Noncommutative generalized Catalan numbers and their generating functions

M. Aigner (1999) introduced generalized Catalan numbers:

$$A = \begin{pmatrix} s_0 & 1 & 0 & 0 & 0 & \dots \\ 1 & s_1 & 1 & 0 & 0 & \dots \\ 0 & 1 & s_2 & 1 & 0 & \dots \\ & & \dots & \dots & & \end{pmatrix}$$

$C_n(A) := (A^n)_{00}$, $n \geq 0$.

When $s_0 = 1$, $s_i = 2$ for $i \geq 1$ we get Catalan numbers

When $s_i = 1$ for $i \geq 0$ we get Motzkin numbers

When $s_i = 0$ for $i \geq 0$ we get $\binom{2n}{n}$, $n \geq 0$

Aigner also showed that

$$\begin{vmatrix} C_0(A) & C_1(A) & \dots & C_n(A) \\ C_1(A) & C_2(A) & \dots & C_{n+1}(A) \\ \dots & & & \dots \\ C_n(A) & C_{n+1}(A) & \dots & C_{2n}(A) \end{vmatrix} = 1$$

and computed determinants for Hankel matrices started with $C_1(A)$.

Generalizations:

1) Different types of matrices

2) Matrices over noncommutative rings

Example 1:

$$A = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & 1 & \dots & 1 & 1 & 0 & 0 & \dots \\ 1 & 1 & 1 & \dots & 1 & 1 & 1 & 0 & \dots \\ & & \dots & \dots & & & & & \end{pmatrix}$$

(k ones in the upper row, $k \geq 2$)

In this case

$$C_n(A) = \frac{1}{kn+1}\binom{kn+1}{n}$$

- kth Fuss-Catalan numbers

Example 2:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ \dots & & & & & \dots \\ 0 & \dots & 0 & \dots & 1 & \dots \\ 1 & \dots & 0 & \dots & \dots & \dots \\ 0 & 1 & 0 & \dots & \dots & \dots \\ & \dots & & & \dots & \dots \end{pmatrix}$$

(k zeros before 1 in the left column, $k \geq 2$)

In this case $C_{km}(A)$'s are kth Fuss-Catalan numbers and $C_n(A) = 0$ if $n \neq km$

Our general approach is based on finding equations for generating functions

$$\phi(t) = \sum_{n \geq 0} C_n(A) t^n$$

where $C_n(A) = (A^n)_{00}$ using technique of quasideterminants

Let A be a square matrix over an associative ring.

Notations: A^{ij} is the submatrix of A with ith row and jth column removed; r_i be the ith row without a_{ij} and c_j be the jth column without a_{ij}.

Then

$$|A|_{ij} = a_{ij} - r_i (A^{ij})^{-1} c_j$$

Examples for $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$$|A|_{11} = a_{11} - a_{12} a_{22}^{-1} a_{21}, \quad |A|_{12} = a_{12} - a_{11} a_{21}^{-1} a_{22},$$

$$|A|_{21} = a_{21} - a_{22} a_{12}^{-1} a_{11}, \quad |A|_{22} = a_{22} - a_{21} a_{11}^{-1} a_{12}$$

If $A^{-1} = (b_{ij})$ then $b_{pq} = |A|_{qp}^{-1}$.

In the commutative case

$$|A|_{ij} = (-1)^{i+j} \frac{\det A}{\det A^{ij}}$$

It is also convenient to box the entries a_{ij} for $|A|_{ij}$, i.e.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & \boxed{a_{22}} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}_{22}$$

Definition. Let $A = (a_{ij})$, $i, j \geq 0$ over a ring. We call $C_n(A) := (A^n)_{00}$ the nth Catalan number defined by matrix A.

First applications of quasideterminants

Let

$$A = \begin{pmatrix} a_{00} & a_{01} & 0 & 0 & 0 & \dots \\ a_{10} & a_{11} & a_{12} & 0 & 0 & \dots \\ 0 & a_{21} & a_{22} & a_{21} & 0 & \dots \\ & & \dots & \dots & & \end{pmatrix}$$

Then

$$\begin{vmatrix} C_0(A) & C_1(A) & \dots & C_n(A) \\ C_1(A) & C_2(A) & \dots & C_{n+1}(A) \\ \dots & & & \dots \\ C_n(A) & C_{n+1}(A) & \dots & \boxed{C_{2n}(A)} \end{vmatrix} =$$

$$= a_{01}a_{12} \dots a_{n-1,n} \cdot a_{n,n-1} \dots a_{21}a_{10}$$

$$\begin{vmatrix} C_1(A) & C_1(A) & \dots & C_{n+1}(A) \\ C_2(A) & C_3(A) & \dots & C_{n+2}(A) \\ \dots & & & \dots \\ C_{n+1}(A) & C_{n+2}(A) & \dots & \boxed{C_{2n+1}(A)} \end{vmatrix} =$$

$$= a_{01}a_{12}\dots a_{n-1,n} \cdot |A|_{nn} \cdot a_{n,n-1}\dots a_{21}a_{10}$$

Second application of quasideterminants

Let $A = (a_{ij})$, $i, j \geq 0$. Set $A(k) = (a_{ij}), i, j \geq k$.

