Thresholds

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PLAN

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- In this talk I will introduce **the Kahn-Kalai Conjecture**, a central conjecture in probabilistic combinatorics, and discuss **our results** related to it.
- We first start with a brief introduction to **random graphs**, to begin with a concrete picture in mind. (Also, historically random graph theory was the starting point for our story.) **The setting of our work is much more general.**

I. Random Graphs

Random graph $G_{n,p}$

- $G_{n,p}$ (Erdős-Rényi random graph):
  - vertex set $= [n] := \{1, 2, \ldots, n\}$; think n (finite but) large.
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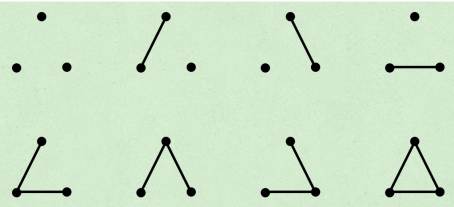

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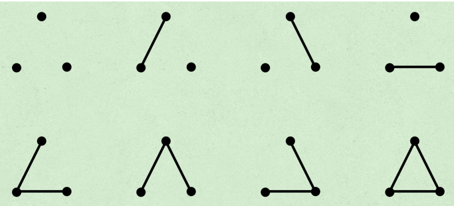

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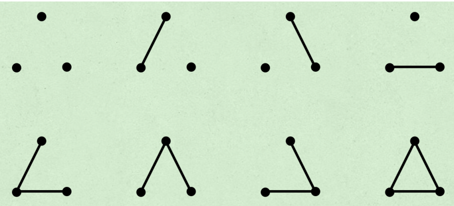

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- whp (with high probability): "$G_{n,p}$ does A whp" means

$$\mathbb{P}(G_{n,p} \text{ does } A) \to 1 \text{ as } n \to \infty.$$

Evolution of $G_{n,p}$

$\asymp$: same order

One **striking** thing about $G_{n,p}$ is: appearance and disappearance of certain properties are "abrupt." $\rightarrow$ **thresholds**

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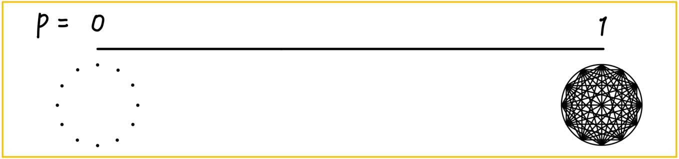

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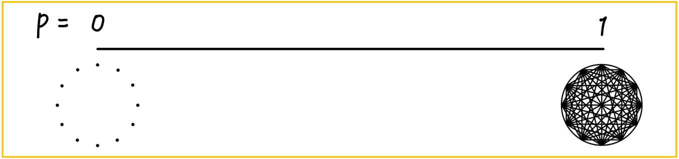

E.g. (typical) maximum size of connected components of $G_{n,p}$

$$\begin{cases} \lesssim \log n & \text{if } np < 1-\epsilon \\ \asymp n & \text{if } np > 1+\epsilon \end{cases}$$

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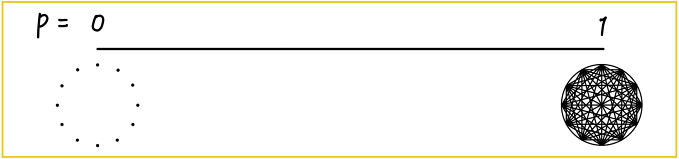

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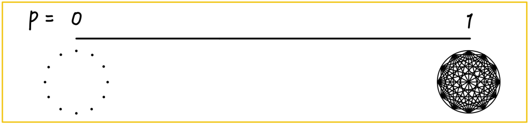
Central interest in Probabilistic Combinatorics

Find thresholds for various properties!

Thresholds for various properties of random graphs



Some past results on finding thresholds

- Small subgraphs (Erdős-Rényi '59, Bollobás '81)
- Connectivity (Erdős-Rényi '59)
- Perfect matchings (Erdős-Rényi '66)
- Long paths and cycles (Ajtai-Komlós-Szemerédi '81, de la Vega '79)
- Hamilton cycles (Pósa '76, Bollobás '84, Ajtai-Komlós-Szemerédi '85)
- Ramsey properties (Rödl-Ruciński '95)
- Clique factors (Johansson-Kahn-Vu '08)
- Spanning trees (Krivelevich '10, Montgomery '19)

...

