Gosper’s algorithm and Bell numbers

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Rutgers Experimental Mathematics Seminar

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$$1, 3, 7, 15, 31, 63, 127, 255, \ldots$$

$$1, 1, 2, 5, 15, 52, 203, 877, \ldots$$

$$0, 1, 1, 3, 9, 31, 121, 523, \ldots$$

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$$A(n) : 0, 1, 1, 3, 9, 31, 121, 523, \ldots$$

The $A(n)$ are called *Gould numbers*. Remember that!

Sums

$$\sum_{k=0}^{n-1} 2^k =$$

$$\sum_{k=0}^{n-1} F_k =$$

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$$\sum_{k=0}^{n-1} \frac{1}{k!} = ???$$

Unfortunately, not every sum has a closed form answer.

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$$\sum_{k=0}^{n} \frac{k^4-15}{k!} = -\frac{n^3+4n^2+9n+15}{n!}$$

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$1, 2, 5, 15, 52, \ldots$

A function $f(n)$ is *hypergeometric* if $f(n+1)/f(n)$ is rational in n.

"Closed form" is usually short-hand for hypergeometric.

The question is to determine when $\sum_k f(k)$ itself is hypergeometric.

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Gosper's miracle algorithm

Given a hypergeometric function $f(k)$, produce an explicit hypergeometric $s(n)$ such that

$$s(n+1) - s(n) = f(n)$$

or output a proof that no such $s(n)$ exists.

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Fact

If

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Just factor out an $s(n)$:

$$s(n) = \frac{1}{s(n+1)/s(n) - 1} f(n).$$

Hypergeometric terms have two parts: *pure* and *polynomial*.

The “pure” part cancels in the quotient.

The “polynomial” part is everything else. (Very roughly.)

$$f(n) = \underbrace{(n-3)}_{\text{polynomial}} \cdot \underbrace{\frac{n!}{2^n}}_{\text{pure}}$$

$$\frac{f(n+1)}{f(n)} = \frac{n-2}{n-3}\frac{n+1}{2}$$

Decompose the quotient with this idea:

$$\frac{f(n+1)}{f(n)} = \frac{a(n)}{b(n)} \frac{c(n+1)}{c(n)}$$

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Gosper's equation

A hypergeometric antidifference $s(n)$ exists if and only if there is a *polynomial* $x(n)$ such that

$$x(n+1)a(n) - x(n)b(n-1) = c(n).$$

In that case,

$$s(n) = \frac{b(n-1)}{c(n)} x(n).$$

Example

When is $f(n) = p(n)n!$ summable?

$$\frac{f(n+1)}{f(n)} = \frac{n+1}{1}\frac{p(n+1)}{p(n)}$$

$$a(n) = n+1 \qquad b(n) = 1 \qquad c(n) = p(n)$$

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When $p(n)$ lies in the vector space

$$\{x(n+1)(n+1) - x(n) \mid x(n) \in F[n]\}.$$

First few basis terms:

$$\{n,\ n^2+1,\ n^3-1,\ \ldots\}$$

Every “pure” hypergeometric term has a vector space of “good polynomial parts.”

$$\sum_{k=0}^{n} \frac{k-1}{k!} = -\frac{1}{n!}$$

$$\sum_{k=0}^{n} \frac{k^2-2}{k!} = -\frac{n+2}{n!}$$

$$\sum_{k=0}^{n} \frac{k^3-5}{k!} = -\frac{n^2+3n+5}{n!}$$

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These are a basis for the "good polynomial parts" for $f(k) = \frac{1}{k!}$.

How do you get this basis?