Set $b_k = |A(k)|_{kk}^{-1}$.

Recall that A is a lower Hessenberg matrix if $a_{ij} = 0$ for $j - i > 1$

Theorem. Let A be a lower Hessenberg matrix such that all $|A(k)|_{kk}$ and $a_{k,k+1}$ are invertible. Then

$$1 = \sum_{k \geq 0} (-1)^k b_0 \cdot a_{01} \cdot b_1 \cdot a_{12} \cdots a_{k-1,k} \cdot b_k \cdot a_{k0}$$

Let $P = (p_{ij})$, $i, j \geq 0$ be a Hessenberg matrix. Define generating functions for generalized Catalan numbers defined by $P(k)$, $k \geq 0$ by

$$\phi_k(t) = \sum_{n \geq 0} (P(k)^n)_{kk} t^n$$

In particular, $\phi_0(t)$ is the generating function for Catalan numbers defined by matrix P.

Corollary.

$$\phi_0(t) = 1 + \sum_{k\geq 0} t^{k+1}\phi_0(t)\cdot p_{01}\cdot\phi_1(t)\cdot p_{12}\cdots p_{k-1,k}\cdot\phi_k(t)\cdot p_{k0}$$

Various Special Cases:

Assume that P is a uni-Hessenberg matrix and there exists a homomorphism θ such that $\theta(p_{ij}) = p_{i+1,j+1}$ for all i, j. Set $\phi(t) = \phi_0(t)$. Then

$$\phi(t) = 1 + \sum_{k\geq 0} t^{k+1}\phi(t)\cdot\theta(\phi(t))\cdots\theta^k(\phi(t))\cdot p_{k0}$$

For Hessenberg matrix P with constant diagonals we get

$$\phi(t) = 1 + \sum_{k\geq 0} t^{k+1}\phi(t)\cdot p_{01}\cdot\phi(t)\cdot p_{01}\cdot\phi(t)\cdot p_{01}\ldots\phi(t))\cdot p_{k0}$$

So, to solve the above equation we construct semi-infinite Hessenberg matrix P with constant diagonals and set

$$\phi(t) = \sum_{n\geq 0}(P^n)_{00}t^n$$

If, additionally, P is a three-diagonal matrix then

$$\phi(t) = 1 + t\phi(t)p_{00} + t^2\phi(t)p_{01}\phi(t)p_{10}$$

If $p_{01} = p_{00} = p_{10} = 1$ the last equation is the well-known characteristic equation for the generating function for Motzkin numbers.

Let P be a two-diagonal uni-Hessenberg matrix with $p_{k-1,0} = z_1$

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ & & \dots & & \\ z_1 & 0 & 0 & \dots & \\ 0 & z_2 & 0 & 0 & \dots \\ & & \dots & & \end{pmatrix}$$

Let $\theta(z_i) = z_{i+1}$ then

$$\phi(t) = 1 + \phi(t)\theta(\phi(t))\dots\theta^k(\phi(t))z_1t^k$$

One can see that $\phi(t) = \sum_{n\geq 0} C_n^{(k)}t^{nk}$. Set $u = t^k$ and $\Phi(u) = \sum C_n^{(k)}u^n$, then

$$\Phi(u) = 1 + u\Phi(u)\theta(\Phi(u))\dots\theta^k(\Phi(u))z_1\ .$$

Specialization $z_i \mapsto 1$, $i \geq 0$ leads to the classical equation for generating functions for k-Fuss-Catalan numbers, i.e. the specialization of $C_n^{(k)}$ is the nth k-Fuss-Catalan number

Another model for Fuss-Catalan numbers

Let x_i, $i \geq 0$ be free variables. We define homomorphism θ on Laurent polynomials in x_i's by setting $\theta(x_i) = x_{i+1}$.

Let

$$P = \begin{pmatrix} x_1x_0^{-1} & 1 & 0 & 0 & \dots \\ x_2x_0^{-1} & x_2x_1^{-1} & 1 & 0 & \dots \\ x_3x_0^{-1} & x_3x_1^{-1} & x_3x_2^{-1} & 1 & \dots \\ & & \dots & \dots & \end{pmatrix}$$

Consider matrix

$$E = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots \\ & & \dots & \dots & & \end{pmatrix}$$

We call $C_n(E^sP)x_0$, $s \geq 0$ the $(s+2)$-th **noncommutative Fuss-Catalan number** $\mathcal{FC}_n(s+2)$.