$\rightarrow$ Study has been done mostly for specific properties.

II. Thresholds

Basic definitions

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  $\to X_p$: random CNF formula

- $\mathcal{F} \subseteq 2^X$ is an **increasing property** if

$$B \supseteq A \in \mathcal{F} \Rightarrow B \in \mathcal{F}$$

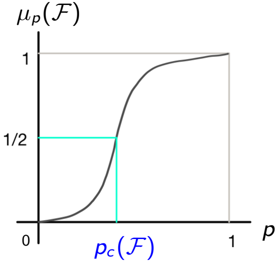
  e.g.1. $\mathcal{F} = \{\text{connected}\}$; $\mathcal{F} = \{\text{contain a triangle}\}$

  e.g.2. $\mathcal{F} = \{\text{not satisfiable}\}$

Thresholds

Fact.

For any increasing property $\mathcal{F}$ ($\neq \emptyset, 2^X$), $\mu_p(\mathcal{F})$ ($= \mathbb{P}(X_p \in \mathcal{F})$) is continuous and strictly increasing in p.



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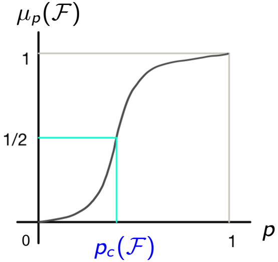

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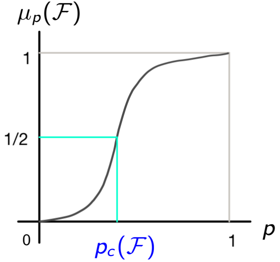
- cf. Erdős-Rényi: $p_0 = p_0(n)$ is **a threshold function** for $\mathcal{F}_n$ if

$$\mu_p(\mathcal{F}_n) \to \begin{cases} 0 & \text{if } p \ll p_0 \\ 1 & \text{if } p \gg p_0 \end{cases}$$

* $p_c(\mathcal{F}_n)$ is always an Erdős-Rényi threshold (Bollobás-Thomason '87).

Study of thresholds



- **Location** of thresholds:
  - historically most of interesting work was on thresholds for specific properties.
  - **The Kahn-Kalai Conjecture** ('06) suggests a general bound.
- **Sharpness** of thresholds: Kahn-Kalai-Linial ('88), Friedgut-Kalai ('96), Friedgut ('99) all based on Fourier analysis

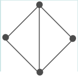
III. The Kahn-Kalai Conjecture

"It would probably be more sensible to conjecture that it is **not** *true."*

\- Kahn and Kalai (2006)

Question.

What drives $p_c(\mathcal{F})$?

Example 1. Containing a copy of H 

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- $X = \binom{[n]}{2}$ (so $X_p = G_{n,p}$); $\mathcal{F}_H$: contain a copy of H

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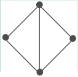

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- Usual suspect: expectation calculation

$$\mathbb{E}[\#\ H\text{'s in } G_{n,p}] \asymp n^4 p^5 \to \begin{cases} 0 & \text{if} \quad p \ll n^{-4/5} \\ \infty & \text{if} \quad p \gg n^{-4/5} \end{cases}$$

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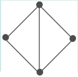
Example 1. Containing a copy of H 

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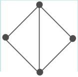

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Dream

$\mathbb{E}$ predicts $p_c(\mathcal{F})$?

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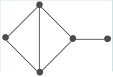

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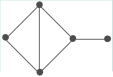

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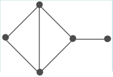

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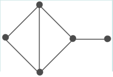

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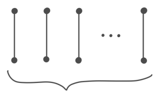
Erdős-Rényi ('60), Bollobás ('81)

(Rough:) For **fixed** graph H,

$$p_c(\mathcal{F}_H) \asymp \text{"threshold for } \mathbb{E}\text{" of the "densest" subgraph of H}$$

Example 3. Containing a perfect matching



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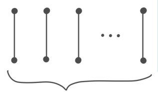

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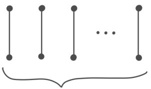

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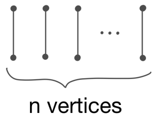

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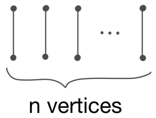

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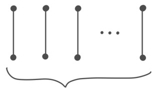

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- **Coupon collector problem**: Each box of cereal contains a random coupon, and there are n different types of coupons.

