The Meta C-finite Ansatz

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Cassini's identity

Let $F(n)$ be the nth Fibonacci number.

$$F(n-1)F(n+1) = F(n)^2 + (-1)^n$$

0, 1, 1, 2, 3, 5, 8

$3^2 = 9$

$2 \cdot 5 = 10$

$5^2 = 25$

$3 \cdot 8 = 24$

Cassini's identity

Induction?

$$F(n-1)F(n+1) - F(n)^2$$

$$= F(n-1)\left[F(n) + F(n-1)\right] - F(n)^2$$

$$= \ldots$$

$$= -\left\{F(n-2)F(n) - F(n-1)^2\right\}$$

$$\square$$

The C-finite ansatz

Definition (C-finite sequences)

A sequence $a(n)$ is *C-finite* if it satisfies a linear recurrence with constant coefficients. That is,

$$a(n) = \sum_{k=1}^{d} c_k a(n-k).$$

(We call d the *order* of $a(n)$.)

✓ $F(n) = \underline{1\,F(n-1) + 1\,F(n-2)}$

✗ $n! = n\,(n-1)!$

Equivalent definitions

For differential equators:

$$y(x) = \sum_{k \geq 0} \frac{a(k)}{k!} x^k$$

satisfies a linear differential equation with constant coefficients.

For generatingfunctionologists:

$$f(x) = \sum_{k \geq 0} a(k) x^k$$

is a rational function.

Examples

Perrin numbers (A001608):

$$P(0) = 3 \quad P(1) = 0 \quad P(2) = 2$$
$$P(n+3) = P(n+1) + P(n).$$

(If p is prime, then p divides $P(p)$.)

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Pell numbers (A000129):

$$\lambda(0) = 1 \quad \lambda(1) = 0$$
$$\lambda(n+2) = 2\lambda(n+1) + \lambda(n).$$

($\lambda(n-1)/\lambda(n)$ is a good approximation to $\sqrt{2} - 1$.)

Closure-properties

If $a(n)$ and $b(n)$ are C-finite with order d_a and d_b, then so are the following:

Sequence	Order $\leq$
$a(mn+i)$	d_a
$a(n) \pm b(n)$	$d_a + d_b$
$a(n)b(n)$	$d_a d_b$
$\sum_{k=0}^{n} a(k)b(n-k)$	$d_a + d_b$

$$F(n)^2 - F(n-1)F(n+1)$$

The C-finite ansatz

Cassini's identity again:

$$\underbrace{\boxed{\underbrace{\underbrace{F(n-1)}_{2}\,\underbrace{F(n+1)}_{2}}_{\le 4} - \underbrace{F(n)^2}_{\le 4} - \underbrace{(-1)^n}_{1}}}_{\le 4+4+1=9} = 0$$

Check $n=1,\dots,9$. $\square$ ✓

Conjecture = Proof

Some more identities:

$$\sum_{k=0}^{n} F(k) = F(n+2) - 1$$

$$\sqrt{5}F(n) = \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$\sum_{k=0}^{n} F(k)^2 = F(n)F(n+1)$$

Every “finitely-generated” C-finite identity is routine to prove.

If you guess it, it is almost certainly true.

Fibonacci recurrences

The closure properties of C-finite sequences are *effective*.

Computing a recurrence for, say, $F(2n)$ is easy.

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$F(2n+2) =$

$$\begin{aligned} F(2n) &= x \\ F(2(n+1)) &= x + y \\ F(2(n+2)) &= F(2n+2) + F(2n+3) \\ &= 2x + 3y. \end{aligned}$$

$$F(2(n+2)) - 3F(2(n+1)) + F(2n) = 0$$

$$F(2n) = 3F(2(n-1)) - F(2(n-2))$$

Fibonacci recurrences

Here are the first few recurrences for $n \mapsto F(mn)$:

$$
\begin{aligned}
F_n &= F_{n-1} + F_{n-2} \\
F_{2n} &= 3F_{2(n-1)} - F_{2(n-2)} \\
F_{3n} &= 4F_{3(n-1)} + F_{3(n-2)} \\
F_{4n} &= 7F_{4(n-1)} - F_{4(n-2)} \\
F_{5n} &= 11F_{5(n-1)} + F_{5(n-2)}.
\end{aligned}
$$

1, 3, 4, 7, 11

$(-1)^{m+1}$

$L(1) = 1$

$L(2) = 3$

Lucas numbers!

$L(n) = L(n-1) + L(n-2)$

A general recurrence

Conjecture:

$$F(mn) = L(m)F(m(n-1)) + (-1)^{m+1}F(m(n-2)),$$

where $L(m)$ is the mth Lucas number.

Less routine! “Finite checking” is harder[1].

[1] But it works.

Finite-checking works anyway!

Theorem

If $a(n)$ is C-finite with order d, then

$$n \mapsto \quad a(mn) = \sum_{k=1}^{d} c_k(m)\, a(m(n-k))$$

where $c_k(m)$ is C-finite and has order not exceeding $\binom{d}{k}$.

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In particular,

$$F(mn) = c_1(m) F(m(n-1)) + c_2(m) F(m(n-2))$$

for *some* C-finite sequences $c_1(m)$ and $c_2(m)$.

