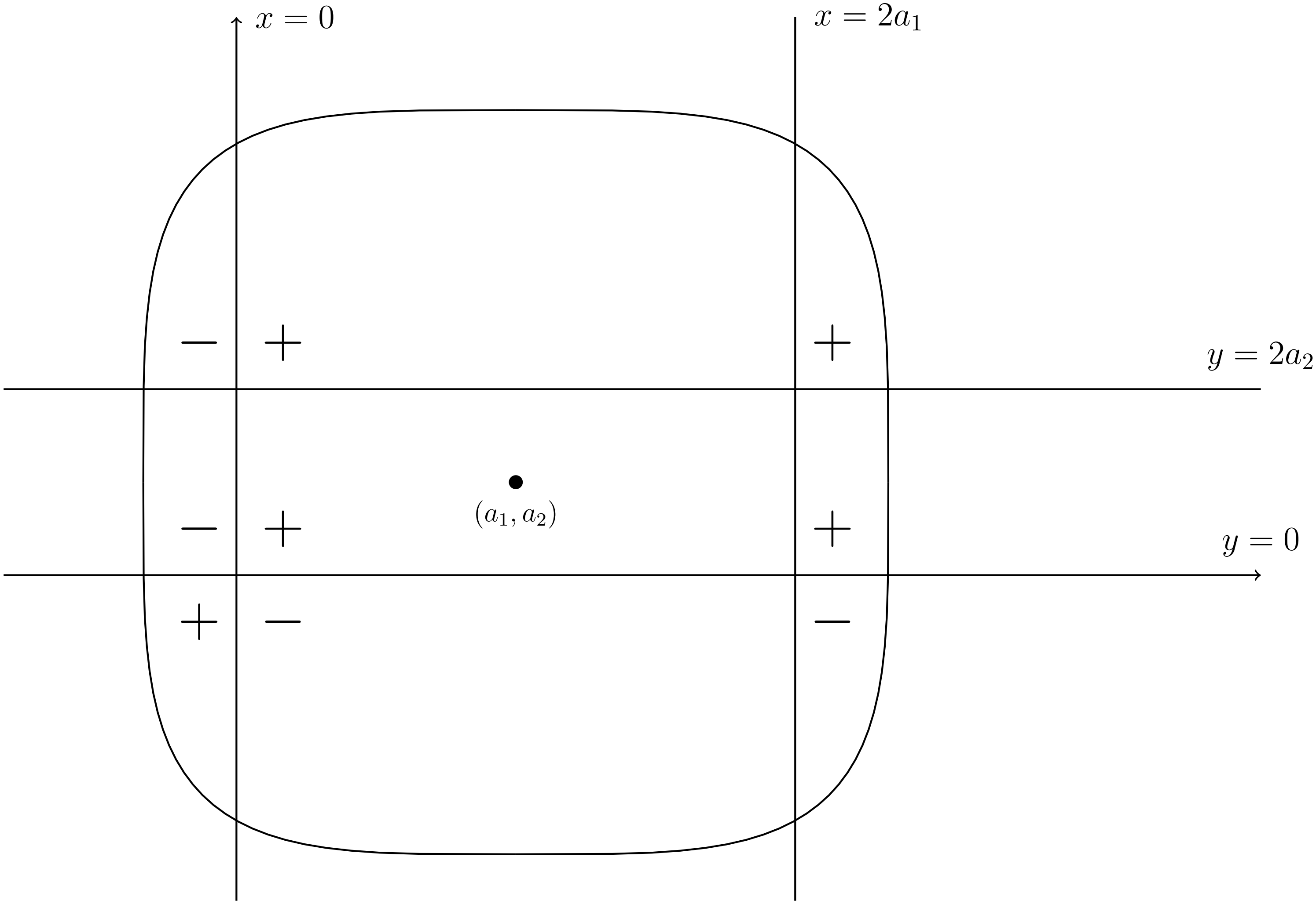
$\mathcal{H}$ is a product arrangement where the factors are from the types A_1, B_n for $n \geq 2$, D_{2m} for $m \geq 2$, E_7, E_8, F_4, H_3, H_4 and $I_2(2k)$ for $k \geq 2$.

Missing: A_n for $n \geq 2$, D_{2m+1} for $m \geq 2$, E_6 and $I_2(2k+1)$ for $k \geq 2$.

