Determinant solutions for nonlinear differential equations

Let f be a function and

$$W_n = \begin{pmatrix} f & f' & f'' & \dots & f^{(n-1)} \\ f' & f'' & f''' & \dots & f^{(n)} \\ & & \dots & & \\ f^{(n-1)} & f^{(n)} & f^{(n+1)} & \dots & f^{(2n-2)} \end{pmatrix}$$

In 1862 J.J. Sylvester discovered that

$$\tau_n = |W_n|, n = 0, 1, 2, \dots$$

satisfy the system of differential equations

$$(\tau_n)''\tau_n - (\tau_n')^2 = \tau_{n+1}\tau_{n-1}$$

(Here $\tau_0 = 1$.)

Similar equations were later used by G. Darboux. In 1967 the Sylvester equations were reborn as Toda equations or Toda lattice.

Toda wrote a Hamiltonian for a chain of particles with nearest neighbor interaction, and his model implies the Sylvester type equations. There are now hundreds of papers discussing various aspects of Toda lattice.

Logarithmic form

The Sylvester equations can be written as

$$\log \tau_n{}'' = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}$$

or, setting $\theta_n = \tau_n \tau_{n-1}^{-1}$, $n \geq 1$, as

$$(\log \theta_1)'' = \theta_2 \theta_1^{-1},$$

$$(\log \theta_n)'' = \theta_{n+1}\theta_n^{-1} - \theta_n \theta_{n-1}^{-1}, \; n \geq 2$$

In this form the system admits a noncommutative solution.

Let A be an algebra over a field and $D : A \to A$ be a derivation, i.e. a linear map satisfying

$$D(ab) = D(a)b + aD(b) \forall a, b \in A$$

Consider the system

$$D(D(\theta_1)\theta_1^{-1}) = \theta_2\theta_1^{-1},$$

$$D(D(\theta_n)\theta_n^{-1}) = \theta_{n+1}\theta_n^{-1} - \theta_n\theta_{n-1}^{-1},\ n \geq 2$$

Noncommutative case

We recall now a "replacement" for determinants in the noncommutative case.

Let $A = (a_{ij})$,$i, j = 1, 2, \ldots, n$ be a matrix over an associative unital ring R. Denote by A^{ij} the $(n-1)\times(n-1)$ submatrix of A obtained by deleting the i-th row and j-th column.

Let r_i be the row matrix $(a_{i1}, \ldots, \hat{a}_{ij}, \ldots, a_{in})$ and c_j be the column matrix with entries $(a_{1j}, \ldots, \hat{a}_{ij}, \ldots, a_{nj})$.

We define the **quaisdeterminant** of A as

$$|A|_{ij} = a_{ij} - r_i (A^{ij})^{-1} c_j$$

if the matrix A^{ij} is invertible. Other notation

$$|A|_{ij} = \begin{vmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \dots & & \dots & & \dots \\ a_{i1} & \dots & \boxed{a_{ij}} & \dots & a_{in} \\ \dots & & \dots & & \dots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{vmatrix}$$

If the inverse matrix $A^{-1} = (b_{ij})$ exists then $b_{ij} = |A|_{ji}^{-1}$ provided that the quasideterminant is invertible.

If R is commutative then

$$|A|_{ij} = (-1)^{i+j} \frac{\det A}{\det A^{ij}}$$

for any i and j.

Let R be a division ring with a derivation D. Let $f \in R$ and the quasideterminants

$$\theta_n = \begin{vmatrix} f & Df & \dots & D^{n-1}f \\ Df & D^2f & \dots & D^nf \\ \dots & \dots & \dots & \dots \\ D^{n-1}f & D^nf & \dots & \boxed{D^{2n-2}f} \end{vmatrix}$$

are defined an invertible.

Set $\theta_1 = f$. In 1992 I. Gelfand and V. R. showed that θ_n, $n = 1, 2, \dots$ satisfy the noncommutative version of the Sylvester-Toda equations.

