Modular arithmetic with trinomial moduli

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Rutgers Experimental Mathematics Seminar

Joint work with Mits Kobayashi, Natalya Ter–Saakov, and Eugene Zima

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Why? Plug in $x = 2^c$ into the identity

$$(x^5 - x + 1)(-x^2) + (x^5 - x^3 + 1)(x^2 + 1) = 1.$$

How do you find this identity? Extended GCD algorithm.

$$(2^{6} - 2^{2} + 1)^{-1} \equiv 2^{4} - 2^{2} + 1 \pmod{2^{6} - 2^{5} + 1}$$

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What went wrong?

$$(x^6 - x^2 + 1)(-x^5 + 4x^4 - 5x^3 + x^2 - 3x + 2) + (x^6 - x^5 + 1)(x^5 - 3x^4 + 2x^3 + x^2 + 3x + 5) = 7.$$

The 7 ruins us!

$$(x^5 - x + 1, x^5 - x^3 + 1) \rightarrow \text{good inverse pattern!}$$
$$(x^6 - x^2 + 1, x^6 - x^5 + 1) \rightarrow \text{bad inverse pattern!}$$

What's the difference?

$$\mathrm{res}(x^5 - x + 1, x^5 - x^3 + 1) = 1$$
$$\mathrm{res}(x^6 - x^2 + 1, x^6 - x^5 + 1) = 7,$$

where $\mathrm{res}(f, g)$ is the *resultant* of f and g.

For the non-experts: Resultants are like determinants.

If f and g are monic, then

$$\mathrm{res}(f,g) = \pm \prod_{\substack{f(z)=0 \\ g(w)=0}} (z-w)$$

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Important facts:

- Resultants can be computed without knowing any roots.
- If f and g have integer coefficients, then $\operatorname{res}(f, g)$ is an integer.
- $\operatorname{res}(f, g) = 0$ iff f and g have a common factor.

$$\mathrm{res}(x^5 - x + 1, x^5 - x^3 + 1) = 1$$
$$\mathrm{res}(x^6 - x^2 + 1, x^6 - x^5 + 1) = 7$$

The inverse sequence

$$(2^{cn} - 2^{ck} + 1)^{-1} \bmod (2^{cn} - 2^{cj} + 1)$$

will be "nice" if and only if $\mathrm{res}(x^n - x^k + 1, x^n - x^j + 1)$ is $\pm$ a power of 2.

Definition

$x^n - x^k + 1$ and $x^n - x^j + 1$ *dyadically resolve* if their resultant is a signed power of 2.

Original motivation: Faster Chinese remaindering for integer computations.

The "trinomial moduli"

$$2^n - 2^k + 1 \qquad (n \text{ fixed}, 0 < k < n)$$

have nice binary properties, but we need nice inverse sequences!

Basic questions

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2. Are there arbitrarily large sets of moduli that have pairwise "good" inverses?

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Basic questions (new)

1. When do $x^n - x^k + 1$ and $x^n - x^j + 1$ dyadically resolve?
2. Are there arbitrarily large sets of dyadically resolving trinomials?
3. How can we efficiently find these sets?

Definition

Let $T(n)$ be the graph with vertices $\{1, 2, 3, \ldots, n-1\}$ that contains the edge $\{k, j\}$ if and only if $x^n - x^k + 1$ and $x^n - x^j + 1$ dyadically resolve.

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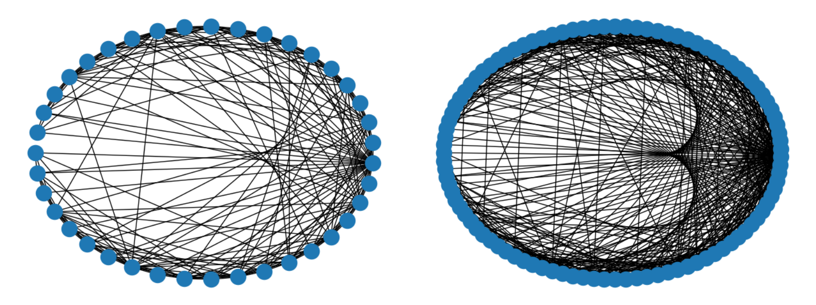

$T(40)$ and $T(100)$

What's going on here?

Symmetry? Heart? Circle?

Queston 1

When is $\mathrm{res}(x^n - x^k + 1, x^n - x^j + 1)$ a signed power of 2?