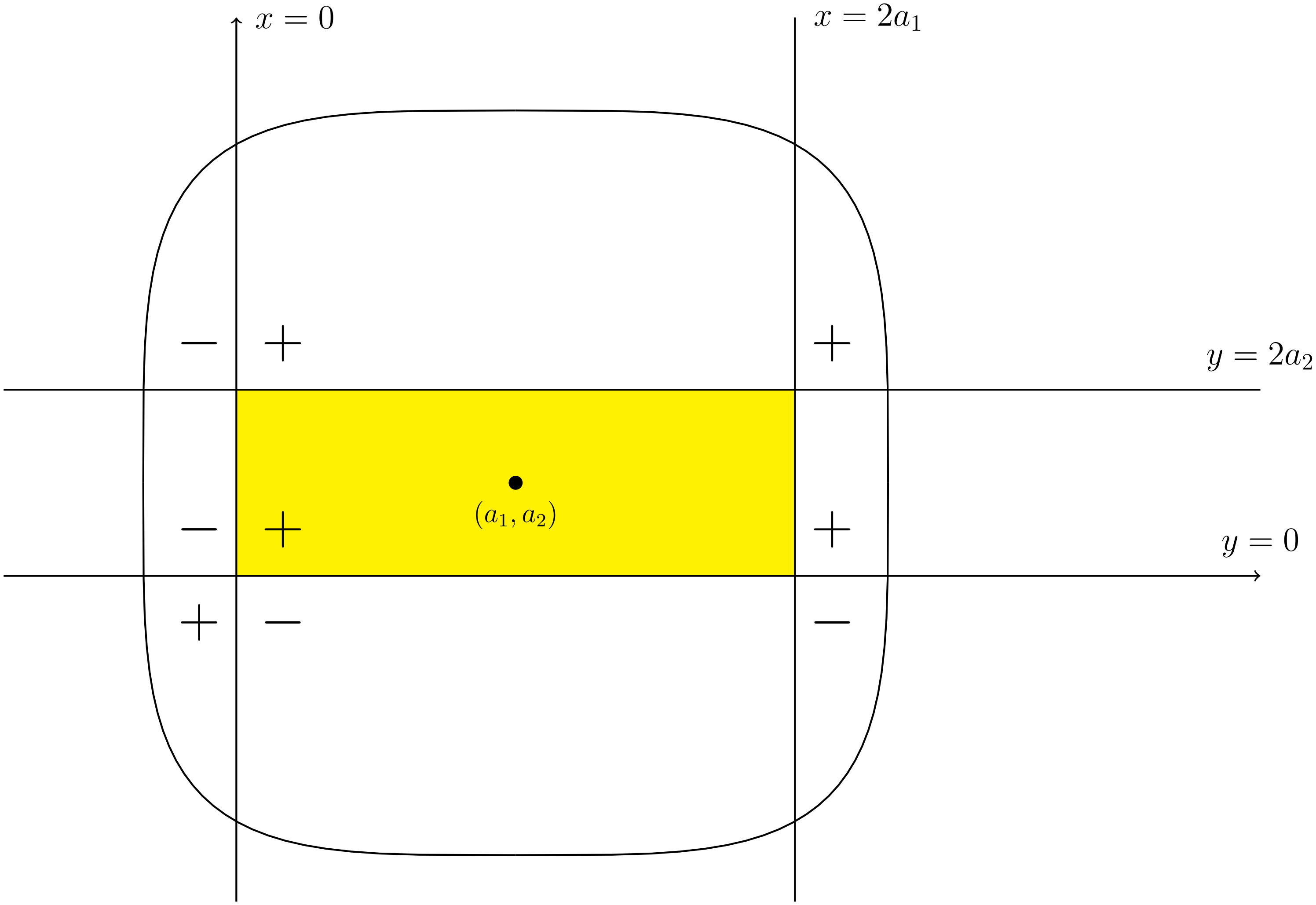
What happens with A_1^n?



Cut also with $x_i = 2a_i$.

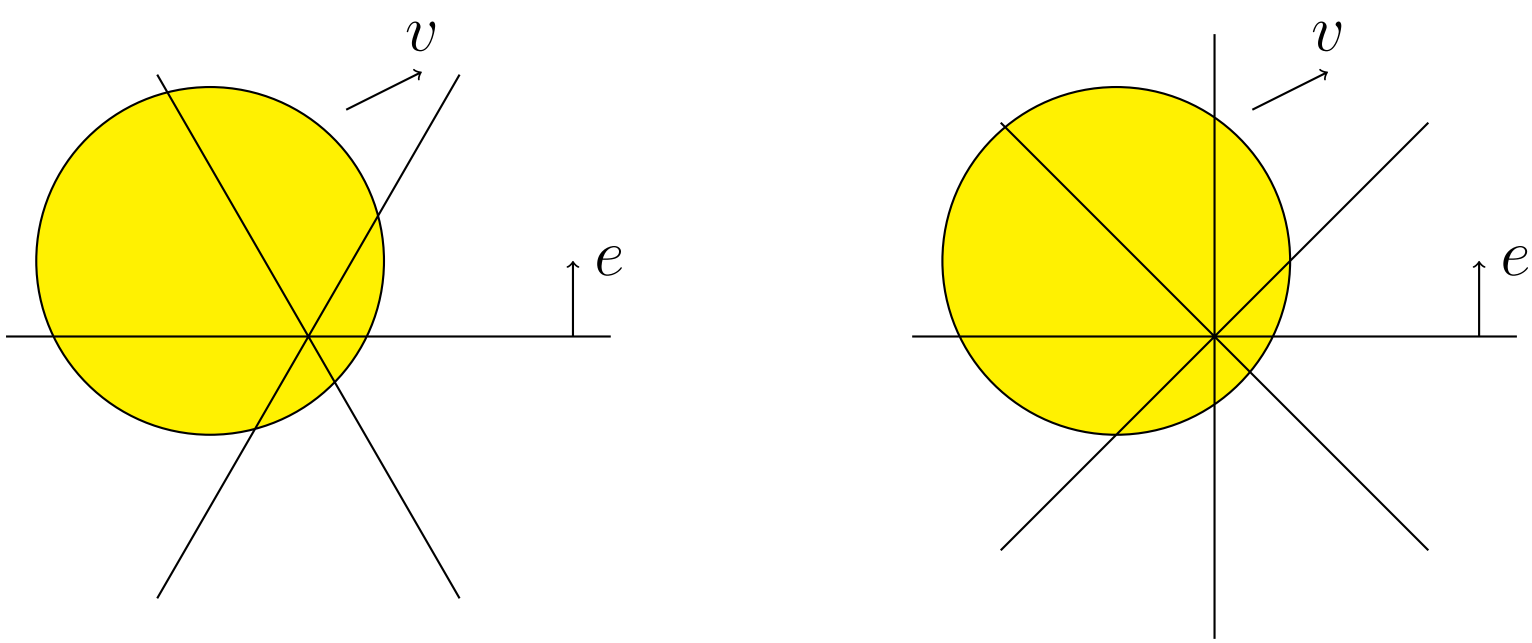


$$P(A_1^n, K + (a_1, \ldots, a_n)) = 2^n \cdot a_1 \cdots a_n$$



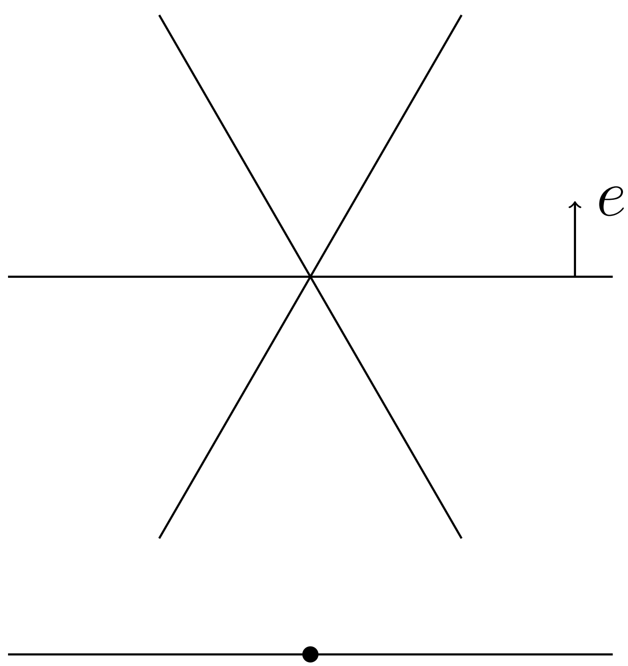
$$\frac{d}{dt}P(\mathcal{H}, K + t \cdot v)$$

