Sharing Pizza in $n$ Dimensions

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Pizza
Pizza
Pick any point
Cut with four equidistributed lines
Pizza Theorem: Alternating sum of the areas is equal to 0.
History

1967 Upton stated the problem in *Mathematics Magazine*

1968 Goldberg solved the problem and stated the result for $2k$ equidistributed lines, $k \geq 2$

1994 Carter and Wagon gave a dissection proof

1999 Hirschhorn, Hirschhorn, Hirschhorn, Hirschhorn and Hirschhorn considered $p$ people sharing pizza

2009 Mabry and Deiermann showed the result fails for an odd number of equidistributed lines

2012 Frederickson provided dissection proofs
Classical proof
Higher dimensions?

$V$ real vector space of dimension $n$ with inner product $(\cdot, \cdot)$

Index set $E$ finite set of unit vectors such that $E \cap (-E) = \emptyset$

Hyperplane $H_e = \{v \in V : (v, e) = 0\}$

Hyperplane arrangement $\mathcal{H} = \{H_e\}_{e \in E}$
A chamber $T$ is a connected component of $V - \bigcup_{e \in E} H_e$

$\mathcal{T}$ set of all chambers

Pick $T_0$ base chamber

Sign $(-1)^T = (-1)^k$ where $k$ is the number of hyperplanes separating $T$ from $T_0$
Pizza quantity

\[ P(\mathcal{H}, K) = \sum_{T \in \mathcal{T}} (-1)^T \text{Vol}(K \cap T) \]
$\mathcal{H}$ is a Coxeter arrangement if

– the group $W$ generated by the orthogonal reflections in the hyperplanes of $\mathcal{H}$ is finite and
– the arrangement is closed under all such reflections
$\mathcal{H}_i$ arrangement in $V_i$

$\mathcal{H}_1 \times \mathcal{H}_2$ arrangement in $V_1 \times V_2$ with hyperplanes

$$\{H \times V_2 : H \in \mathcal{H}_1\} \cup \{V_1 \times H : H \in \mathcal{H}_2\}$$

$\mathcal{H}_1$ and $\mathcal{H}_2$ Coxeter $\implies \mathcal{H}_1 \times \mathcal{H}_2$ Coxeter
Type $A_n$

\[ V = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1 + x_2 + \cdots + x_{n+1} = 0\} \]

\[ \mathcal{H} = \{x_i = x_j : 1 \leq i < j \leq n + 1\} \]

Symmetries of the $n$-dimensional simplex

\[ A_1 \]

\[ A_1^n = A_1 \times A_1 \times \cdots \times A_1 \]

\[ = \{x_i = 0 : 1 \leq i \leq n\} \]
Type $B_n$ (and type $C_n$) $\quad n \geq 2$

\[ V = \mathbb{R}^n \]
\[ \mathcal{H} = \{ x_i = 0 : 1 \leq i \leq n \} \cup \{ x_i = \pm x_j : 1 \leq i < j \leq n \} \]

Symmetries of the $n$-dimensional cube and crosspolytope

Type $D_n$ $\quad n \geq 4$

\[ V = \mathbb{R}^n \]
\[ \mathcal{H} = \{ x_i = \pm x_j : 1 \leq i < j \leq n \} \]

$D_2 = A_1^2 \quad D_3 = A_3$
Type $E_6$, $E_7$ and $E_8$

Type $F_4$

\[ V = \mathbb{R}^4 \]

\[ \mathcal{H} = \{ x_i = 0 : 1 \leq i \leq 4 \} \]
\[ \cup \{ x_i = \pm x_j : 1 \leq i < j \leq 4 \} \]
\[ \cup \{ x_1 \pm x_2 \pm x_3 \pm x_4 = 0 \} \]

$F_4 = \text{symmetries of the 24-cell}$
Type $G_2$

$G_2 = I_2(6)$

Type $H_3$ and $H_4$

$H_3$ = symmetries of the dodecahedron and the icosahedron

$H_4$ = symmetries of the 120-cell and 600-cell

Do not arise from root systems
Type $I_2(k)$  
$k \geq 2$

$I_2(k)$ = symmetries of the $k$-gon

$I_2(k)$ consists of $k$ lines

$I_2(2) = A_1^2$  
$I_2(3) = A_2$  
$I_2(4) = B_2$
\[ \mathbb{B}(a, R) = \{ x \in V : \|x - a\| \leq R \}. \]