What are the edges of $T(n)$?

$$\begin{pmatrix} 0 & 1 & 3 & 1 & 3 & 31 & 9 & 8 & 3 \\ 1 & 0 & 1 & 1 & 4 & 1 & 31 & 16 & 8 \\ 3 & 1 & 0 & 1 & 3 & 1 & 3 & 31 & 9 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 31 \\ 3 & 4 & 3 & 1 & 0 & 1 & 3 & 4 & 3 \\ 31 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 9 & 31 & 3 & 1 & 3 & 1 & 0 & 1 & 3 \\ 8 & 16 & 31 & 1 & 4 & 1 & 1 & 0 & 1 \\ 3 & 8 & 9 & 31 & 3 & 1 & 3 & 1 & 0 \end{pmatrix}$$

$$M(k, j) = \mathrm{res}(x^{10} - x^k + 1, x^{10} - x^j + 1)$$

There are not usually “formulas” for resultants, so this could be hard.

Looks like mostly powers of 2?

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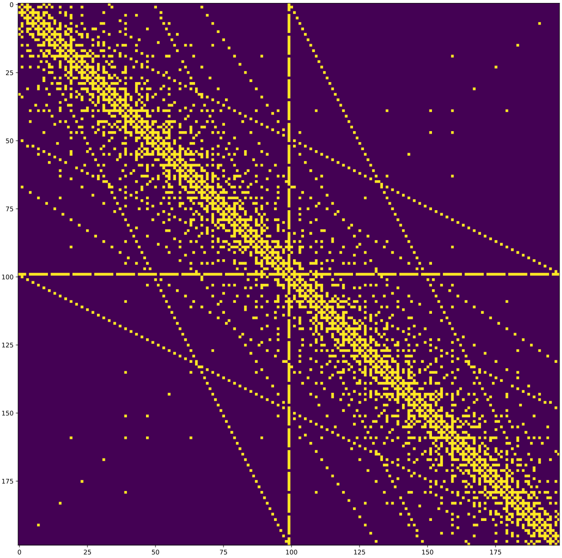
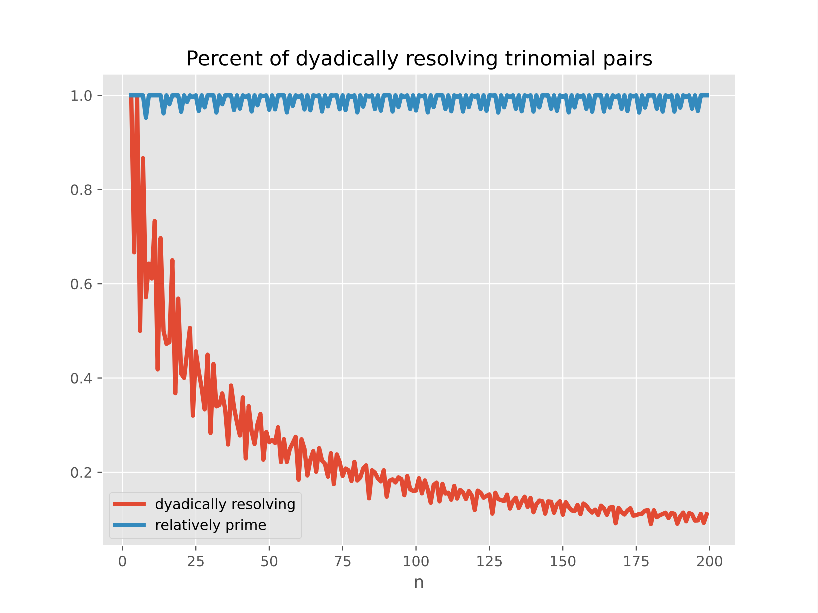


Adjacency matrix of $T(200)$.



Very few powers of 2! But lots of relatively prime pairs?

Theorem (RDB, Kobayashi, Ter-Saakov, Zima)

If $g(x) := \gcd(x^n - x^k + 1, x^n - x^j + 1) \neq 1$, *then:*

- n *is even;*
- $k - j$ *is divisible by* 6*; and*
- $g(x)$ *is a product of cyclotomic polynomials whose orders are multiples of* 6.

Approximately 97% of all pairs of trinomials for large *n* are relatively prime.

We have no corresponding statement for dyadically resolving pairs.

The proportion *should* go to 0.

To understand how complicated this might be, look at this evaluation:

$$\mathrm{res}(x^{900} - x^{22} + 1, x^{900} - x^{72} + 1) = 1125899839733761.$$

Where does this number come from?