You get *a* basis by setting $x(n) = -n^k$ in Gosper's equation:

$$p_k(n) = n^{k+1} - (n+1)^k.$$

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\end{aligned}$$

Subtract the correct multiples to simplify:

$$\begin{aligned}
p_0(n) &= n - 1 \\
p_1(n) + p_0(n) &= n^2 - 2 \\
p_2(n) + p_1(n) + 3p_0(n) &= n^3 - 5
\end{aligned}$$

$$\begin{bmatrix} n-1 \\ n^2-2 \\ n^3-5 \\ n^4-15 \\ n^5-52 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 3 & 1 & 1 & 0 & 0 & \cdots \\ 9 & 4 & 1 & 1 & 0 & \cdots \\ 31 & 14 & 5 & 1 & 1 & \cdots \\ \vdots \end{bmatrix} \begin{bmatrix} n-1 \\ n^2-n-1 \\ n^3-n^2-2n-1 \\ n^4-n^3-3n^2-3n-1 \\ n^5-n^4-4n^3-6n^2-4n-1 \\ \vdots \end{bmatrix}$$

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$$

The constants on the left are the row sums:

$$
\begin{aligned}
1 &= 1 \\
2 &= 1+1 \\
5 &= 3+1+1 \\
{\color{red}15} &= {\color{red}9+4+1+1} \\
52 &= 31+14+5+1+1
\end{aligned}
$$

These are bell numbers!

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 3 & 1 & 1 & 0 & 0 & \cdots \\ 9 & 4 & 1 & 1 & 0 & \cdots \\ 31 & 14 & 5 & 1 & 1 & \cdots \\ \vdots & & & & & \end{bmatrix}$$

This matrix was first studied by Gould and Quaintance (2007).

It's defined by $B_{dd} = 1$ and

$$B_{(d+1)j} = \sum_k \binom{d}{k} B_{kj}.$$

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The row sums $b(d) = \sum_j B_{dj}$ satisfy

$$b(d+1) = \sum_k \binom{d}{k} b(k).$$

This explains the bell numbers.

Gould numbers

The entries in the first column are the *Gould numbers* $A(n)$.

$$\sum_{k\geq 0} \frac{A(n)}{n!} x^n = e^{e^x - 1} \int_0^x e^{1-e^t}\, dt.$$

$A(n)$: number of partitions of $\{1, 2, ..., n\}$ where the last block is a singleton. (Blocks are arranged in order of their least element.)

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$A(3) = 3$:

$$\{1\}, \{2\}, \{3\}$$

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$$\{1, 3\}, \{2\}$$

Amazingly, it turns out that

$$\lim_{n\to\infty} \frac{A(n)}{b(n)} = \int_0^\infty \frac{e^{-t}}{t+1}\, dt \approx 0.6.$$

This integral is called the Gompertz constant.

Conjecture by Gould and Quaintance.

Proof by Asakly and friends (2014).

The closed form

There are some combinatorial things here.

With the change of basis matrix, we can "evaluate" the sum:

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Implied bell number identity

For any $n \geq 1$ and any integer $d \geq 0$,

$$b(d) = \frac{\sum_{k=0}^{n-1} k^d n^{\underline{n-k}} + \sum_{j\geq 1} B_{dj} n^{\underline{j}}}{\sum_{k=0}^{n-1} n^{\underline{n-k}}}$$

What happens if you let $n \to \infty$?

Recap

The sum

$$\sum_{k=0}^{n} \frac{k^d - b(d)}{k!}$$

can be expressed in closed form if and only if $b(d)$ is the dth bell number.

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Proposition

The hypergeometric term

$$\frac{z^k(k^d - c(d))}{a^{\overline{k}}}$$

is Gosper summable if and only if

$$c(d) = [x^d/d!]e^{ze^x - z - (a-1)x}$$

That exponential generating function is related to the bell numbers:

$$e^{ze^x - z - (a-1)x} = B(x)^z e^{(1-a)x},$$

where

$$B(x) = \sum_{n \geq 0} \frac{b(n)}{n!} x^n = e^{e^x - 1}.$$

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This implies some neat things. For example,

$$\sum_{k=0}^{n} \frac{(k^d - c(d))2^k}{k!}$$

has a closed form answer if and only if

$$c(d) = \sum_k \binom{n}{k} b(k) b(n-k).$$

Summary

Using Gosper's algorithm as a key tool:

1. Make bad sums good.
2. Find patterns.
3. Prove them.