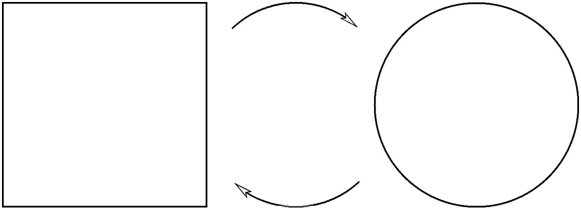
A Two Variable Vandermonde Decomposition of q-binomials Emerging From a Complex Dynamics Problem

Rodrigo Pérez

IU Indianapolis

Experimental Mathematics Seminar, 9/26/24

Riemann Map to Siegel Disks



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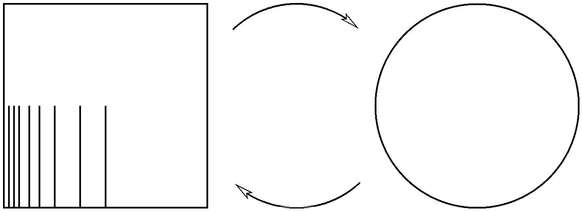

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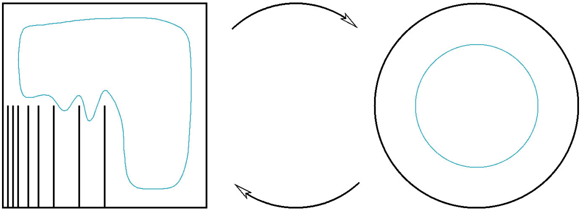

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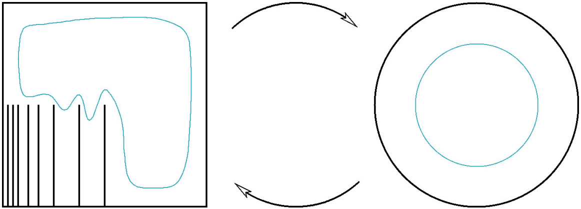

$$f : R \longrightarrow R$$
$$z \mapsto \varphi\big(\lambda \cdot \varphi^{-1}(z)\big)$$

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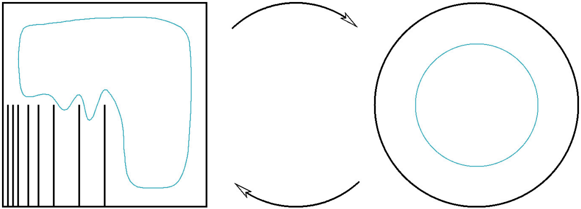

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Can f be polynomial?

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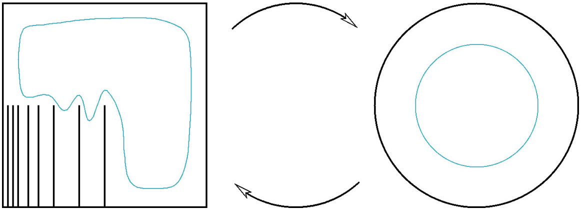

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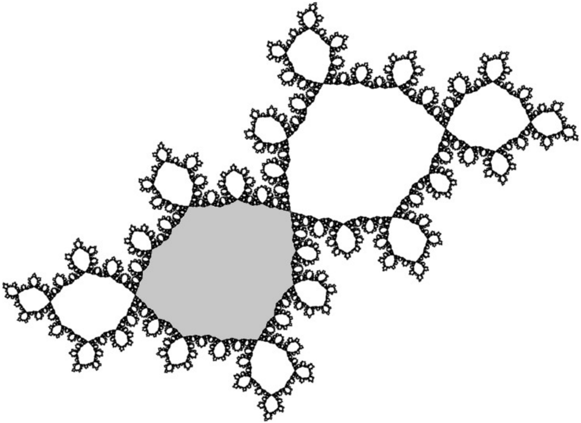
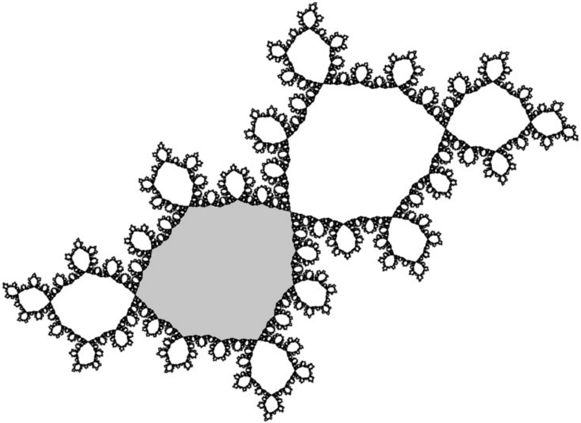
Can f be polynomial? What is the geometry of R?