Special case

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Special case

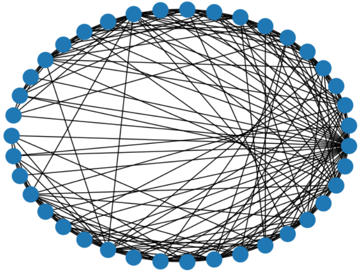
If $k - j$ divides n, then

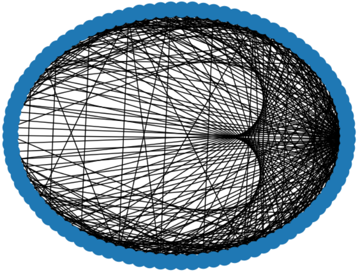
$$\mathrm{res}(x^n - x^k + 1, x^n - x^j + 1) = \pm \left(\prod_{m \mid \frac{k-j}{\gcd(k,j)}} \Phi_m(2) \right)^{\gcd(k,j)}$$

where Φ_m is the mth cyclotomic polynomial.

$$\begin{aligned}\mathrm{res}(x^{900} - x^{22} + 1, x^{900} - x^{72} + 1) &= 1125899839733761 \\ &= (\Phi_5(2)\Phi_{25}(2))^2.\end{aligned}$$

We know of essentially no other formulas!





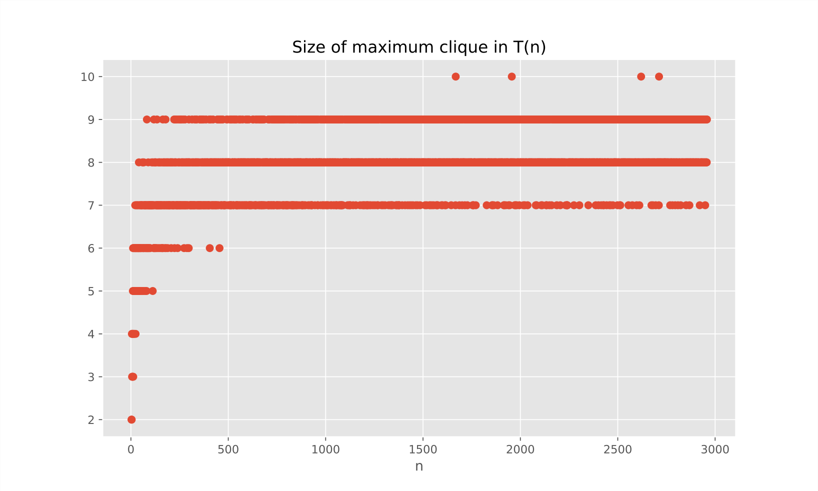
$T(40)$ and $T(100)$

Questions 2 and 3

What is the largest set of *pairwise dyadically resolving* trinomials with degree n?

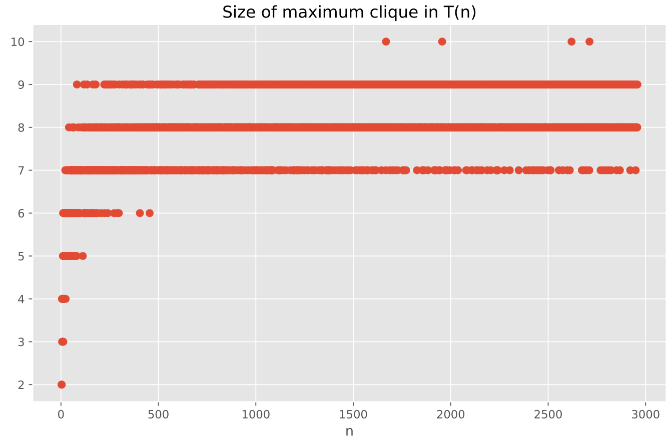
What is the largest clique in $T(n)$?

Computing maximum cliques is fast!



It took approximately 10 years of CPU time to produce this graph.

Clique growth looks slow, but…



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Theorem

The clique number of $T(n)$ goes to ∞ as $n \longrightarrow \infty$.

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Our proof is constructive, but the exponents are big:

$$\{1\}$$

$$\{1, 2\}$$

$$\{2, 3, 4\}$$

$$\{12, 15, 16, 18\}$$

$$\{720, 760, 765, 768, 780\}$$

$$\{48372480, 48434496, 48435465, 48435712, 48436128, 48439664\}$$

The last set implies that there is a clique of size 6 in $T(n)$ for $n > 48439664$.

But there's a clique of size 6 in $T(22)$!

We do not know the true growth rate of the clique numbers.

We have not found a *reasonable* clique of size 11.

clique size k	smallest n
2	3
3	5
4	5
5	10
6	11
7	22
8	41
9	82
10	1668
11	≥ 2985

The best estimate we have is the following.

