Hardinian arrays

Robert Dougherty-Bliss (with Manuel Kauers)

October 26, 2023

1, 2, 3, 4, 5, 6, 7, 8, 9, 1, 2, 4, 5, 6, 8, 9, 11, 21, 32, 33, 43, 44, 74

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A326344:

> *Begin with 1. Thereafter, if n is prime, a(n) is the next prime after a(n-1), but written backwards. If n is not prime, a(n) is the next composite after a(n-1), written backwards.*

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By accident, a(9) = 9, so

a(10) = backwards(nextcomposite(9)) = 1.

Discussion		
Wed Sep 11	12:11	**Michel Marcus**: apparently 10^8 terms without getting 4-digits : but let's wait for some confirmation and longer runs
	13:34	**Michel Marcus**: apparently a(n) does not go beyond 909
	17:11	**Max Tohline**: Yeah, 909's definitely the limit in my dataset (first hits at n = 21752, then 8 more times in the next 100000 terms). But if there were the exact-right series of prime gaps, could it exceed that? I don't know how to prove it can't get higher than 909.

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Discussion		
Thu Sep 12	15:55	**Rémy Sigrist**: added proof that the sequence is bounded

A proof came within 24 hours.

Conjectures in the OEIS get in front of a lot of people!

K&K

Kauers and Koutschan recently had a great sequence of thoughts.

- There are lots of interesting conjectures in the OEIS.
- There is more OEIS data than anyone can process alone.
- A program could search for promising conjectures.

Guessing recurrences

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Specifically, recurrences of the form

$$p_d(n)a(n+d) + p_{d-1}(n)a(n+d-1) + \cdots + p_0(n)a(n) = 0$$

for some polynomials $p_i(n)$.

D-finite

In this case, $a(n)$ is called *D-finite*.

How guessing works

The normal way to guess a recurrence of the form

$$(c_{11}n + c_{10})a(n+1) + (c_{01}n + c_{00})a(n) = 0$$

is to plug in $n = 0, 1, 2, 3$ and set up a system of equations:

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is to plug in $n = 0, 1, 2, 3$ and set up a system of equations:

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 2 & 2 \\ 2 & 4 & 5 & 10 \\ 5 & 15 & 14 & 42 \end{bmatrix} \begin{bmatrix} c_{11} \\ c_{10} \\ c_{01} \\ c_{00} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Then you hope that you have a solution.

(I made these numbers up.)

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They tried their guessing procedure on the entire OEIS.

This produced:

- A lot of junk.
- Some known or easy recurrences.
- About 20 *interesting* recurrences that no one knew.

Highlight

Let $a(n)$ be the number of permutations of n copies of $\{1, 2, 3, 4, 5\}$ such that two neighboring entries differ by at most 1.

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For $n = 2$, we have examples like

$$(2, 1, 1, 2, 3, 3, 4, 5, 4, 5)$$

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Theorem (Kauers and Koutschan)

$a(n)$ satisfies the recurrence given on the following slide.

Theorem (Kauers and Koutschan)

$$
\begin{aligned}
&3n^3(1+n)(1+3n)(2+3n)(3281160+13324928n+\cdots+13113n^8)a(n)\\
&-(1+n)^2(14722560+163505952n+\cdots+878571n^{12})a(n+1)\\
&+2(2+n)^2(20370096+207973548n+\cdots+668763n^{12})a(n+2)\\
&-(2+n)^2(3+n)^4(10512+90060n+\cdots+13113n^8)a(n+3)=0.
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\end{aligned}$$

Obviously the proof is not “by hand.”

Leftovers

K&K left some open conjectures.

- Restricted permutations
- Graph enumeration
- Permanents
- Weird matrix things

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- Restricted permutations
- Graph enumeration
- Permanents
- Weird matrix things (We are here!)