Quadratic Golden Siegel Disk



$$f(z) = z^2 - (0.39054 + 0.58678i)$$

Quadratic Golden Siegel Disk



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Rotation number $= \varphi$ (i.e., rotates by angle $2\pi\varphi$)

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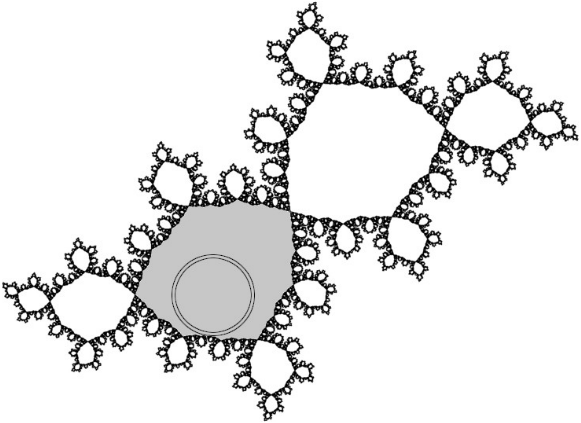

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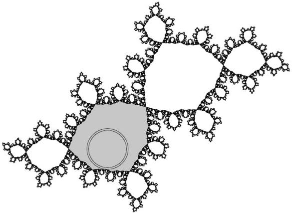
Rotation number $= \varphi$ (i.e., rotates by angle $2\pi\varphi$)

Conjecture (L. Carleson, ca. 1990). The distance from the center of rotation to the boundary of the disk is $\frac{1}{4}$ (attained at the critical value).

Our Approach (joint with M. Aspenberg, Lund University)



Try and compute the exact radius of convergence of the conjugating map.

$$\varphi(z) = z + \sum_{n=2} a_n z^n$$

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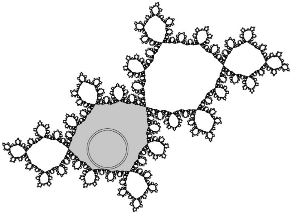

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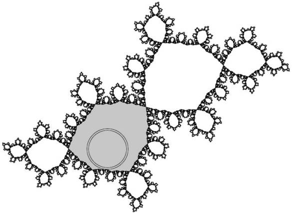

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$$a_n = \left(\frac{\lambda^n}{1-\lambda^n}\right) \cdot \sum_{r=n/2}^{n-1} \binom{r}{n-r} a_r, \text{ where } \lambda = \exp(2\pi i \varphi).$$

Initial Combinatorial Setting I

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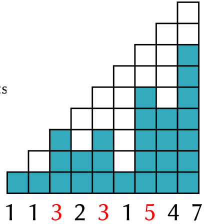
$$a_n = \left(\prod_{j=1}^{n-1} \frac{1}{1-\lambda^j}\right) \cdot \sum_{n\text{-paths}} \lambda^{\Sigma \text{ descents}}$$

Bounded for Diophantine rotation #s ↗



Σ descents $= 3 + 3 + 5 = 11$

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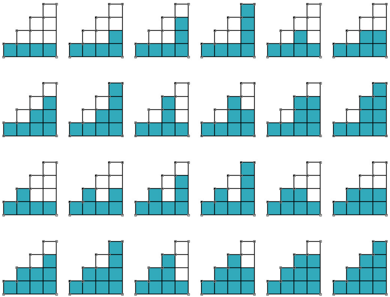

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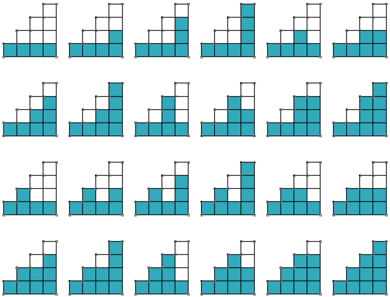

$$\sum_{\text{4-paths}} \lambda^{\Sigma \text{ descents}} = 14 + 4\lambda^2 + 6\lambda^3$$

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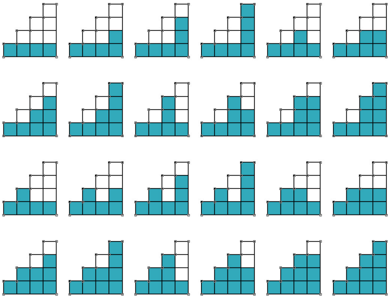

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Here, $14 = C_4$ since Catalan numbers count paths without descents.

