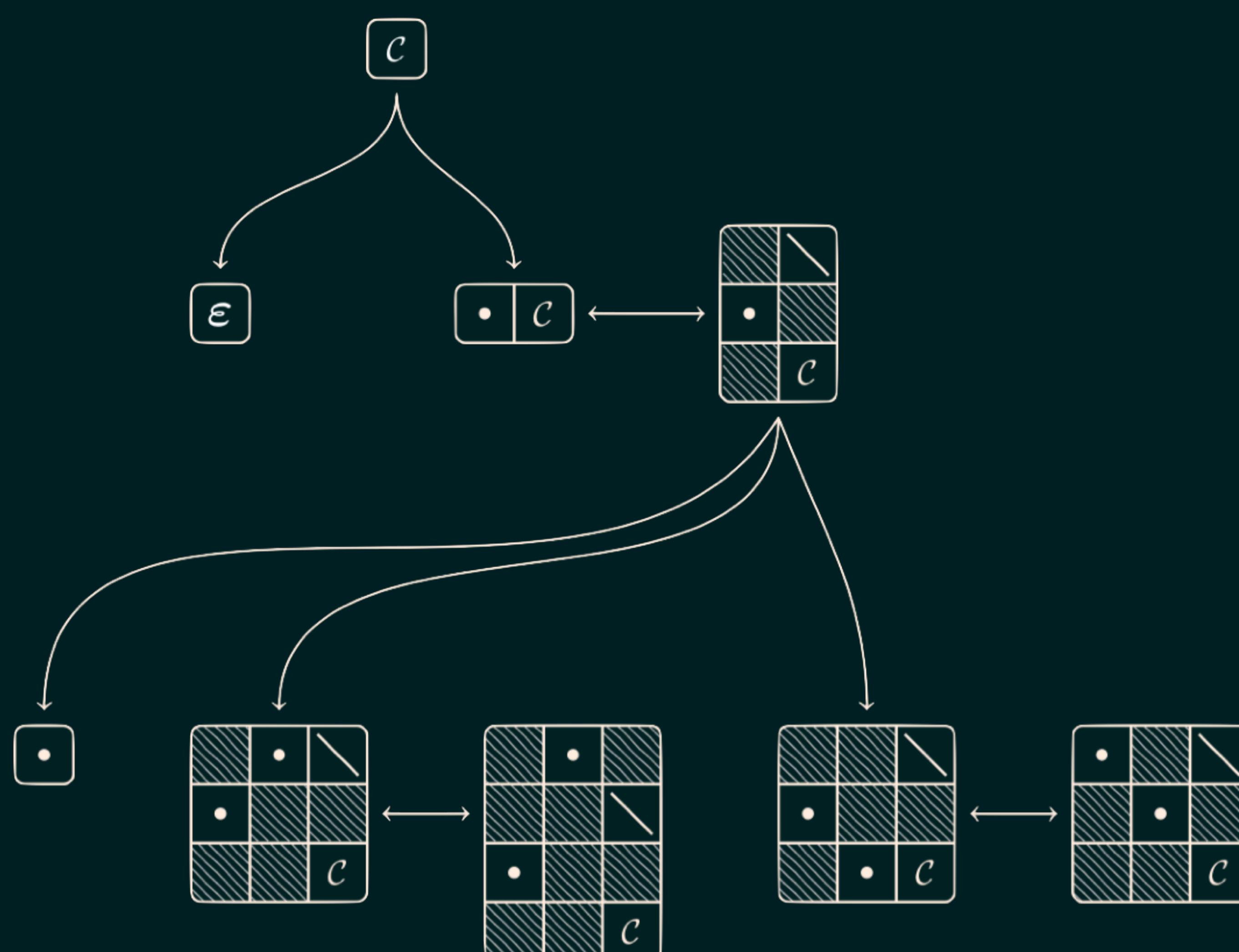
Combinatorial Exploration

an algorithmic framework for enumeration

Jay Pantone
Marquette University

with:
Michael Albert
Christian Bean
Anders Claesson
Émile Nadeau
Henning Ulfarsson



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[Submitted on 15 Feb 2022]

Combinatorial Exploration: An algorithmic framework for enumeration

Michael H. Albert, Christian Bean, Anders Claesson, Émile Nadeau, Jay Pantone, Henning Ulfarsson

Combinatorial Exploration is a new domain-agnostic algorithmic framework to automatically and rigorously study the structure of combinatorial objects and derive their counting sequences and generating functions. We describe how it works and provide an open-source Python implementation. As a prerequisite, we build up a new theoretical foundation for combinatorial decomposition strategies and combinatorial specifications.
We then apply Combinatorial Exploration to the domain of permutation patterns, to great effect. We rederive hundre
new ones. These results c
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concept, showing exampl
of alternating sign matrice