Factorization of differential operators

Let R be a division algebra, D a derivation of R and $a_1, \dots, a_n \in R$. Let

$$P(D) = D^n + a_1 D^{n-1} + \dots + a_n$$

be a differential operator with solutions

$$P(D)f_i = 0, \ i = 1, \dots, n$$

We call the set of solutions f_i, $i = 1, \dots, n$ **nondegenerate** if the **quasi-wronskians**

$$w_i = \begin{vmatrix} f_1 & f_2 & \dots & f_i \\ Df_1 & Df_2 & \dots & Df_i \\ \dots & \dots & \dots & \dots \\ D^{i-1}f_1 & D^{i-1}f_2 & \dots & \boxed{D^{i-1}f_i} \end{vmatrix}$$

are defined and invertible for $i = 1, \dots, n$.

Theorem (P. Etingof, I. Gelfand, V.R.)

$P(D) = (D - b_n)(D - b_{n-1}) \cdots (D - b_1)$

where

$$b_i = (Dw_i)w_i^{-1}, \ i = 1, \dots, n$$

Generalized Toda lattice and Painlevé equations

In 2001 Kajiwara, Masuda, Noumi, Ohta, and Yamada solved a generalized Toda system

$$\tau_n'' - (\tau_n')^2 = \tau_{n+1}\tau_{n-1} - \phi\psi,$$

for $n \in \mathbf{Z}$ with the conditions $\tau_1 = \phi$, $\tau_0 = 1$, $\tau_{-1} = \psi$.

Let $a_0 = \phi$, $b_0 = \psi$ and

$$a_n = a_{n-1}' + \psi \sum_{i+j=n-2} a_i a_j$$

$$b_n = b_{n-1}' + \phi \sum_{i+j=n-2} b_i b_j$$

where $i, j \geq 0$ and $n \geq 1$.

Then the Hankel determinants $\tau_0 = 1$,

$$\tau_n = \det((a_{i+j-2}))_{1\leq i,j\leq n} \quad \text{for } n > 0$$

$$\tau_n = \det((b_{i+j-2}))_{1\leq i,j\leq |n|} \quad \text{for } n < 0$$

satisfy the generalized Toda lattice.

Toda lattice and Painlevé-II equation

Recall that the Painlevé-II equation is written as

$$P_{II}(\alpha): \; u'' = 2u^3 - 4xu + 4(\alpha + \frac{1}{2})$$

Kajiwara and Masuda showed that if ϕ, ψ satisfy relations

$$\psi''\psi^{-1} = \phi''\phi^{-1} = 2x - 2\phi\psi$$

and

$$\phi'\psi - \psi'\phi = 2\alpha$$

then

$$u_n = \big(\log \frac{\tau_{n+1}}{\tau_n}\big)'$$

satisfies $P_{II}(\alpha + n)$, $n \in \mathbf{Z}$.

Noncommutative version (V.R., V. Rubtsov)

Set $Df = f'$ and consider

$$a_n = a'_{n-1} + \sum_{i+j=n-2} a_i \psi a_j$$
$$b_n = b'_{n-1} + \sum_{i+j=n-2} b_i \phi b_j$$

where $i, j \geq 0$ and $n \geq 1$.

Construct Hankel matrices $A_n = (a_{i+j})$, $B_n = b_{i+j}$, $i, j = 0, 1, \ldots, n$

Set $\theta_{p+1} = |A_p|_{pp}, \quad \eta_{-q-1} = |B_q|_{qq}$.

Theorem. Then $\theta_1 = \phi$, $\theta_0 = \psi^{-1}$,

$$(\theta'_n \theta_n^{-1})' = \theta_{n+1}\theta_n^{-1} - \theta_n \theta_{n-1}^{-1}$$

for $n \geq 1$.

