(Non)commutative integrable systems and Catalan numbers

Solutions of Volterra chain:

$$u_n' = u_n(u_{n+1} - u_{n-1}),\ n \geq 1, \quad u_0 = 0$$

V. Adler and A. Shabat in 2018 suggested the following approach:

Let f be a function. For $k \geq 0$ set

$$w_{2k} = \begin{vmatrix} f & f' & \dots & f^{(k)} \\ f' & f'' & \dots & f^{(k+1)} \\ \vdots & \vdots & & \vdots \\ f^{(k)} & f^{(k+1)} & \dots & f^{(2k)} \end{vmatrix},$$

$$w_{2k+1} = \begin{vmatrix} f' & f'' & \dots & f^{(k+1)} \\ f'' & f''i & \dots & f^{(k+2)} \\ \vdots & \vdots & & \vdots \\ f^{(k+1)} & f^{(k+2)} & \dots & f^{(2k+1)} \end{vmatrix}$$

Then $u_1 = \frac{w_1}{w_0}$, $u_2 = \frac{w_2}{w_1}$, $u_n = \frac{w_{n-3} \cdot w_n}{w_{n-2} \cdot w_{n-1}}$, $n \geq 3$

If

$$f=\sum_{k\geq 0}\frac{c_k}{k!}t^k$$

is exponential generating function for Catalan numbers $c_n=\frac{1}{n+1}\binom{2n}{n}$ then
$f^{(n)}(0)=c_n$ and $w_{2k}(0)=w_{2k+1}(0)=1$.
It implies that initial conditions for functions u_n are given by $u_n(0)=1$, $n\geq 1$.
Conversely, under these initial conditions f is the exponential generating function for Catalan numbers.

Noncommutative Volterra Chain

Let A be a differential algebra with derivation $D : A \to A$ satisfying Leibniz rule $(uv)'=u'v+uv'$, $u'=Du$.

Noncommutative Volterra chain is given by

$$u_n'=u_{n+1}u_n-u_nu_{n+1}$$

To describe its solutions we need a **theory of quasideterminants**.

Let A be a square matrix over an associative ring.

Let A^{ij} be the submatrix of A with ith row and jth column removed; r_i be the ith row without a_{ij} and c_j be the jth column without a_{ij}.

Then

$$|A|_{ij} = a_{ij} - r_i(A^{ij})^{-1}c_j$$

Examples for $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$$|A|_{11} = a_{11} - a_{12}a_{22}^{-1}a_{21}, \ \ |A|_{12} = a_{12} - a_{11}a_{21}^{-1}a_{22},$$
$$|A|_{21} = a_{21} - a_{22}a_{12}^{-1}a_{11}, \ \ |A|_{22} = a_{22} - a_{21}a_{11}^{-1}a_{12}$$

If $A^{-1} = (b_{ij})$ then $b_{pq} = |A|_{qp}^{-1}$.
In the commutative case

$$|A|_{ij} = (-1)^{i+j}\frac{\det A}{\det A^{ij}}$$

It is also convenient to box the entries a_{ij} for $|A|_{ij}$, i.e.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & \boxed{a_{22}} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}_{22}$$

Let f be an element of a differential algebra. For any $k \geq 0$ set

$$z_{2k} = \begin{vmatrix} f & f' & \dots & f^{(k)} \\ f' & f'' & \dots & f^{(k+1)} \\ & \dots & \dots & \\ f^{(k)} & f^{(k+1)} & \dots & \boxed{f^{(2k)}} \end{vmatrix}$$

$$z_{2k+1} = \begin{vmatrix} f' & f'' & \dots & f^{(k+1)} \\ f'' & f''' & \dots & f^{(k+2)} \\ & \dots & \dots & \\ f^{(k+1)} & f^{(k+2)} & \dots & \boxed{f^{(2k+1)}} \end{vmatrix}$$

In the commutative case $z_n = \frac{w_n}{w_{n-2}}$.

Theorem. Set $u_n = z_n z_{n-1}^{-1}$, $n \geq 1$, $u_0 = 0$. Then

$$u_n' = u_{n+1}u_n - u_n u_{n-1}$$

In the commutative case our u_n's are equal to:

$$u_n = \frac{w_n}{w_{n-2}} \cdot \left(\frac{w_{n-1}}{w_{n-3}}\right)^{-1} = \frac{w_{n-3}w_n}{w_{n-2}w_{n-1}},\ n \geq 3$$

Consider now differential algebra of series $\sum a_n t^n$ over a noncommutative ring, derivation D is defined by d/dt.

Let

$$f = \sum \frac{C_k}{k!} t^k$$

be exponential generating function for noncommutative Catalan numbers (A. Berenstein, V. R.).

Noncommutative Catalan numbers:

Let $x_0, x_1, x_2, \ldots$ be free noncommuting variables. Then the solutions of equations

$$\begin{vmatrix} C_0 & C_1 & \ldots & C_k \\ C_1 & C_2 & \ldots & C_{k+1} \\ & \ldots & \ldots & \\ C_k & C_{k+1} & \ldots & \boxed{C_{2k}} \end{vmatrix} = x_{2k},$$

$$\begin{vmatrix} C_1 & C_2 & \dots & C_{k+1} \\ C_2 & C_3 & \dots & C_{2k+2} \\ & \dots & \dots & \\ C_{k+1} & C_{k+2} & \dots & \boxed{C_{2k+1}} \end{vmatrix} = x_{2k+1}$$

are Laurent polynomials:

$C_0 = x_0$, $C_1 = x_1$, $C_2 = x_2 + x_1 x_0^{-1} x_1$,

$C_3 = x_3 + x_2 x_1^{-1} x_2 + x_2 x_0^{-1} x_1 + x_1 x_0^{-1} x_2 + x_1 x_0^{-1} x_1 x_0^{-1} x_1$, ...