Subjects: Combinatorics (math.
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References & Citations

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We would like to thank the many undergraduate and masters students with whom we had invaluable discussions and who contributed to this work, including (with references to their theses when applicable): Annija Apine, Ragnar Páll Árdal [13], Arnar Bjarni Arnarsson [14, 15], Alfur Birkir Bjarnason [15], Jon Steinn Eliasson [75], Unnar Freyr Erlendsson [15, 76], Kolbeinn Páll Erlingsson [87], Bjarki Ágúst Guðmundsson, Björn Gunnarsson [87], Sigurður Helgason [126], Kristmundur Ágúst Jónsson [87], Tómas Ken Magnússon [111], James Robb [126], Óðinn Hjaltason Schiöth, Murray Tannock, and Sigurjón Freyr Viktorsson [15].

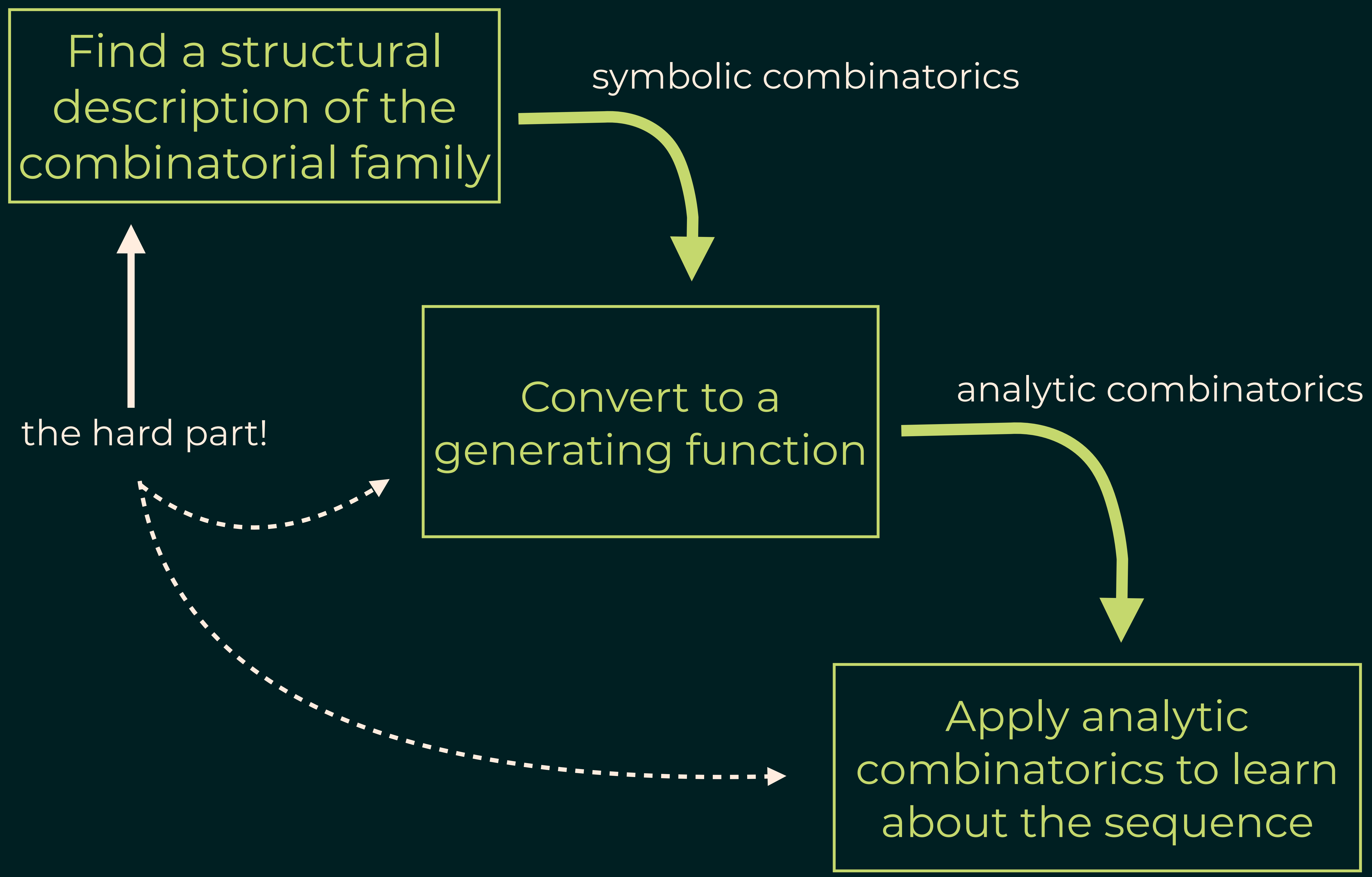
- A <u>combinatorial family</u> is a set of objects defined by some property.