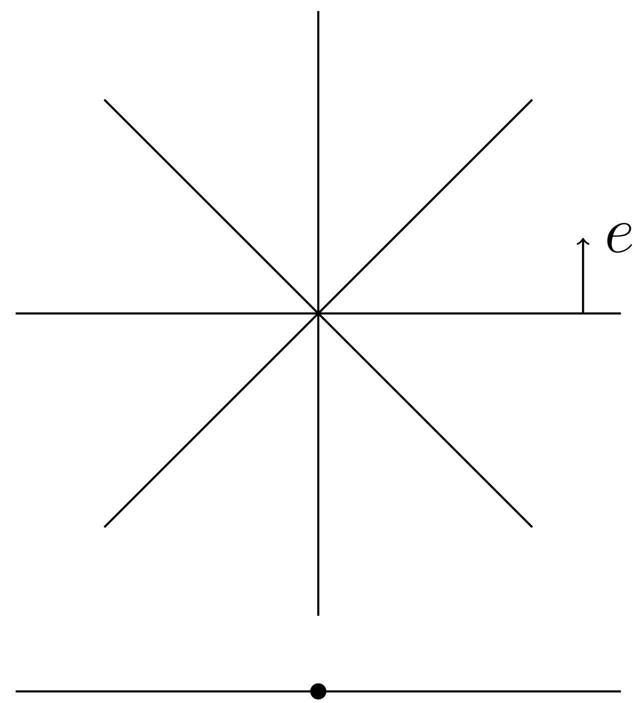
How much of $K + t \cdot v$ passes over the hyperplane $H_e \in \mathcal{H}$?



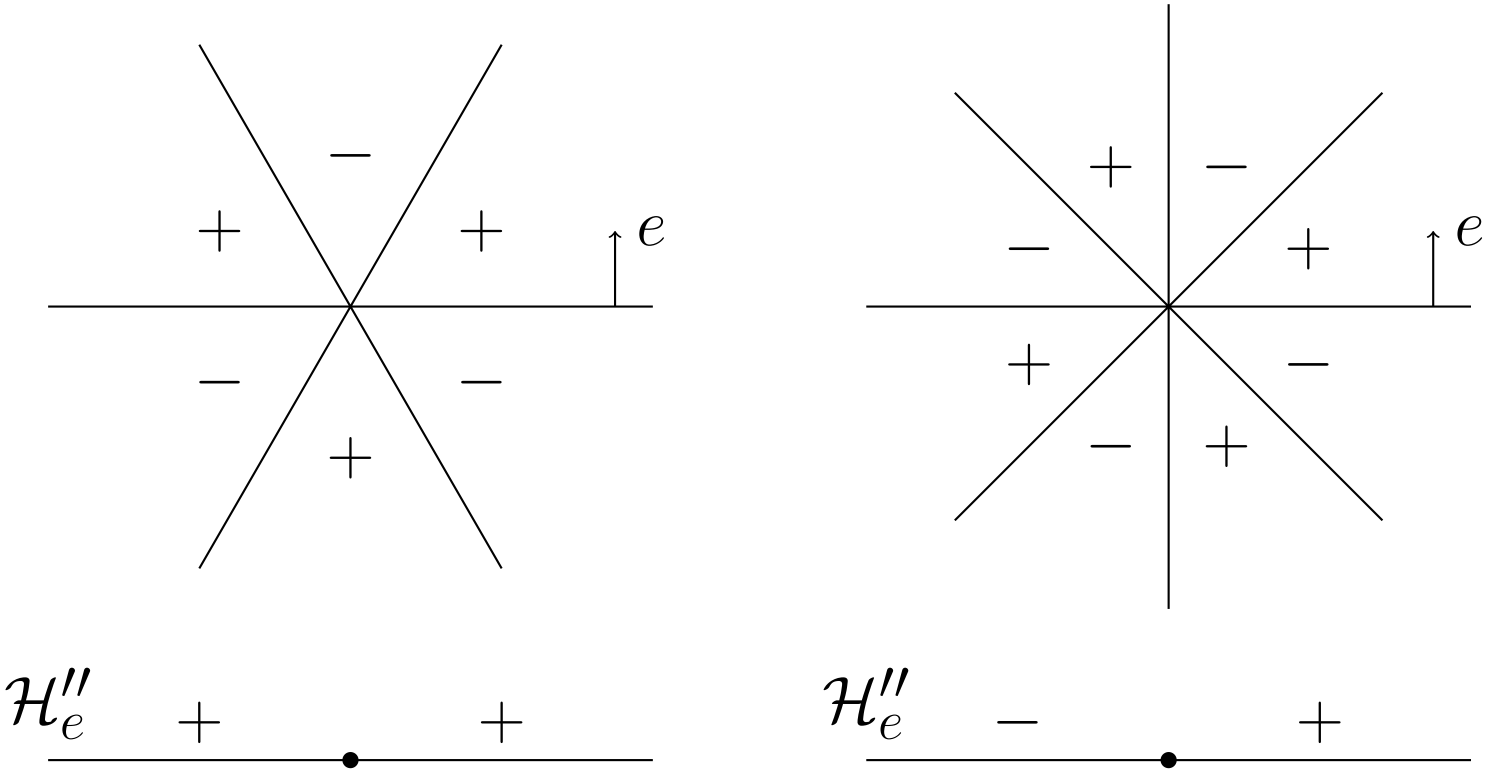
The restricted arrangement $\mathcal{H}''_e$ in H_e