Definition (R.H. Hardin)

Let $H_1(n, k)$ be the number of $n \times k$ matrices which obey the following rules:

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Example for $H_1(6,5)$

$$\begin{bmatrix} 0 & 1 & 2 & 2 & 3 \\ 1 & 1 & 2 & 2 & 3 \\ 2 & 2 & 2 & 3 & 3 \\ 3 & 3 & 3 & 3 & 4 \\ 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 \end{bmatrix}$$

Example for $H_1(6,5)$

$$\begin{bmatrix} \color{red}{0} & 1 & 2 & 2 & 3 \\ 1 & 1 & 2 & 2 & 3 \\ 2 & 2 & 2 & 3 & 3 \\ 3 & 3 & 3 & 3 & 4 \\ 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 \end{bmatrix}$$

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$$\begin{bmatrix} \color{red}{0} & 1 & 2 & 2 & 3 \\ 1 & \color{red}{1} & 2 & 2 & 3 \\ 2 & 2 & 2 & 3 & 3 \\ 3 & 3 & 3 & 3 & 4 \\ 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 & \color{red}{4} \end{bmatrix}$$

Example for $H_1(6, 5)$

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Results

Hardin conjectured

$$H_1(n, n) = \frac{1}{3}(4^{n-1} - 1),$$

and also that $H_1(n, k)$ is a linear polynomial in n for $n \geq k$.

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Theorem (RDB, Kauers)

For $n \geq k \geq 1$,

$$H_1(n,k) = 4^{k-1}(n-k) + \frac{1}{3}(4^{k-1} - 1).$$

The diagonal case

$$\begin{bmatrix} 0 & 1 & 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 2 & 2 & 3 & 4 & 5 \\ 2 & 2 & 2 & 2 & 3 & 4 & 5 \\ 2 & 2 & 3 & 3 & 3 & 4 & 5 \\ 3 & 3 & 3 & 3 & 3 & 4 & 5 \\ 4 & 4 & 4 & 4 & 4 & 4 & 5 \\ 5 & 5 & 5 & 5 & 5 & 5 & 5 \end{bmatrix}$$

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Every valid array can be partitioned into “regions” for each value.

$H_1(n, n)$ is the number of tuples of nonintersecting paths from the first column to the first row.

Path counting

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Theorem (Gessel–Viennot)

Fix n distinct start points x_k and n distinct end points y_k.

Let A be the $n \times n$ matrix where A_{ij} is the number of lattice paths from x_i to y_j.

The determinant of A gives the number of tuples of n non-intersecting paths which take x_i to y_i.

Plan of attack: Find A and compute its determinant.

The matrices

$$\begin{bmatrix} 0 & 1 & 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 2 & 2 & 3 & 4 & 5 \\ 2 & 2 & 2 & 2 & 3 & 4 & 5 \\ 2 & 2 & 3 & 3 & 3 & 4 & 5 \\ 3 & 3 & 3 & 3 & 3 & 4 & 5 \\ 4 & 4 & 4 & 4 & 4 & 4 & 5 \\ 5 & 5 & 5 & 5 & 5 & 5 & 5 \end{bmatrix}$$

There are actually several matrices, because start and stop points are not fixed.

The first row and column each have exactly one “unused” position, so there is a matrix for each pair of position choices.

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$$A = \left(\binom{i+j}{i} \right)_{0 \le i,j < n}$$

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Specifically, A_i^j is A with the i-th row and j-th column deleted.

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Theorem (RDB, Kauers)

$H_r(n, n)$ is D-finite for all $r \geq 1$.

Conjectures

For sufficiently large n:

$$H_2(n,1) = \frac{1}{2}n^2 - \frac{3}{2}n + 1$$
$$H_2(n,2) = 4n^2 - 20n + 25$$
$$H_2(n,3) = 40n^2 - 279n + 497$$
$$H_2(n,3) = 480n^2 - 4354n + 10098$$
$$H_2(n,4) = 6400n^2 - 71990n + 206573$$
$$H_2(n,5) = 90112n^2 - 1212288n + 4150790$$
$$H_2(n,6) = 1306624n^2 - 20460244n + 81385043$$

Similar conjectures for all $H_r(n,k)$.

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- Linz has really great public transit.