**Theorem [Goldberg]** Let \( \mathcal{H} \) be the dihedral arrangement \( I_2(2k) \) in \( \mathbb{R}^2 \) for \( k \geq 2 \). For every \( a \in \mathbb{R}^2 \) such that \( 0 \in \mathbb{B}(a, R) \), the pizza quantity for the ball \( \mathbb{B}(a, R) \) vanishes:

\[ P(\mathcal{H}, \mathbb{B}(a, R)) = 0. \]

**Theorem [Frederickson]** Let \( \mathcal{H} \) be of type \( A_1 \times I_2(2k) \) in \( \mathbb{R}^3 \) for \( k \geq 2 \). For every \( a \in \mathbb{R}^3 \) such that \( 0 \in \mathbb{B}(a, R) \), the pizza quantity for the ball \( \mathbb{B}(a, R) \) vanishes:

\[ P(\mathcal{H}, \mathbb{B}(a, R)) = 0. \]
A set $K \subseteq V$ is stable under the group $W$ if

$$w(K) = K$$

for all $w \in W$
Lemma. \( \mathcal{H} \) Coxeter arrangement with group \( W \). If \( K \) is stable under \( W \) and \( a \in H \in \mathcal{H} \) then
\[
P(\mathcal{H}, K + a) = 0
\]
Theorem. Let $\mathcal{H}$ be a Coxeter arrangement on $V$ such that the map $-\mathrm{id}_V$ belongs to the Coxeter group $W$ and where we furthermore assume that $\mathcal{H}$ is not of type $A_1^n$. Then the pizza quantity of $\mathcal{H}$ vanishes, that is, for a convex set $K$ containing the origin that is the translate of a set stable under reflections in the hyperplanes of the arrangement $\mathcal{H}$, the identity $P(\mathcal{H}, K) = 0$ holds.
$- \text{id}_V \in W$

$$\iff$$

$\mathcal{H}$ is a product arrangement where the factors are from the types $A_1, B_n$ for $n \geq 2$, $D_{2m}$ for $m \geq 2$, $E_7, E_8, F_4, H_3, H_4$ and $I_2(2k)$ for $k \geq 2$.

Missing: $A_n$ for $n \geq 2$, $D_{2m+1}$ for $m \geq 2$, $E_6$ and $I_2(2k+1)$ for $k \geq 2$. 
What happens with $A_1^n$?
Cut also with $x_i = 2a_i$. 
\[ P(A_1^n, K + (a_1, \ldots, a_n)) = 2^n \cdot a_1 \cdots a_n \]
\[
\frac{d}{dt} P(\mathcal{H}, K + t \cdot v)
\]

How much of \( K + t \cdot v \) passes over the hyperplane \( H_e \in \mathcal{H} \)?
The restricted arrangement $\mathcal{H}''_e$ in $H_e$

$$\mathcal{H}''_e = \{H_e \cap H_f : f \in E - \{e\}\}$$
Consider the signs

\[ C_\mathcal{H}^e + + + \\
- - - \]

\[ H_e^e - - + \\
+ + + \]

\[ H_e'' - - + \\
+ + + \]
Let $V' \subseteq V$ be a subspace of codimension 2. The *intersection multiplicity* of $V'$ is

$$\text{imult}(V') = |\{e \in E : H_e \supseteq V'\}|$$

For $e \in E$ the *even restricted arrangement* $\mathcal{H}_e$ is

$$\mathcal{H}_e = \{H_e \cap H_f : f \in E - \{e\}, \text{imult}(H_e \cap H_f) \equiv 0 \mod 2\}$$
Consider the signs

\[ H_e \]

\[ + - + \]

\[ - + - \]

\[ e \]

\[ H_e \]

\[ - + + \]

\[ + - - \]

\[ e \]
\[
\frac{d}{dt} P(\mathcal{H}, K + tv) = 2 \sum_{e \in E} (-1)^{Z_0(e) \circ e} \cdot (v, e) \cdot P(\mathcal{H}_e, (K + tv) \cap H_e)
\]

where \(Z_0(e)\) is a base chamber in \(\mathcal{H}_e\)
**Proposition.** If $K$ is a convex set stable under $W$ of the Coxeter arrangement $\mathcal{H}$ then $K \cap H_e$ is a convex set stable by the Coxeter group of the even restricted arrangement $\mathcal{H}_e$.