$$\mathcal{H}''_e = \{H_e \cap H_f : f \in E - \{e\}\}$$





Consider the signs



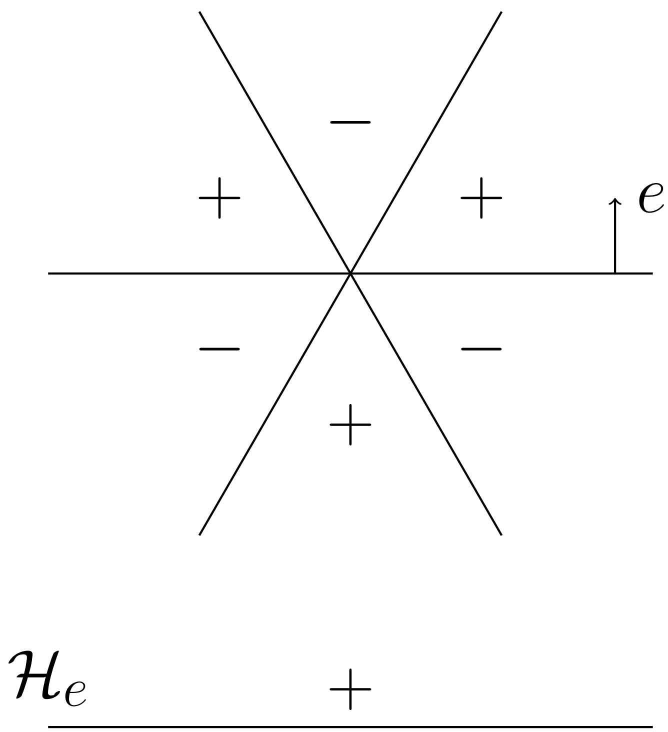
Let $V' \subseteq V$ be a subspace of codimension 2.
The *intersection multiplicity* of V' is

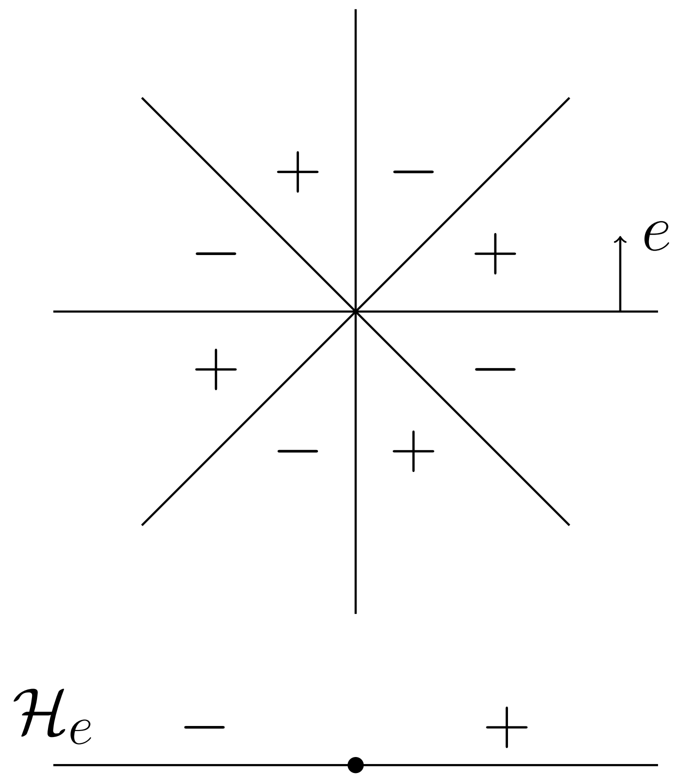
$$\text{imult}(V') = |\{e \in E : H_e \supseteq V'\}|$$

For $e \in E$ the *even restricted arrangement* $\mathcal{H}_e$ is

$$\mathcal{H}_e = \{H_e \cap H_f : f \in E - \{e\}, \text{imult}(H_e \cap H_f) \equiv 0 \bmod 2\}$$

Consider the signs





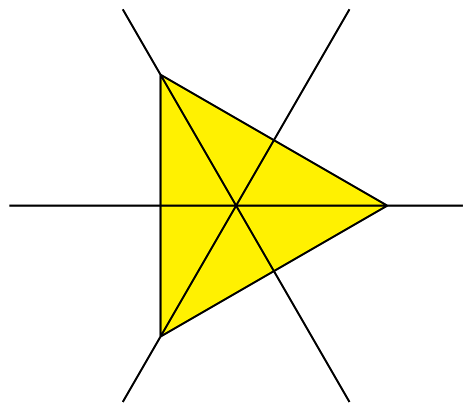
$$\frac{d}{dt} P(\mathcal{H}, K + tv)$$

$$= \ 2 \cdot \sum_{e \in E} (-1)^{Z_0(e) \circ e} \cdot (v, e) \cdot P(\mathcal{H}_e, (K + tv) \cap H_e)$$

where $Z_0(e)$ is a base chamber in $\mathcal{H}_e$

Proposition. If K is a convex set stable under W of the Coxeter arrangement $\mathcal{H}$ then $K \cap H_e$ is a convex set stable by the Coxeter group of the even restricted arrangement $\mathcal{H}_e$.

Not true for the restricted arrangement $\mathcal{H}_e''$



Proposition. If K is a translate of convex set stable under W of the Coxeter arrangement $\mathcal{H}$ then $K \cap H_e$ is the translate of a convex set stable by the Coxeter group of the even restricted arrangement $\mathcal{H}_e$.

At this point we can prove the theorem case by case…

…need better idea!