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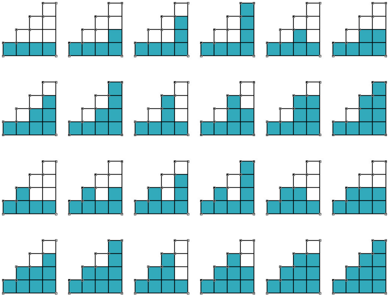

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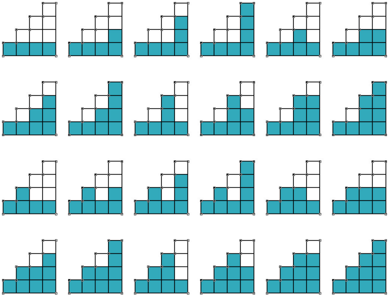

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Strategy: Collect paths in Catalan equivalence families and prove that the contribution of each family is bounded.

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Treat a path like a Lehmer code.

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␣ 2 ␣ ␣ ␣ ␣ ␣ 1 ␣

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␣ 2 ␣ ␣ ␣ ␣ ␣ 1 ␣
␣ 2 ␣ ␣ ␣ ␣ ␣ 1 3

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␣ 2 4 ␣ 5 ␣ ␣ 1 3

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␣ 2 ␣ ␣ ␣ ␣ ␣ 1 3

␣ 2 4 ␣ ␣ ␣ ␣ 1 3

␣ 2 4 ␣ 5 ␣ ␣ 1 3

6 2 4 ␣ 5 ␣ ␣ 1 3

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␣ ␣ ␣ ␣ ␣ ␣ ␣ 1 ␣
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␣ 2 ␣ ␣ ␣ ␣ ␣ 1 3
␣ 2 4 ␣ ␣ ␣ ␣ 1 3
␣ 2 4 ␣ 5 ␣ ␣ 1 3
6 2 4 ␣ 5 ␣ ␣ 1 3
6 2 4 ␣ 5 7 ␣ 1 3
```

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␣ 2 4 ␣ ␣ ␣ ␣ 1 3
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```

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␣	2	␣	␣	␣	␣	␣	1	3
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␣	2	␣	␣	␣	␣	␣	1	3
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␣	2	4	␣	5	␣	␣	1	3
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␣	2	␣	␣	␣	␣	␣	1	3
␣	2	4	␣	␣	␣	␣	1	3
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␣	2	␣	␣	␣	␣	␣	1	3
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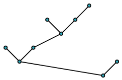
Notice that paths with no descents translate to permutations with no dissents:

$$111344679 \longrightarrow 324657819$$

Catalan Equivalence Class II

The two sample paths 113231547, 111344679 in the previous slide were selected because their corresponding permutations have the same binary tree structure:



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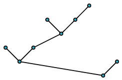

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Each tree equivalence class of permutations contains exactly one representative without dissents, namely the one obtained by labeling vertices in ascending order while traversing the tree in clockwise fashion starting at the root.

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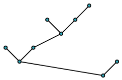

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Each tree equivalence class of permutations contains exactly one representative without dissents, namely the one obtained by labeling vertices in ascending order while traversing the tree in clockwise fashion starting at the root. This permutation contributes 1 to the coefficient sum;

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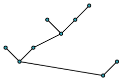

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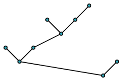

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Revised Goal. Given a binary tree, collect the contributions of all equivalent permutations, and prove that their sum is bounded.

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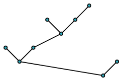

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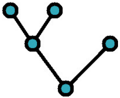
Revised Goal. Given a binary tree, collect the contributions of all equivalent permutations, and prove that their sum is bounded.
Since there are C_n such trees, there are $O(4^n)$ equivalence classes. If each provides a bounded contribution to a_n, the coefficients a_n will have exponential growth 4^n.

Catalan Equivalence Class III

Example. The tree below represents 8 permutations:

43512	53412	42513	52413
32514	52314	32415	42315



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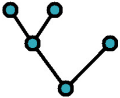
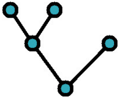


Their dissents are, respectively