A description of C_n's: consider paths from $(0,0)$ to (n,n) which lie below the diagonal (Catalan paths). To each point $p = (p_1, p_2)$ on a plane we associate its *content* $c(p) := p_1 - p_2$. If P is a Catalan path and $p \in P$ then $c(p) \geq 0$.

To each Catalan path P from $(0,0)$ to (n,n) we assign an element M_P by

$$M_P = \prod x_{c(p)}^{\sigma(p)} \ ,$$

where the product is over all corners $p \in P$ (taken in the natural order) and

$$\sigma(p) = \begin{cases} 1 & \text{if } p \text{ is southeast} \\ -1 & \text{if } p \text{ is northwest} \end{cases} .$$

Let $\mathcal{P}_n$ be the set of all Catalan paths from $(0,0)$ to (n,n). Then the *noncommutative Catalan number* C_n is given by

$$C_n = \sum_{P \in \mathcal{P}_n} M_P \ .$$

Theorem. A series f defines solutions for Volterra chain $u_n' = u_{n=1}u_n - u_n u_{n-1}$ if and only if $u_n(0) = x_n x_{n-1}^{-1}$.

Under specialization $x_k \mapsto q^{k(k-1)/2}$ noncommutative Catalan numbers are equal to q-Catalan numbers $c_{q,1}$ and $u_n(0) = q^{n-1}$.

Toda-Sylvester chain

Commutative case:

$$\tau_n'' = (\tau_n')^2 + \tau_{n+1}\tau_{n-1},\ n \geq 1,\ \tau_0 = 1$$

Solutions:

$$\tau_k = w_{k-1} = \begin{vmatrix} f & f' & \dots & f^{(k-1)} \\ f' & f'' & \dots & f^{(k)} \\ \vdots & \vdots & & \vdots \\ f^{(k-1)} & f^{(k+1)} & \dots & f^{(2k-2)} \end{vmatrix}$$

Noncommutative case

Set $\theta_n = \tau_n \tau_{n-1}^{-1}$. Then

$$\left(\theta_n' \theta_n^{-1}\right)' = \theta_{n+1}\theta_n^{-1} - \theta_n \theta_{n-1}^{-1},\ n \geq 1,\ \theta_0 = 0$$

Noncomm. solution (I. Gelfand, V. R., 1992):

$$\theta_k = \begin{vmatrix} f & f' & \dots & f^{(k-1)} \\ f' & f'' & \dots & f^{(k)} \\ \vdots & \vdots & & \vdots \\ f^{(k-1)} & f^{(k+1)} & \dots & \boxed{f^{(2k-2)}} \end{vmatrix}$$

If f is exponential generating function for noncommutative C_n then $\theta_n(0) = x_{2n-2}$.

Generalized Catalan numbers

Definition. Let $A = (a_{ij})$, $i, j \geq 0$ over a ring. We call $C_n(A) := (A^n)_{00}$ the nth Catalan number defined by matrix A.

The idea belongs to Aigner(1999), he considered 3-diagonal matrices with $a_{i,i\pm1} = 1$.

Let

$$A = \begin{pmatrix} a_{00} & a_{01} & 0 & 0 & 0 & \dots \\ a_{10} & a_{11} & a_{12} & 0 & 0 & \dots \\ 0 & a_{21} & a_{22} & a_{21} & 0 & \dots \\ & & \dots & \dots & & \end{pmatrix}$$

with (non)commuting entries. Then

$$\begin{vmatrix} C_0(A) & C_1(A) & \dots & C_n(A) \\ C_1(A) & C_2(A) & \dots & C_{n+1}(A) \\ \dots & & & \dots \\ C_n(A) & C_{n+1}(A) & \dots & \boxed{C_{2n}(A)} \end{vmatrix} =$$

$$= a_{01}a_{12}\dots a_{n-1,n} \cdot a_{n,n-1}\dots a_{21}a_{10}$$

It shows that when f is an exponentional generating function for "numbers" $C_k(A)$ initial condictions $\theta_n(0)$ are nice monomials in $a_{i,i\pm1}$'s.

For shifted Hankel matrices the situation is more complicated:

$$\begin{vmatrix} C_1(A) & C_1(A) & \dots & C_{n+1}(A) \\ C_2(A) & C_3(A) & \dots & C_{n+2}(A) \\ \dots & & & \dots \\ C_{n+1}(A) & C_{n+2}(A) & \dots & \boxed{C_{2n+1}(A)} \end{vmatrix} =$$

$$= a_{01}a_{12}\dots a_{n-1,n}\cdot|A^{(n)}|_{nn}\cdot a_{n,n-1}\dots a_{21}a_{10}$$

where $A^{(n)} = (a_{ij})$, $i, j \le n$.

In this case initial conditions for solutions of Volterra equations are expressed through quasideterminants $|A^{(n)}|_{nn}$.