$c_1(m)$: order ≤ 2.

$c_2(m)$: order ≤ 1.

Conjecture = Proof II

$$F(mn) = L(m)F(m(n-1)) + (-1)^{m+1}F(m(n-2)),$$

$$\begin{aligned}
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This is a proof!

What’s the idea?

It’s just $r^{mn} = (r^m)^n$.

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Every C-finite sequence can be written as

$$a(n) = \sum_k p_k(n) r_k^n.$$

$p_k(n)$: polynomials.

r_k: roots of the characteristic polynomial of $a(n)$.

Characteristic polynomials

Let polynomials in x act on $a(n)$ by

$$x^i a(n) = a(n+i).$$

The characteristic polynomial is the polynomial for $a(n)$'s recurrence.

$$F(n+2) = F(n+1) + F(n)$$
$$\Updownarrow$$
$$x^2 - x - 1$$

There's a map between closed forms and characteristic polynomials.

$$x^2 - x - 1 = (x - \varphi)(x - \bar{\varphi})$$

$$F(n) = \alpha \varphi^n + \beta \bar{\varphi}^n$$

What's the idea?

$$a(n) = \sum_k p_k(n) r_k^n.$$

$$n \mapsto a(mn) = \sum_k p_k(mn) r_k^{mn}$$

$$r_k \mapsto r_k^m \qquad = \sum_k \widetilde{p_k}(n) \left(r_k^m\right)^n$$

What's the idea?

The characteristic polynomial of $a(mn)$ is (a multiple of)

$$(x - r_1^m)(x - r_2^m)\cdots(x - r_d^m).$$

Coefficients:

C-finite w.r.t. m

$$\begin{aligned} x^d &: 1 \\ x^{d-1} &: (-1)^d(r_1^m + r_2^m + \cdots + r_d^m) \\ x^{d-2} &: (-1)^d((r_1 r_2)^m + (r_1 r_3)^m + \cdots) \\ &\vdots \end{aligned}$$

Coefficients are C-finite with respect to m by the closure properties!

Something non-Fibonacci

Perrin sequence again:

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Result:

$$P(mn) = P(m)P(m(n-1)) + c_2(m)P(m(n-2)) + P(m(n-3)),$$

where $c_2(m) = \text{A078712}(m)$.

Using recurrences

Recurrences give us access to:

- Summation identities
- Generating function identities
- Asymptotics

We have *uniform* recurrences in two variables.

Summation identities

Here's an earlier identity:

$$\sum_{0 \le k < n} F(k) = F(n+1) - 1.$$

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Here's an earlier identity:

$$\sum_{0 \le k < n} F(k) = F(n+1) - 1.$$

Here is its older brother:

$$\sum_{0 \le k < n} F(mk) = \frac{F(m(n+1)) + (1 - L(m))F(mn) - F(m)}{L(m) - (-1)^m - 1}.$$

This is easy to get from a (not well-known) recurrence trick.

(Ask me later if you want details!)

Perrin sum

Little more complicated for the Perrin numbers:

$$\sum_{0\le k<n} P(mk)$$
$$= \frac{P(m(n+2)) + (1-P(m))P(m(n+1)) + (1-P(m)-c_2(m))P(mn)}{P(m)+c_2(m)}$$
$$- \frac{P(2m) + (1-P(m))P(m) + 3(1-P(m)-c_2(m))}{P(m)+c_2(m)}.$$

Products

These polynomial arguments apply to other settings.

As long as you can handle the following deep identities:

$$r^{mn} = (r^m)^n \quad \text{and} \quad r_1^n r_2^n = (r_1 r_2)^n.$$

$$n \mapsto F(in)\,F(jn)$$

Products

These polynomial arguments apply to other settings.

As long as you can handle the following deep identities:

$$r^{mn} = (r^m)^n \quad \text{and} \quad r_1^n r_2^n = (r_1 r_2)^n.$$

Theorem

Let $a(n)$ be C-finite of order d, and set $P_{ij}(n) = a(in)a(jn)$. Then, $P_{ij}(n)$ satisfies

$$P_{ij}(n) = \sum_{k=1}^{D} c_k(i,j) P_{ij}(n-k)$$

where each $c_k(i,j)$ is C-finite in i and j, symmetric in i and j, and has order not exceeding $\binom{d^2}{k}$.

One example

Let $P_{ij}(n) = F(in)F(jn)$.

Then:

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Then:

$$\begin{aligned} P_{ij}(n) = {} & ((-1)^j L(i-j) - L(i+j)) P_{ij}(n-1) \\ & - (-1)^j (L(i-j)L(i+j) + 2(-1)^i) P_{ij}(n-2) \\ & + (-1)^i (L(i+j)(-1)^j + L(i-j)) P_{ij}(n-3) \\ & - P_{ij}(n-4). \end{aligned}$$

There are implicit (messy!) summation identities.

Review

If $a(n)$ is C-finite, then there are "meta" C-finite patterns.

- Uniform recurrences for $a(mn)$
- Uniform summation identities for $\sum_k a(mk)$
- Uniform generating functions $\sum_k a(mk)x^k$ (Not shown, but easy!)
- Uniform recurrences for $a(in)a(jn)$
- $\ldots$