    - walks in the plane that never collide with themselves

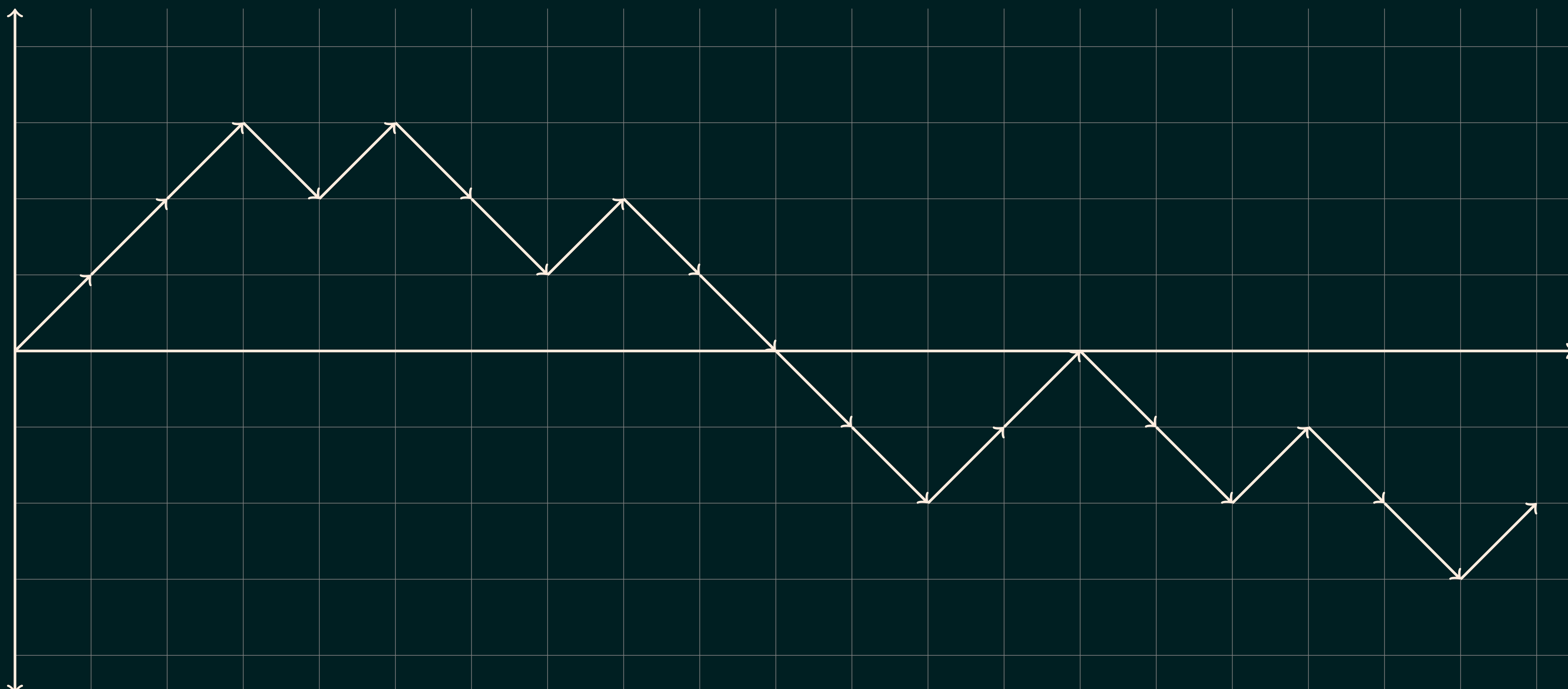
    - permutations whose entries never form certain patterns

    - polyominoes whose columns are all convex

- Questions:
  - How many are there of each size?
    - explicit formula, generating function, polynomial-time algorithm
  - How does the counting sequence grow asymptotically as $n \to \infty$?
  - How can I sample an object of size n uniformly at random?
  - How can I build the objects of size n from the objects of smaller size?


Find a structural description of the combinatorial family
symbolic combinatorics
Convert to a generating function
analytic combinatorics
Apply analytic combinatorics to learn about the sequence
the hard part!