Definition. Call an hyperplane arrangement $\mathcal{H} = \{H_e\}_{e \in E}$ *even* if:

(i) $\mathcal{H}$ has type A_1, or

(ii) there exists $e \in E$ such that $\mathcal{H}_e$ is even

Equivalently,

(ii) for all $e \in E \neq \emptyset$ we have $\mathcal{H}_e$ is even

For Coxeter arrangements:

$$-\operatorname{id}_V \in W$$

$$\Longleftrightarrow$$

$\mathcal{H}$ is a product arrangement where the factors are from the types A_1, B_n for $n \geq 2$, D_{2m} for $m \geq 2$, E_7, E_8, F_4, H_3, H_4 and $I_2(2k)$ for $k \geq 2$.

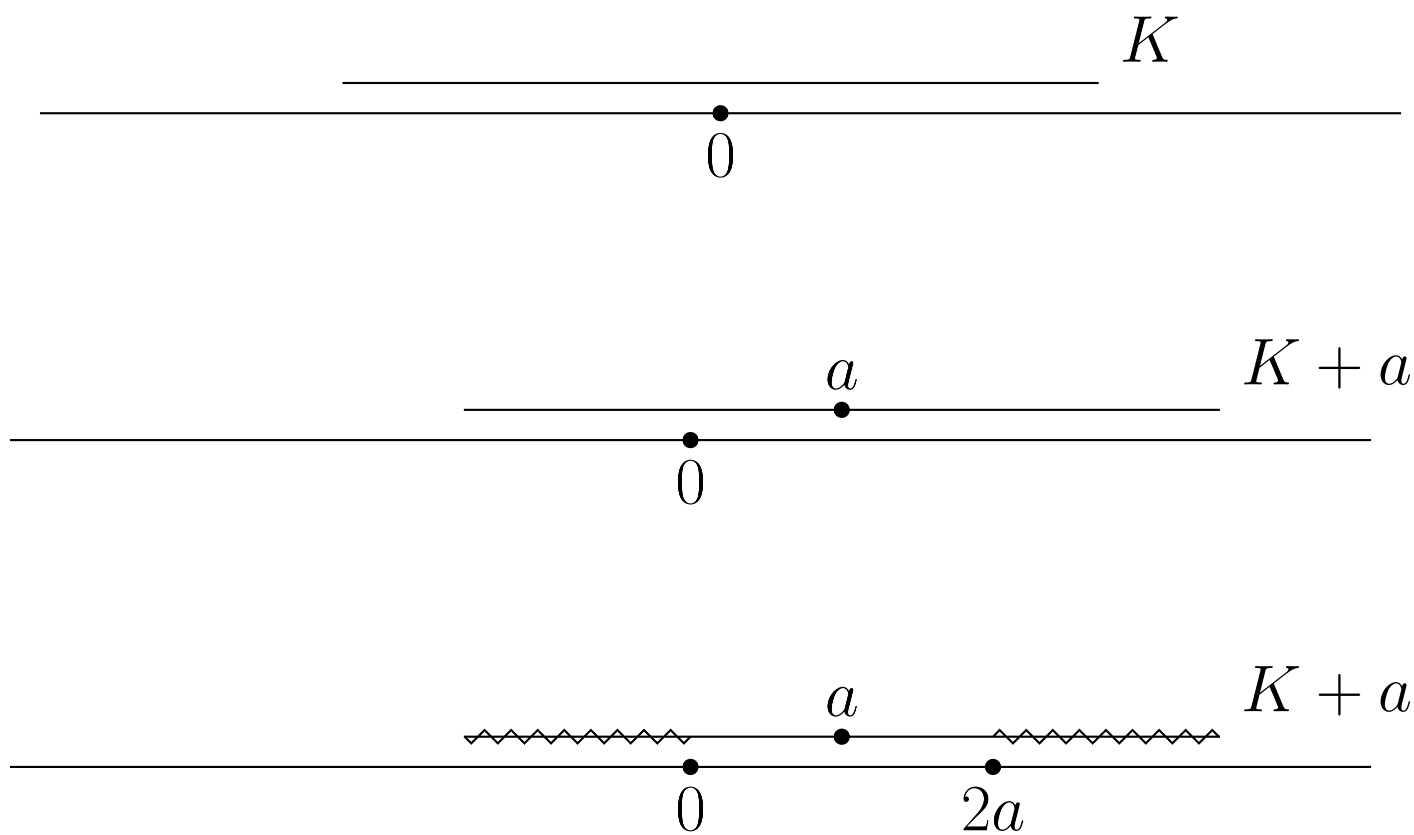
$$\Longleftrightarrow$$

$\mathcal{H}$ is even

Theorem. $\mathcal{H}$ an n-dimensional even Coxeter arrangement. $K \subseteq V$, stable set by the Coxeter group W. Assume $0 \in K + a$. Then the pizza quantity $P(\mathcal{H}, K + a)$ is a polynomial homogenous of degree n in the variable $a = (a_1, \dots, a_n)$.

Remark.
As long as $0 \in K + a$, $P(\mathcal{H}, K + a)$ is independent of K

Proof. $n = 1 \Longrightarrow \mathcal{H} = A_1 \Longrightarrow P(\mathcal{H}, K + a) = 2a$



Induction step:

$$P(\mathcal{H},K+a) - P(\mathcal{H},K)$$
$$= 2 \cdot \sum_{e \in E} (-1)^{Z_0(e) \circ e} \cdot (a,e) \cdot \int_0^1 P(\mathcal{H}_e, (K+ta) \cap H_e)\, dt$$

Polynomials in a:

(a,e) homogenous of degree 1

$P(\mathcal{H}_e, (K+ta) \cap H_e)$ homogenous of degree $n-1$

$P(\mathcal{H},K) = 0$

Proof of Pizza Theorem.

Consider the hypersurface

$$X = \{a \in V : P(\mathcal{H}, K + a) = 0\}$$

X is hypersurface of degree n.