Question.

How many boxes of cereal do we (typically) need to buy to collect all n coupons?

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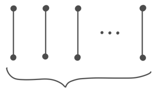

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How many boxes of cereal do we (typically) need to buy to collect all n coupons?

Answer $\asymp n \log n$

One more example: perfect hypergraph matchings

- Now, $X = \binom{[n]}{r}$
- $X_p =$ random r-uniform hypergraph $\mathcal{H}^r_{n,p}$

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- $p_c(\mathcal{F}) \asymp \log n/n^2$ (Johansson-Kahn-Vu '08) * $\log n$ **gap again**

What drives $p_c(\mathcal{F})$?

- We have some **trivial lower bounds** on p_c:
  - Ex 1, 2 (contain H/$\tilde{H}$): "threshold for $\mathbb{E}$"
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  - threshold for bounded degree spanning trees ("tree conjecture"; Montgomery '19)

$p_{\mathsf{E}}(\mathcal{F})$: the expectation threshold

- For abstract $\mathcal{F}$, it's unclear whose expectation we want to compute, so need a careful definition for the "threshold for $\mathbb{E}$."

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Observation

$p_c(\mathcal{F}) \geq q$ if $\exists\ \mathcal{G} \subseteq 2^X$ such that

1. "$\mathcal{G}$ covers $\mathcal{F}$": $\forall A \in \mathcal{F}\ \ \exists B \in \mathcal{G}$ such that $A \supseteq B$
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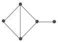
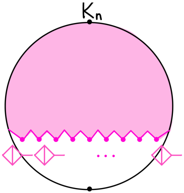
e.g. in Ex 2, $X = \binom{[n]}{2}$, $\mathcal{F}$: contain a copy of $\tilde{H}$ 



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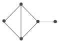
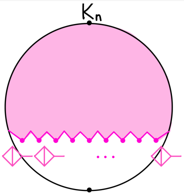
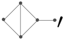
e.g. in Ex 2, $X = \binom{[n]}{2}$, $\mathcal{F}$: contain a copy of $\tilde{H}$ 



- $\mathcal{G}_1 = \{$all (labeled) copies of $\tilde{H}$  $'s\}$

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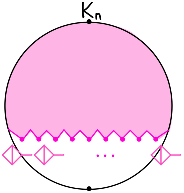
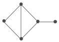
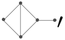


e.g. in Ex 2, $X = \binom{[n]}{2}$, $\mathcal{F}$: contain a copy of $\tilde{H}$ 

- $\mathcal{G}_1 = \{$all (labeled) copies of $\tilde{H}$  $'s\}$

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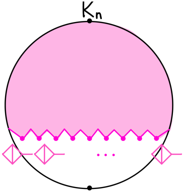
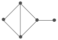
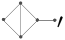


e.g. in Ex 2, $X = \binom{[n]}{2}$, $\mathcal{F}$: contain a copy of $\tilde{H}$ 

- $\mathcal{G}_1 = \{$all (labeled) copies of $\tilde{H}$  's$\}$

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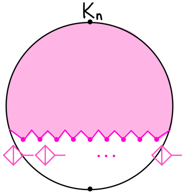

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The Kahn-Kalai Conjecture ('06)

There exists a universal $K > 0$ such that for every finite X and increasing $\mathcal{F} \subseteq 2^X$,

$$(p_{\text{E}}(\mathcal{F}) \leq)\; p_c(\mathcal{F}) \leq K p_{\text{E}}(\mathcal{F}) \log |X|$$

Results and Proof Sketch

Conj of Talagrand: fractional version of Kahn-Kalai Conj

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- Weaker than KKC, but in all known applications, $p_{\mathsf{E}}(\mathcal{F}) \asymp p^*_{\mathsf{E}}(\mathcal{F})$
- Proof inspired by Alweiss-Lovett-Wu-Zhang

"Erdős-Rado Sunflower Conjecture"

New result

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- Proofs inspired by ALWZ (sunflower) and FKNP (fractional Kahn-Kalai) but implementation different

Thank you!