Similarly, $\eta_0 = \phi^{-1}$, $\eta_{-1} = \psi$,

$$(\eta_{-n}^{-1}\eta_{-n})' = \eta_{-n}^{-1}\eta_{-n-1} - \eta_{-n+1}^{-1}\eta_{-n}$$

for $n \geq 1$.

Noncommutative Painlevé-II equation

Set $Df = f'$.

$$P_{II}(\alpha):\ u'' = 2u^3 - 2xu - 2ux + 4(\alpha + \frac{1}{2})$$

where $u, x \in R$, $x' = 1$, $\alpha' = 0$.

Note: Noncommutativity brings symmetry.

Theorem. Let

$$\psi^{-1}\psi'' = \phi''\psi^{-1} = 2x - 2\phi\psi$$

$$\psi\phi' - \psi'\psi = 2\alpha$$

Then for Toda solution for $n \in \mathbf{N}$:

$u_n = \theta'_n\theta_n^{-1}$ satisfies noncom $P_{II}(\alpha + n - 1)$

$u_{-n} = \eta_{-n}^{-1}\eta'_{-n}$ satisfies noncom $P_{II}(\alpha - n)$

Toda equations and orthogonal polynomials

Classical staff: $\mu(x)$ is a Borel measure on $\mathbf{R}$ such that *moments*

$$c_i = \int_{\mathbf{R}} x^i d\mu(x)$$

are defined for $i = 0, 1, 2, \ldots$.

Set

$$P_n(x) = \begin{vmatrix} c_0 & c_1 & c_2 & \ldots & c_n \\ c_1 & c_2 & c_3 & \ldots & c_{n+1} \\ & \ldots & & \ldots & \\ c_{n-1} & c_n & c_{n+1} & \ldots & c_{2n-1} \\ 1 & x & x^2 & \ldots & \boxed{x^n} \end{vmatrix}$$

Then

$$\int_{\mathbf{R}} P_m(x) P_n(x) = 0 \text{ if } m \neq n$$

Recurrence relations

$$xP_n(x) = P_{n+1}(x) + a_n P_n(x) + b_n P_{n-1}(x), \; n > 0$$

with $P_0(x) = 1$.

If we replace $\mu(x)$ by $e^{tx}\mu(x)$ then everything depend on t and we arrive to

$$b_n' = a_{n+1} - a_n$$
$$a_n' = a_n(b_n - b_{n-1})$$

for $n > 0$.

After substitutions

$$a_n = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2},\ b_n = \big(\log \frac{\tau_{n+1}}{\tau_n}\big)'$$

we arrive to the Toda lattice.

Noncommutative generalizations

First approach (I. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V.R. and J.-Y. Thibon)

Start with abstract moments from a ring R with an anti-involution $r \to \overline{r}$, $r \in R$.

Define "scalar" product as

$$< \sum \alpha_i x^i, \sum \beta_j x^j >= \sum \alpha_i c_{i+j} \overline{\beta}_j$$

Construct polynomials $P_n(x)$ as above. Then

$$< P_m(x), P_n(x) >= 0 \text{ if } m \neq n$$

And there is an analog of the recurrence relations.

Second approach (Dissertation by L. Miranian, a former student of F.A. Grunbaum) The idea goes to Krein (1949).

Consider the ring of $k \times k$-Hermitian matrices. Introduce measure: $\mu dx = W(x)dx$ with Hermitian weight function $W(x)$.

Then define moments, orthogonal polynomials and find coefficients a_n and b_n.

Insert an infinite set of time variables $t_1, t_2, \ldots$ into the measure:

$$\mu_t(dx) = e^{\sum_{i=1}^{\infty} t_i x^i I} \cdot \mu(x)$$

where I is $k \times k$ identity matrix.

Compute $a_n(t)$ and $b_n(t)$. They satisfy

$$(\partial/\partial_{t_1}) b_n = a_{n+1} - a_n$$

$$(\partial/\partial_{t_1}) a_n = a_n b_n - b_{n-1} a_n$$

which can be interpreted as noncommutative Toda system.