Note that $\mathcal{C}_n = \mathcal{FC}_n(2)$ are noncommutative Catalan numbers introduced by A.B. and V.R. earlier.

Proposition. Set $\phi := \sum_{n\geq 0} C_n(E^sP)t^n$. Let $y_m = x_mx_{m-1}^{-1}$, $m \geq 1$. Then

$$\phi = 1 + t(\theta^{s+1}\phi) \cdot y_{s+1} \cdot (\theta^s\phi) \cdot y_s \cdots (\theta\phi) \cdot y_1 \cdot \phi$$

Combinatorial interpretation of noncommutative Fuss-Catalan numbers

Fuss-Catalan number $\frac{1}{kn+1}\binom{kn+1}{n}$ is the number of paths π on $\mathbf{Z} \times \mathbf{Z}$ from $(0,0)$ to $(n,(k-1)n)$ consisting of moves $\rightarrow$ and $\uparrow$ below $y=(k-1)x$
(Dyck or Catalan paths)

To each corner $p=(p_1,p_2)$ of such path π set
$c(p)=((k-1)p_1-p_2)$, $c(p)\geq 0$.

We call p a *southeast corner* of π if π goes right and up and a *northwest corner* otherwise.

To each path π we assign an element M_π by

$$M_\pi = \prod x_{c(p)}^{w(p)} \ ,$$

the product is over all corners $p \in P$ in the natural order and $w(p)=1$ for southwest corners and $w(p)=-1$ for northwest corners.

Theorem. $\mathcal{FC}_k = \sum M_\pi$
over all Dyck paths π from $(0,0)$ to $(n,(k-1)n)$.

For $k=2$ it describes noncommutative Catalan numbers introduced by A.B. and V.R.

Hankel matrices

$$\begin{vmatrix} \mathcal{C}_i & \mathcal{C}_{i+1} & \dots & \mathcal{C}_{i+n} \\ \mathcal{C}_{i+1} & \mathcal{C}_{i+2} & \dots & \mathcal{C}_{i+n+1} \\ \dots & & & \dots \\ \mathcal{C}_{i+n} & \mathcal{C}_{i+1+1} & \dots & \boxed{\mathcal{C}_{i+2n}} \end{vmatrix} = x_{i+2n}, \quad i = 0, 1$$

Narayana numbers

Denote by $\mathcal{P}_n(k)$ the set of Dyck paths from $(0, 0)$ to (n, n) with k evenly indexed horizontal moves. It is known that Narayana number $N(n, k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ is equal to $|\mathcal{P}_n(k)|$

Denote by $\mathcal{N}(n, k)$ the sum of monomials corresponding to paths from $\mathcal{P}_n(k)$ and call it the *Noncommutative Narayana (n, k)-number*.

$$P' = \begin{pmatrix} x_1x_0^{-1} & 1 & 0 & 0 & \dots \\ x_2x_0^{-1}t & x_2x_1^{-1}t & t & 0 & \ddots \\ x_3x_0^{-1} & x_3x_1^{-1}t & x_3x_2^{-1} & 1 & \ddots \\ x_4x_0^{-1}t & x_4x_1^{-1}t & x_4x_2^{-1}t & x_4x_3^{-1}t & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Then $C_n(P')x_0 = \sum_{k=1}^{n} \mathcal{N}(n, k)t^k$.

Orthogonal polynomials

Let $C_0, C_1, \ldots$ elements of a ring R with an anti-involution $a \mapsto \overline{a}$ and $\overline{C}_i = C_i$ for $i \geq 0$. Define orthogonal polynomials $P_n(t) \in R[t]$ as

$$P_n(t) = \begin{vmatrix} C_0 & C_1 & \ldots & C_{n-1} & 1 \\ C_1 & C_2 & \ldots & C_n & t \\ & & \ldots & & \\ C_n & C_{n+1} & \ldots & C_{2n-1} & \boxed{t^n} \end{vmatrix} .$$

In this definition elements C_i play a role of abstract (noncommutative) moments. Polynomials $P_n(t)$ are orthogonal w. r. t. $\langle at^i, bt^j \rangle = aC_{i+j}\overline{b}$.

For noncommutative Catalan numbers $\mathcal{C}_i$'s

$$P_{n+1}(t) = (t - x_{2n+1}x_{2n}^{-1} - x_{2n}x_{2n-1}^{-1})P_n(t) - x_{2n}x_{2n-2}^{-1}P_{n-1}(t) .$$

One can compute $P_n(t)$ as

$$P_n(t) = \sum_{k=0}^{n} (-1)^{n-k} B(n+k, n-k) t^k$$

where $B(n+k, n-k)$ are noncommutative binomial coefficients introduced by A.B. and V.R.