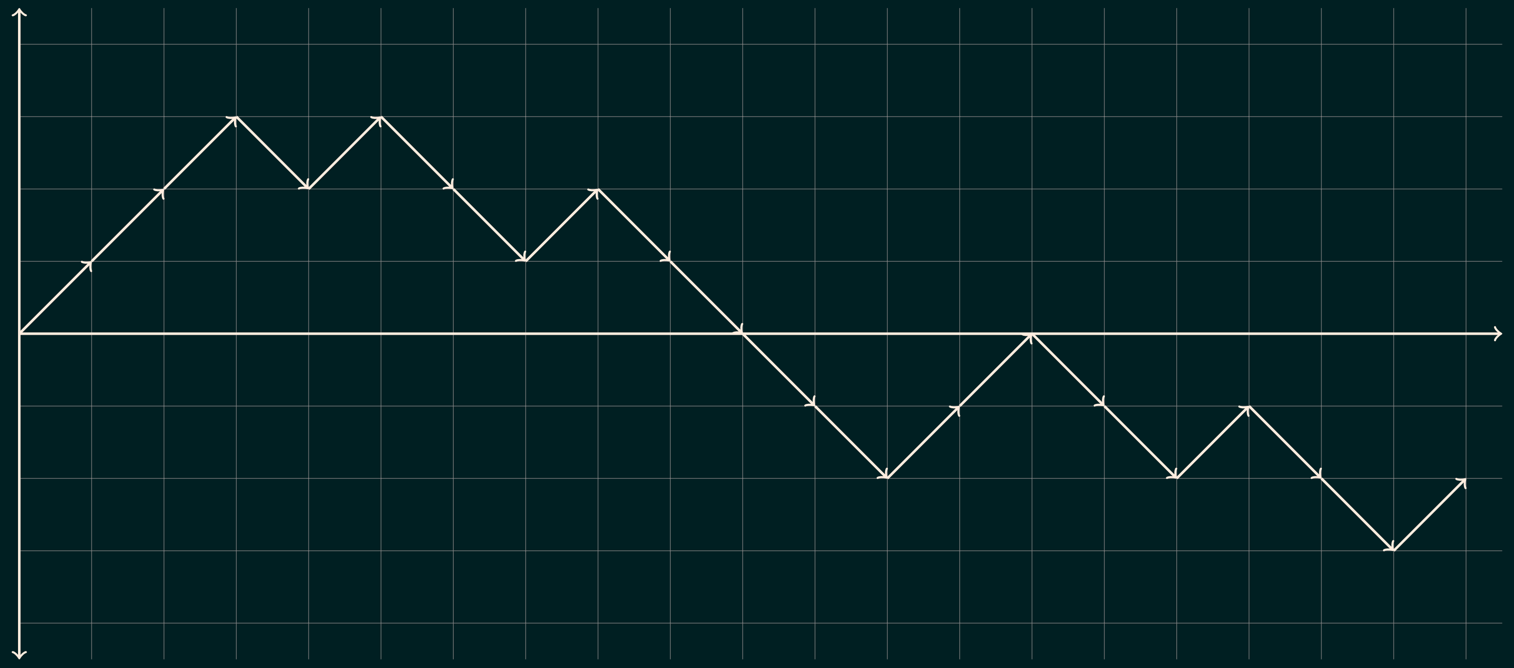
An *up-down walk* is a walk in the plane that starts at the origin and takes only NE and SE steps.