X contains hyperplanes H in Coxeter arrangement $\mathcal{H}$

If $|\mathcal{H}| > n$ then $X = V$ and $P(\mathcal{H}, K + a) = 0$

If $|\mathcal{H}| = n$ then $\mathcal{H}$ has type A_1^n

Returning to balls

Theorem. Let $\mathcal{H}$ be an arrangement in an n-dimensional space V. Assume that $|\mathcal{H}| \equiv n \bmod 2$. Then the function

$$(R, a) \longmapsto P(\mathcal{H}, \mathbb{B}(a, R))$$

on $\{(R, a) \in \mathbb{R} \times V : \|a\| \leq R\}$ is a polynomial homogenous of degree n and only contains even degree terms in R.

$$\begin{aligned} P(\mathcal{H},&\mathbb{B}(a, R)) - P(\mathcal{H}, \mathbb{B}(0, R)) \\ &= 2 \cdot \sum_{e \in E} (-1)^{Z_0(e) \circ e} \cdot (a, e) \cdot \int_0^1 P(\mathcal{H}_e, \mathbb{B}(ta, R) \cap H_e) \, dt \end{aligned}$$

Theorem. Let $\mathcal{H} = \{H_e\}_{e \in E}$ be a Coxeter arrangement in an n-dimensional space V. Assume that $|\mathcal{H}| \equiv n \bmod 2$ and $\|a\| \leq R$.

(i) If $|\mathcal{H}| > n$ then $P(\mathcal{H}, \mathbb{B}(a, R)) = 0$.

(ii) If $|\mathcal{H}| = n$ then

$$P(\mathcal{H}, \mathbb{B}(a, R)) = c \cdot \prod_{e \in E} (a, e)$$

Especially, $P(\mathcal{H}, \mathbb{B}(a, R))$ is independent of R.

Part (ii) applies to A_1^n but also to arrangements such as A_2 times the empty arrangement on $\mathbb{R}$

The even condition implies the parity condition

For balls the pizza theorem also holds for types

- A_n where $n \equiv 0, 1 \bmod 4$
- E_6

For balls the following needs work

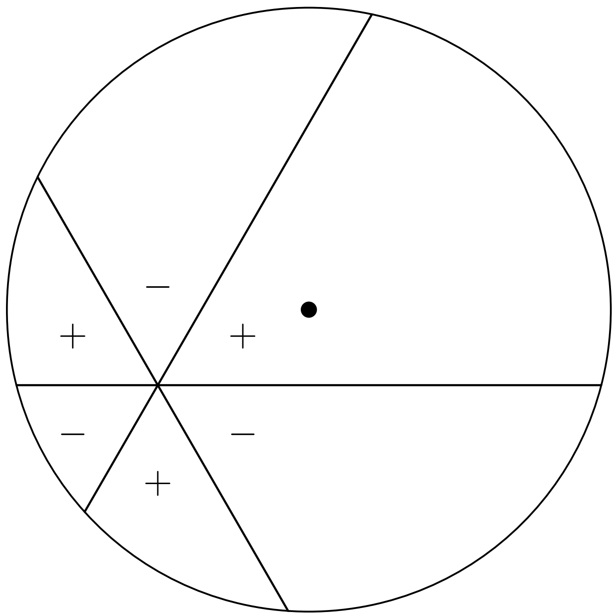
- A_n where $n \geq 3$, $n \equiv 2, 3 \bmod 4$

- D_n where $n \geq 5$, $n \equiv 1 \bmod 2$

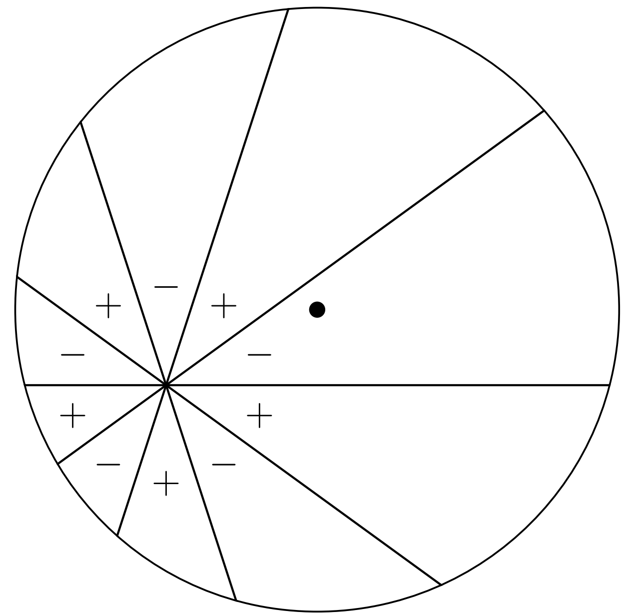
Mabry and Deiermann showed for $\mathcal{H}$ of type $I_2(m)$, $m \geq 3$, m odd, $\|a\| \leq R$ and $a \in T$