Not true for the restricted arrangement $\mathcal{H}''_e$
Proposition. If $K$ is a translate of convex set stable under $W$ of the Coxeter arrangement $\mathcal{H}$ then $K \cap H_e$ is the translate of a convex set stable by the Coxeter group of the even restricted arrangement $\mathcal{H}_e$. 
At this point we can prove the theorem case by case... 

...need better idea!
**Definition.** Call an hyperplane arrangement $\mathcal{H} = \{H_e\}_{e \in E}$ even if:

(i) $\mathcal{H}$ has type $A_1$, or

(ii) there exists $e \in E$ such that $\mathcal{H}_e$ is even

Equivalently,

(ii) for all $e \in E \neq \emptyset$ we have $\mathcal{H}_e$ is even
For Coxeter arrangements:

\[ \text{id}_V \in W \]

\[ \iff \]

\( \mathcal{H} \) is a product arrangement where the factors are from the types \( A_1, B_n \) for \( n \geq 2 \), \( D_{2m} \) for \( m \geq 2 \), \( E_7, E_8, F_4, H_3, H_4 \) and \( I_2(2k) \) for \( k \geq 2 \).

\[ \iff \]

\( \mathcal{H} \) is even
Theorem. \( \mathcal{H} \) an \( n \)-dimensional even Coxeter arrangement. \( K \subseteq V \), stable set by the Coxeter group \( W \). Assume \( 0 \in K + a \). Then the pizza quantity \( P(\mathcal{H}, K + a) \) is a polynomial homogenous of degree \( n \) in the variable \( a = (a_1, \ldots, a_n) \).

Remark. As long as \( 0 \in K + a \), \( P(\mathcal{H}, K + a) \) is independent of \( K \).
Proof. $n = 1 \implies \mathcal{H} = A_1 \implies P(\mathcal{H}, K + a) = 2a$
Induction step:

\[ P(\mathcal{H}, K + a) - P(\mathcal{H}, K) \]

\[ = 2 \cdot \sum_{e \in E} (-1)^{Z_0(e) \circ e} \cdot (a, e) \cdot \int_0^1 P(\mathcal{H}_e, (K + ta) \cap H_e) \, dt \]

Polynomials in \( a \):

\( (a, e) \) homogenous of degree 1

\[ P(\mathcal{H}_e, (K + ta) \cap H_e) \) homogenous of degree \( n - 1 \)

\[ P(\mathcal{H}, K) = 0 \]
Proof of Pizza Theorem.

Consider the hypersurface

\[ X = \{ a \in V : P(\mathcal{H}, K + a) = 0 \} \]

\( X \) is hypersurface of degree \( n \).

\( X \) contains hyperplanes \( H \) in Coxeter arrangement \( \mathcal{H} \)

If \(|\mathcal{H}| > n\) then \( X = V \) and \( P(\mathcal{H}, K + a) = 0 \)

If \(|\mathcal{H}| = n\) then \( \mathcal{H} \) has type \( A_1^n \)
Returning to balls

**Theorem.** Let $\mathcal{H}$ be an arrangement in an $n$-dimensional space $V$. Assume that $|\mathcal{H}| \equiv n \mod 2$. Then the function

$$(R, a) \mapsto P(\mathcal{H}, B(a, R))$$

on $\{(R, a) \in \mathbb{R} \times V : \|a\| \leq R\}$ is a polynomial homogeneous of degree $n$ and only contains even degree terms in $R$.

$$P(\mathcal{H}, B(a, R)) - P(\mathcal{H}, B(0, R))$$

$$= 2 \cdot \sum_{e \in E} (-1)^{Z_0(e) \circ e} \cdot (a, e) \cdot \int_0^1 P(\mathcal{H}_e, B(ta, R) \cap H_e) \, dt$$
Theorem. Let $\mathcal{H} = \{H_e\}_{e \in E}$ be a Coxeter arrangement in an $n$-dimensional space $V$. Assume that $|\mathcal{H}| \equiv n \mod 2$ and $\|a\| \leq R$.

(i) If $|\mathcal{H}| > n$ then $P(\mathcal{H}, B(a, R)) = 0$.

(ii) If $|\mathcal{H}| = n$ then

$$P(\mathcal{H}, B(a, R)) = c \cdot \prod_{e \in E} (a, e)$$

Especially, $P(\mathcal{H}, B(a, R))$ is independent of $R$.