size = # of steps = 20

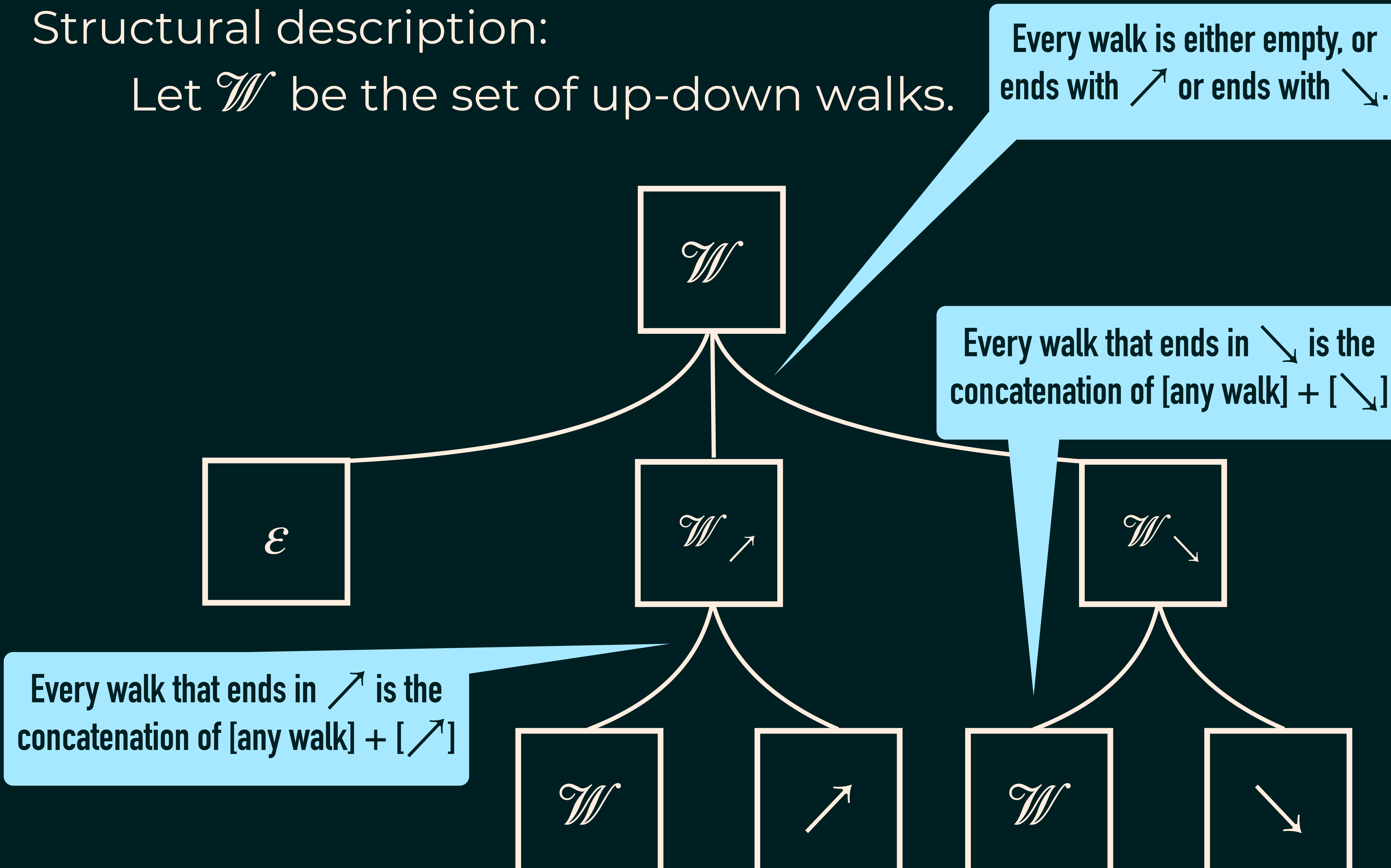
Before we ask questions, we need to understand the structure.

- The set of up-down walks of size n can be built by appending either a NE step or a SE step to every up-down walk of size $n-1$.
- Let's write this structural description in a tree format.



Structural description:

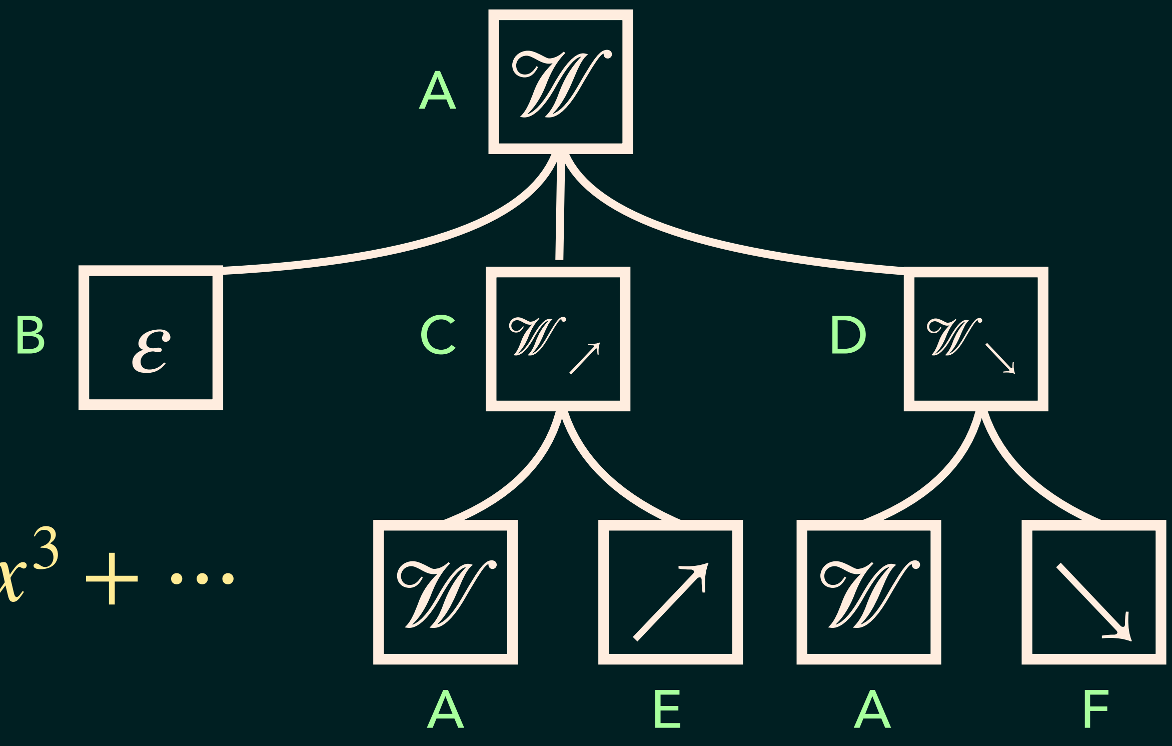
Let $\mathscr{W}$ be the set of up-down walks.