$$(-1)^{(m+1)/2} \cdot (-1)^{T} \cdot P(\mathcal{H}, \mathbb{B}(a, R)) > 0$$

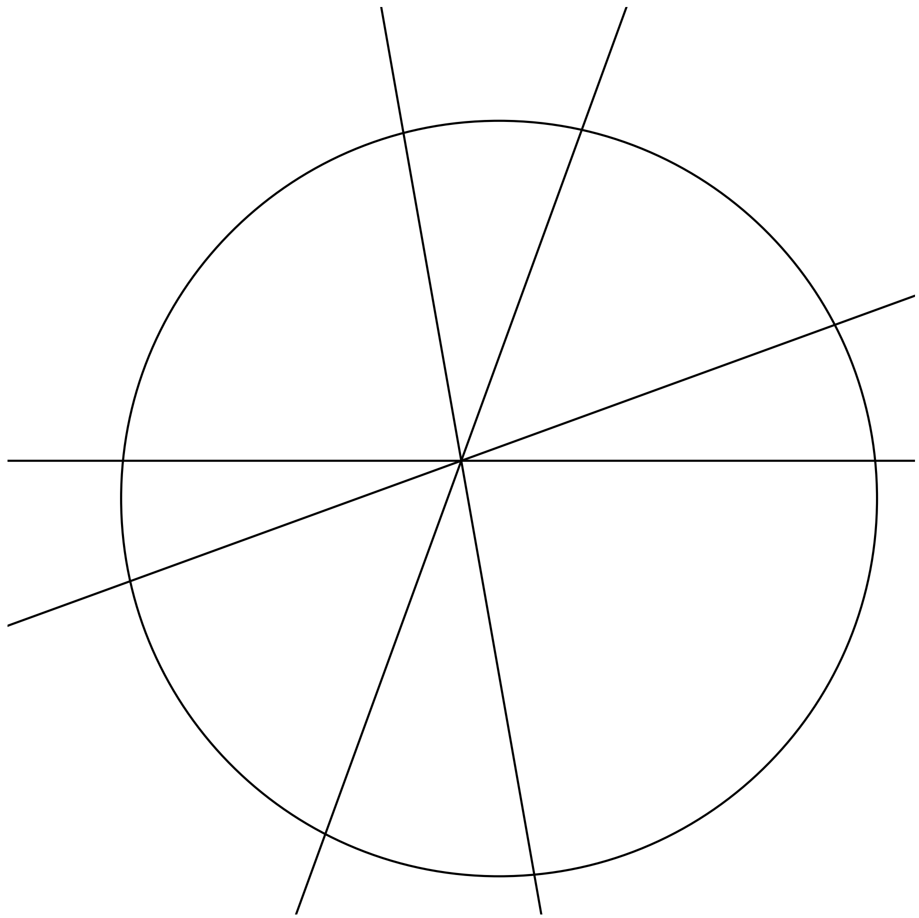
$m \equiv 3 \bmod 4$



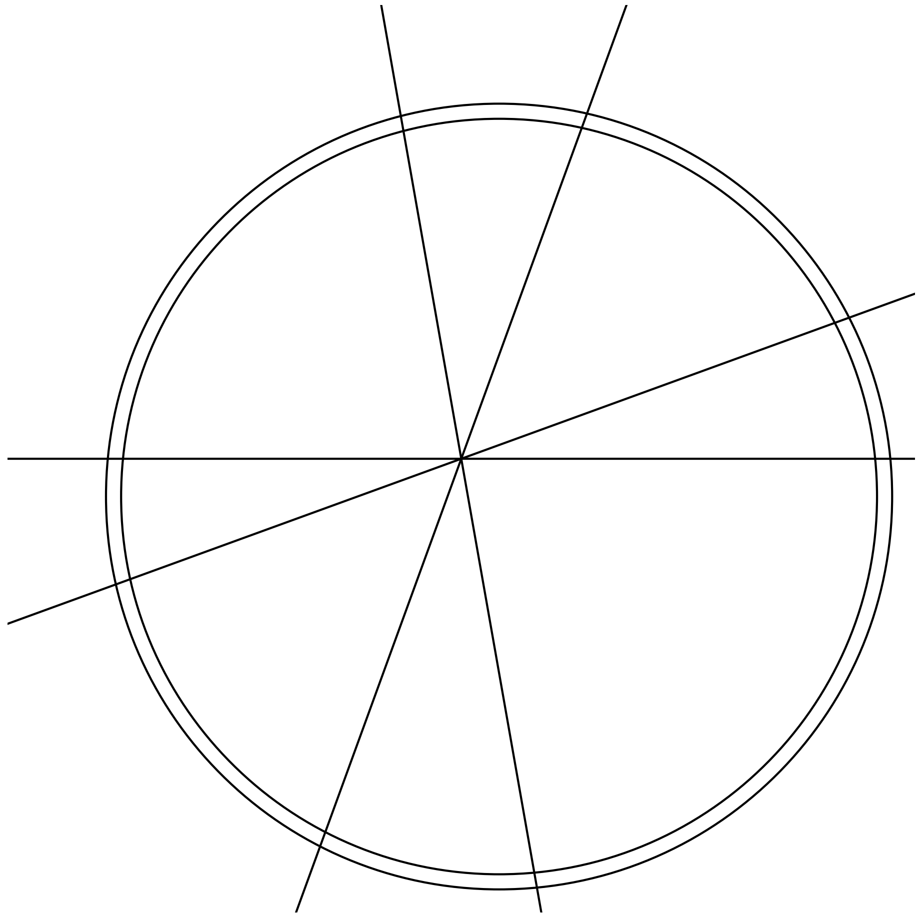
$m \equiv 1 \bmod 4$



$$\frac{d}{dR} P(\mathcal{H}, \mathbb{B}(a, R)) = ?$$

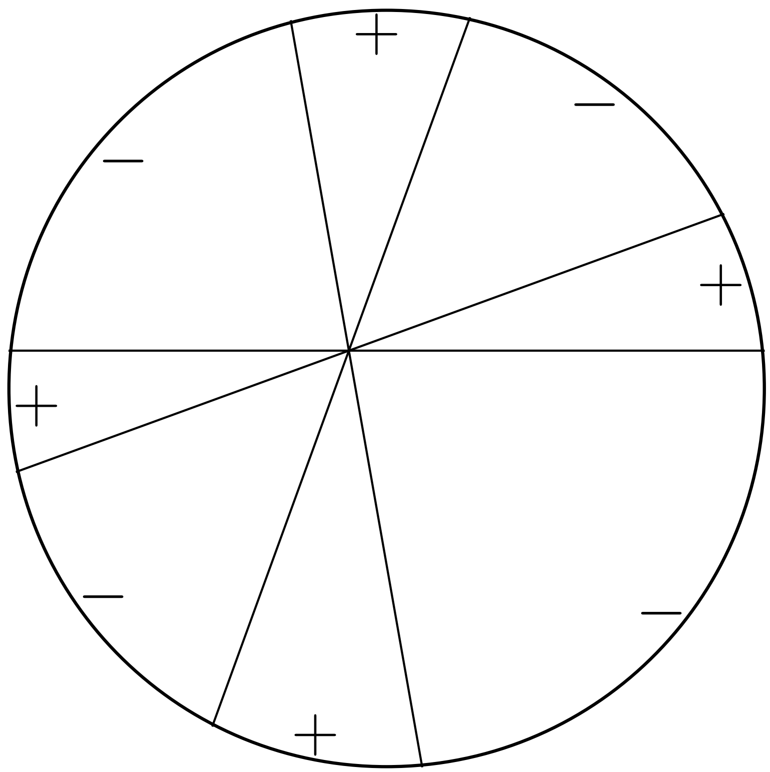


$$\frac{d}{dR}P(\mathcal{H}, \mathbb{B}(a, R)) = ?$$



$$\frac{d}{dR}P(\mathcal{H}, \mathbb{B}(a,R)) = \sum_{T}(-1)^{T} \cdot \mathrm{Vol}_{n-1}(T \cap \mathbb{S}(a,R))$$