2	(2,4)	3	(3,4)
4	4	0	3

Thus the contribution of this tree is $1 + \lambda^2 + 2\lambda^3 + 2\lambda^4 + \lambda^6 + \lambda^7$.

Catalan Equivalence Class IV



Observation. Had we used permutations of $\{0, 1, 2, 3\}$, the true dissents

2	(2,4)	3	(3,4)
4	4	0	3

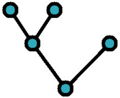
would become

1	(1,3)	2	(2,3)
3	3	0	2

and the contribution of the tree would be

$$1 + \lambda + 2\lambda^2 + 2\lambda^3 + \lambda^4 + \lambda^5 = (1 + \lambda + \lambda^2 + \lambda^3)(1 + \lambda^2)$$

Catalan Equivalence Class IV



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$$1 + \lambda + 2\lambda^2 + 2\lambda^3 + \lambda^4 + \lambda^5 = (1 + \lambda + \lambda^2 + \lambda^3)(1 + \lambda^2)$$

Enticing, since $(1+\lambda^2)$ is the contribution of the left sub-tree, while $(1+\lambda+\lambda^2+\lambda^3)$ is the Gaussian binomial describing the allocation of values on left/right branches.

Main Result

Theorem. (**P**-Aspenberg, '22) If the tree T holds subtrees L, R on its left and right branches, its reduced contribution is

$$\widetilde{P}(T) = P(L) \cdot \widetilde{P}(G) \cdot P(R),$$

where G is the unbranched tree with $m = |L|$ vertices on the left branch, and $n = |R|$ vertices on the right. Moreover, this reduced polynomial is Gaussian:

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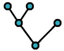
Solution. Use a 2-variable polynomial to keep track of the number of dissents.

For our sample tree, dissents at $0, 1, 2, 2, 3, 3, (1, 3), (2, 3)$ yield

$$P = 1 + (\lambda + 2\lambda^2 + 2\lambda^3)\mu + (\lambda^4 + \lambda^5)\mu^2$$



2-variable Gaussian Binomials

Classically, the binomial $\begin{bmatrix} m+n \\ m \end{bmatrix} = \begin{bmatrix} m+n \\ n \end{bmatrix}$ enumerates paths in an $m \times n$ grid, ranking them by area. We find the same distribution for a different statistic.

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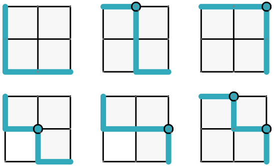
Identify the down/right path with a branched tree that has m vertices on the left and n on the right. Then every corner counts a right-left jump; i.e., a dissent.



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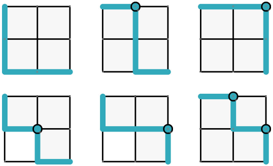

Proposition. $\begin{bmatrix} m+n \\ m \end{bmatrix}$ enumerates paths in the $m \times n$ grid, ranked by corner sum.

In this example, the 2-variable polynomial is

$$P_{\lambda,\mu} = 1 + (1 + 2\lambda^2 + \lambda^3)\mu + (\lambda^4)\mu^2.$$

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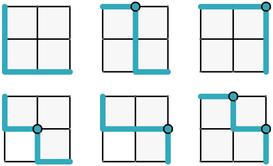

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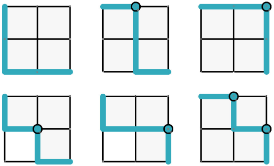

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Substituting $\mu = \lambda$ returns the correct, non-reduced version:

$$P = 1 + \lambda^2 + 2\lambda^3 + \lambda^4 + \lambda^6.$$

Basic Vandermonde

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The last equality is a symmetric version of the standard Chu-Vandermonde.

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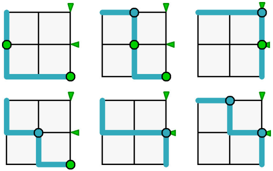

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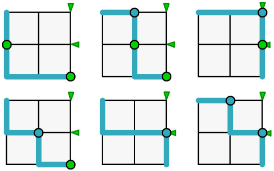

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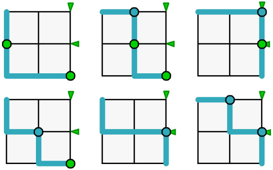

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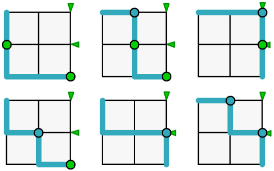

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Main Result Finally! (no proofs...)

Theorem. (**P**-Aspenberg, '22) For any ornament $\mathcal{O} = (H, V)$ on the $m \times n$ grid, with $|H| = d$ and $|V| = r$, the generating function of the bi-statistic (CINDEX, CORNERS) over all paths is the polynomial

$$\begin{bmatrix} m+n \\ m \end{bmatrix}_{\lambda,\mu}^{\mathcal{O}} = \lambda^s \cdot \sum_c \begin{bmatrix} m+r-d \\ c-d \end{bmatrix} \cdot \begin{bmatrix} n+d-r \\ c-r \end{bmatrix} \cdot \lambda^{(c-d)(c-r)} \cdot \mu^c$$

THANK YOU!