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Theorem

The largest clique in $T(n)$ has size no larger than $2\lfloor \log_2 n \rfloor - v_2(n)$, where $v_2(n) = v$ is the largest v such that 2^v divides n.

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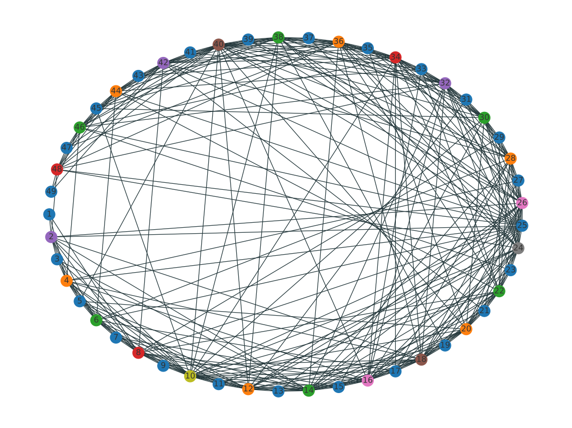

A greedy coloring of $T(50)$ with nine colors.

If a graph can be colored with k colors, then it cannot have a clique of size bigger than k.

How we found the coloring

We found a coloring with the right number of colors purely by experimentation.

The `NetworkX` library does greedy coloring, so we asked nicely.

Colors for $n = 30$:

$$\{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29\}$$

$$\{2, 8, 14, 20, 26\}$$

$$\{4, 10, 16, 22, 28\}$$

$$\{24\}$$

$$\{18\}$$

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$$\{6\}$$

This is 1 mod 2, 2 mod 6, 4 mod 6, then some noise.

The library is slightly too eager, but you can get inspired.

How you prove the coloring

Let's try to compute a resultant. Say that i and k are both odd.

$$\mathrm{res}(x^{10} - x^i + 1, x^n - x^k + 1) = \mathrm{res}(x^{10} - x^i + 1, x^k - x^i)$$

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Because $k - i$ is even, $x + 1$ divides $x^{k-i} - 1$. So,

$$\operatorname{res}(x^{10} - x^i + 1, x + 1) = (-1)^{10} - (-1)^i + 1 = 3$$

divides our resultant.

These cannot dyadically resolve!

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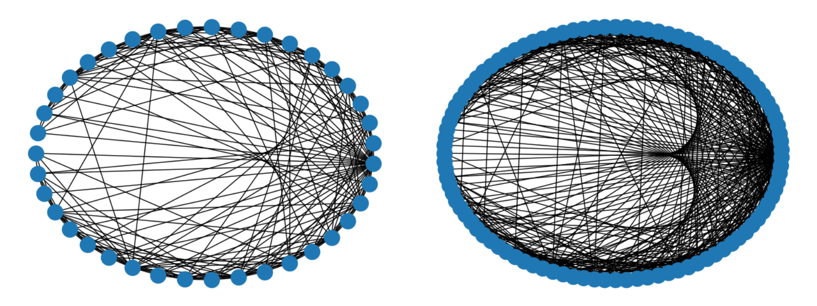
Meaning: $\{1,3,5,7,\ldots\}$ is an independent set in $T(n)$.

Repeat this for different congruence classes.

Symmetries



$T(40)$ and $T(100)$

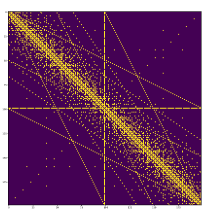
The heart is really there.

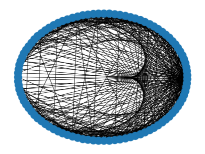
So is the circle.

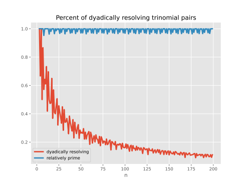
So is the reflectional symmetry.

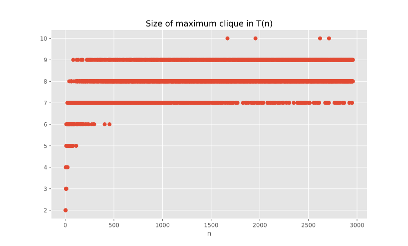
Conjecture: For n large enough, the automorphism group of $T(n)$ is $\mathbb{Z}_2$.

Open questions









True growth of the clique numbers?

Edge density of $T(n)$?

Faster way to compute $T(n)$?

Other moduli shapes: $2^n \pm 2^k \pm 1$, ...