What do we learn from this structural decomposition?

Systems of equations for generating functions!

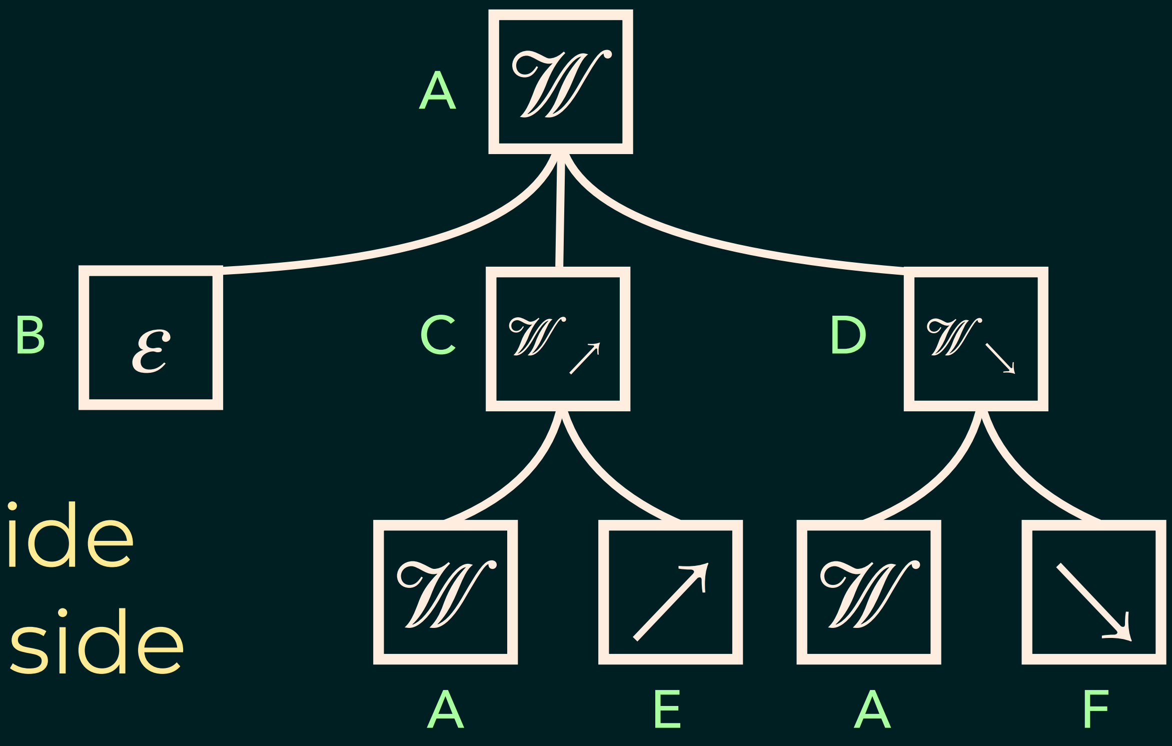
$$
\begin{aligned}
A(x) &= B(x) + C(x) + D(x)\\
B(x) &= 1\\
C(x) &= A(x)E(x)\\
D(x) &= A(x)F(x)\\
E(x) &= x\\
F(x) &= x
\end{aligned}
$$



$$\Longrightarrow A(x) = \frac{1}{1-2x} = 1 + 2x + 4x^2 + 8x^3 + \cdots$$

These structural description trees are just a pictorial way to represent a combinatorial specification.