Part (ii) applies to $A_1^n$ but also to arrangements such as $A_2$ times the empty arrangement on $\mathbb{R}$.
The even condition implies the parity condition

For balls the pizza theorem also holds for types

- $A_n$ where $n \equiv 0, 1 \mod 4$
- $E_6$
For balls the following needs work

- $A_n$ where $n \geq 3$, $n \equiv 2, 3 \mod 4$

- $D_n$ where $n \geq 5$, $n \equiv 1 \mod 2$

Mabry and Deiermann showed for $\mathcal{H}$ of type $I_2(m)$, $m \geq 3$, $m$ odd, $\|a\| \leq R$ and $a \in T$

$$(-1)^{(m+1)/2} \cdot (-1)^T \cdot P(\mathcal{H}, \mathbb{B}(a, R)) > 0$$
\[ m \equiv 3 \mod 4 \]

\[ m \equiv 1 \mod 4 \]
\[ \frac{d}{dR} P(\mathcal{H}, \mathbb{B}(a, R)) = ? \]
\frac{d}{dR} P(\mathcal{H}, \mathbb{B}(a, R)) = ?
\[
\frac{d}{dR} P(\mathcal{H}, \mathbb{B}(a, R)) = \sum_T (-1)^T \cdot \text{Vol}_{n-1}(T \cap S(a, R))
\]

\[
S(a, R) = \{ x \in V : \| x - a \| = R \}
\]
**Theorem.** Let $\mathcal{H}$ be an $n$-dimensional Coxeter arrangement that satisfies the parity condition $|\mathcal{H}| \equiv n \mod 2$ and $|\mathcal{H}| \geq n$. Let $R \geq \|a\|$. Then the alternating sum of the surface volumes of the regions $S(a, R) \cap T$ where $T$ ranges over all chambers of the arrangement $\mathcal{H}$ is zero, that is,

$$\sum_{T \in \mathcal{T}} (-1)^T \cdot \text{Vol}_{n-1}(S(a, R) \cap T) = 0.$$
Hirschhorn\textsuperscript{5}: \( p \) people sharing a pizza.
Dihedral arrangement of type \( I_2(2p) \)
Number of slices \( 4p \)
Every person takes every \( p \)th slice
Distribution is fair
Open problem:

\[ p \geq 3 \text{ people in } d \geq 3 \text{ dimensions} \]

Which arrangements guarantee a fair division of \( \mathbb{B}(a, R) \)?

We know one solution for \( p = d = 4 \).
\[ \mathcal{H}_1 = \{x_i = \pm x_j : 1 \leq i < j \leq 4\} \]
\[ \mathcal{H}_2 = \{x_i = 0 : 1 \leq i \leq 4\} \cup \{x_1 \pm x_2 \pm x_3 \pm x_4 = 0\} \]

Note that \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) both have type \( D_4 \).

Moreover, the type of \( \mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2 \) is \( F_4 \).

\( T \) chamber of \( \mathcal{H} \).

Let \( T_i \) be the unique chamber in \( \mathcal{H}_i \) containing \( T \).

\[ (-1)^T = (-1)^{T_1} \cdot (-1)^{T_2} \]
For $T$ a chamber of $\mathcal{H}$ give the slice $T \cap K$
to person $((-1)^T_1, (-1)^T_2)$

Let $V_{s_1,s_2}$ be the amount person $(s_1, s_2)$ receives.

$\mathcal{H}_1$ satisfies pizza theorem $\implies V_{1,1} + V_{1,-1} = 1/2$ pizza

$\mathcal{H}_2$ satisfies pizza theorem $\implies V_{1,1} + V_{-1,1} = 1/2$ pizza

$\mathcal{H}$ satisfies pizza theorem $\implies V_{1,1} + V_{-1,-1} = 1/2$ pizza

$\implies V_{1,1} = V_{1,-1} = V_{-1,1} = V_{-1,-1} = 1/4$ pizza
Thank you!
Bon appétit!
Reference:

https://arxiv.org/abs/2102.06649

Richard Ehrenborg, Sophie Morel and Margaret Readdy, Pizza and 2-structures, preprint 2021.
https://arxiv.org/abs/2105.07288

(Just Google “Pizza Ehrenborg”)
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