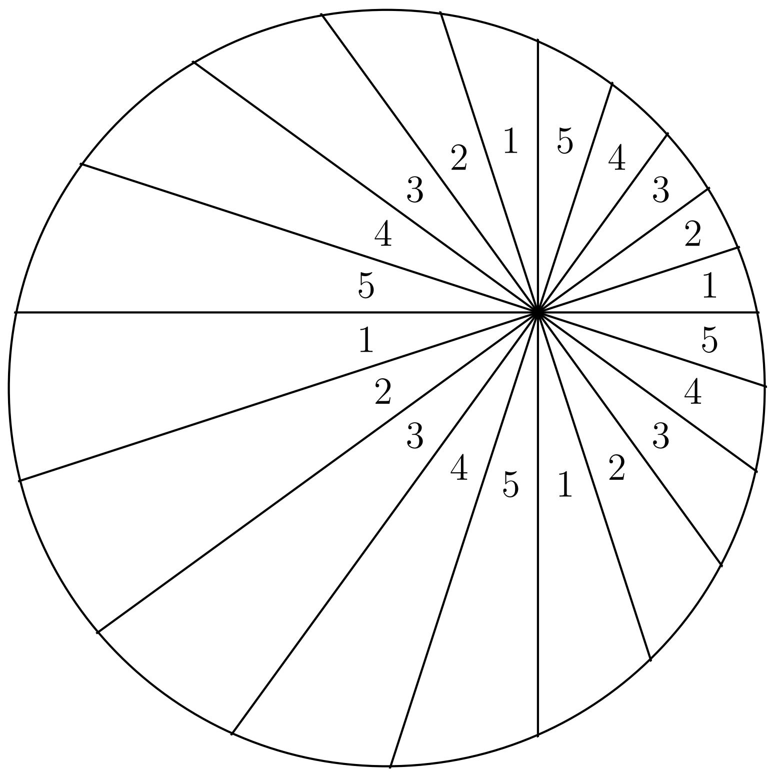
$$\mathbb{S}(a,R) = \{x \in V : \|x-a\| = R\}$$



Theorem. Let $\mathcal{H}$ be an n-dimensional Coxeter arrangement that satisfies the parity condition $|\mathcal{H}| \equiv n \bmod 2$ and $|\mathcal{H}| \geq n$. Let $R \geq \|a\|$. Then the alternating sum of the surface volumes of the regions $\mathbb{S}(a, R) \cap T$ where T ranges over all chambers of the arrangement $\mathcal{H}$ is zero, that is,

$$\sum_{T \in \mathscr{T}} (-1)^T \cdot \mathrm{Vol}_{n-1}(\mathbb{S}(a, R) \cap T) = 0.$$

Hirschhorn[5]: p people sharing a pizza.
Dihedral arrangement of type $I_2(2p)$
Number of slices $4p$
Every person takes every pth slice
Distribution is fair



Open problem:

$p \geq 3$ people in $d \geq 3$ dimensions

Which arrangements guarantee a fair division of $\mathbb{B}(a, R)$?

We know one solution for $p = d = 4$.

$$\mathcal{H}_1 = \{x_i = \pm x_j : 1 \leq i < j \leq 4\}$$
$$\mathcal{H}_2 = \{x_i = 0 : 1 \leq i \leq 4\} \cup \{x_1 \pm x_2 \pm x_3 \pm x_4 = 0\}$$

Note that $\mathcal{H}_1$ and $\mathcal{H}_2$ both have type D_4.

Moreover, the type of $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$ is F_4.

T chamber of $\mathcal{H}$.
Let T_i be the unique chamber in $\mathcal{H}_i$ containing T.

$$(-1)^T = (-1)^{T_1} \cdot (-1)^{T_2}$$

For T a chamber of $\mathcal{H}$ give the slice $T \cap K$ to person $((-1)^{T_1}, (-1)^{T_2})$

Let V_{s_1,s_2} be the amount person (s_1, s_2) receives.

$\mathcal{H}_1$ satisfies pizza theorem $\implies V_{1,1} + V_{1,-1} = 1/2$ pizza

$\mathcal{H}_2$ satisfies pizza theorem $\implies V_{1,1} + V_{-1,1} = 1/2$ pizza

$\mathcal{H}$ satisfies pizza theorem $\implies V_{1,1} + V_{-1,-1} = 1/2$ pizza

$\implies V_{1,1} = V_{1,-1} = V_{-1,1} = V_{-1,-1} = 1/4$ pizza

Thank you!

Bon appétit!

Reference:

Richard Ehrenborg, Sophie Morel and Margaret Readdy, Sharing pizza in n dimensions, to appear in *Transactions of the American Mathematical Society.*
`https://arxiv.org/abs/2102.06649`

Richard Ehrenborg, Sophie Morel and Margaret Readdy, Pizza and 2-structures, preprint 2021.
`https://arxiv.org/abs/2105.07288`

(Just Google "Pizza Ehrenborg")

$\mathcal{H}$	$\mathcal{H}_e$	$\mathcal{H}$	$\mathcal{H}_e$
A_n	A_{n-2}	E_7	D_6
B_2	A_1	E_8	E_7
B_3	B_2 or A_1^2	F_4	B_3
B_n	B_{n-1} or $A_1 \times B_{n-2},$ for $n \geq 4$	H_3	A_1^2
D_4	A_1^3	H_4	H_3
D_5	$A_1 \times A_3$	$I_2(2k)$	A_1
D_n	$A_1 \times D_{n-2},$ for $n \geq 6$	$I_2(2k+1)$	$\emptyset$
E_6	A_5		