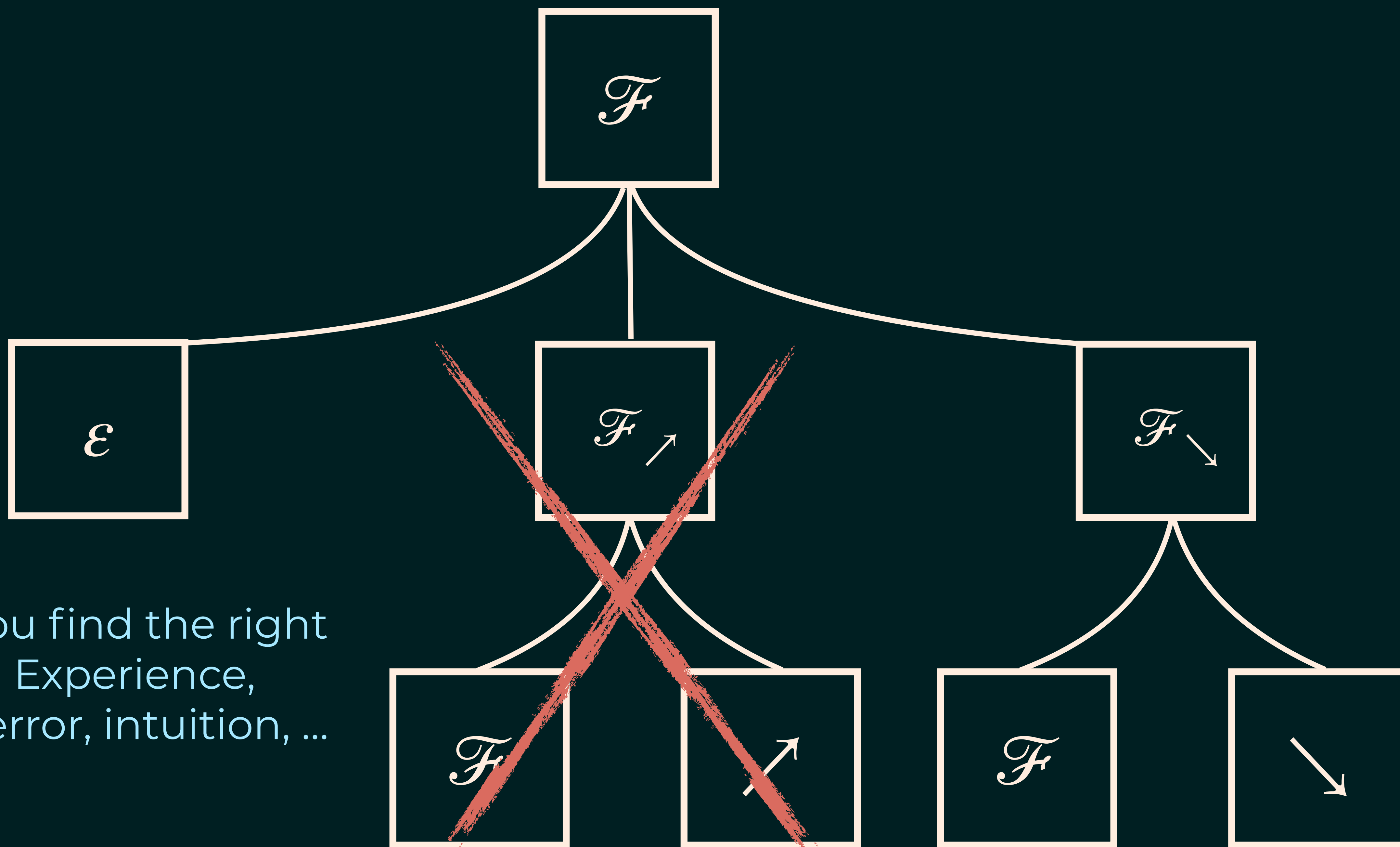
$$
\begin{aligned}
A &\to (B, C, D)\\
B &\to \{\varepsilon\}\\
C &\to (A, E)\\
D &\to (A, F)\\
E &\to \{\ \nearrow\ \}\\
F &\to \{\ \searrow\ \}
\end{aligned}
$$



every symbol on the right-hand side appears on exactly one left-hand side

Slightly more complicated:

$\mathscr{F}$ = the set of walks that don't go up three times in a row



How do you find the right structure? Experience, trial-and-error, intuition, …

Find a structural description of the set of objects

let's automate this part

symbolic combinatorics

Convert to a generating function

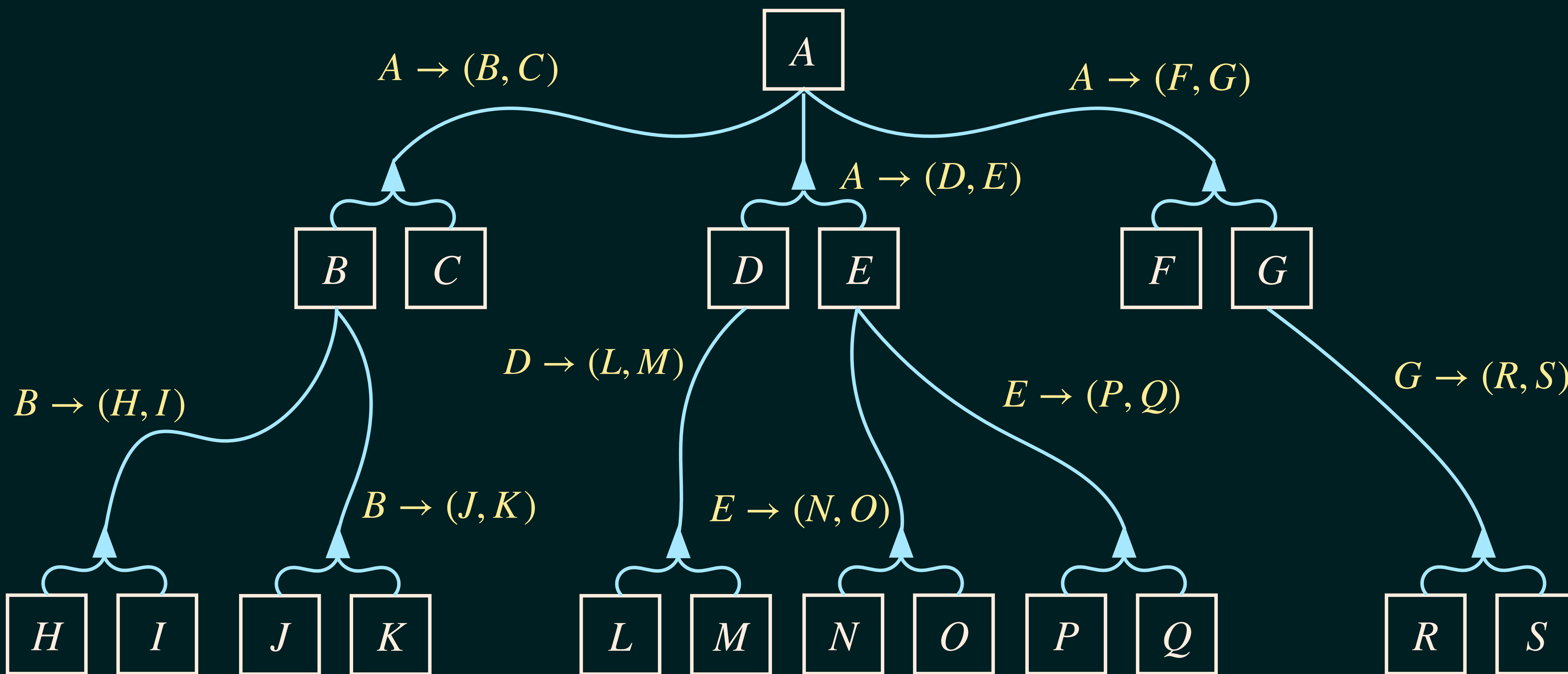
analytic combinatorics

Apply analytic combinatorics to learn about the sequence

Requirements:

- a domain of all objects (up-down walks)
- a representation for the sets of objects that you'll be working with ("$\mathscr{W}_{\nearrow}$" is the set of up-down walks that end with $\nearrow$)
- decomposition strategies to split the sets into (hopefully) simpler sets

develop strategies for a whole domain → apply them to subsets of the domain you want to learn about



this is just a pictorial version of a list of combinatorial rules

when the giant list of rules you're generating contains a subset that is a combinatorial specification, you win!

Caveats:

- This is the main idea, but there's a lot of complicated machinery going on under the hood.
- Many of the internal steps require clever efficient algorithms.
- If you're not careful, the combinatorial specifications you get as output could be tautological.
- ~ 31,000 lines of Python code

To run Combinatorial Exploration on a new type of object, you just need to:

- decide on a good way to represent sets of those objects, and write a Python class for it
- decide on effective decomposition strategies (this is where domain-specific experience comes in handy)
- plug these right into our framework, and hit go
  - Framework: ~ 7,300 lines of code
  - Binary words example: ~ 200 lines of code
  - Permutation Patterns: ~24,000 lines of code

https://github.com/PermutaTriangle/comb_spec_searcher

Domains we've coded:

- permutation patterns (inspired this work)
- set partitions
- Motzkin paths

Domains that seem promising on paper:

- polyominoes
- inversion sequences
- alternating sign matrices

Given a set of permutations B, you can study the set of permutations avoiding the permutations in B as patterns — these sets are called permutation classes.

For the cases where B contains two permutations of length 4, there are essentially 56 different permutation classes.

(https://en.wikipedia.org/wiki/Enumerations_of_specific_permutation_classes)

Their enumerations are all known now, but it took several decades and dozens of papers.

Combinatorial Exploration can enumerate all of them.

- 6/7 avoiding 1 pattern of length 4 — all except Av(1324)
- 56/56 avoiding 2 patterns of length 4
- 317/317 avoiding 3 patterns of length 4
- And all avoiding 4-24 patterns of length 4

- Dozens of known results and dozens of new results, and corrects several wrong results.

https://permpal.com

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The Permutation Pattern Avoidance Library (PermPAL)

PermPAL is a database of algorithmically-derived theorems about permutation classes.

The Combinatorial Exploration framework produces rigorously verified combinatorial specifications for families of combinatorial objects. These specifications then lead to generating functions, counting sequence, polynomial-time counting algorithms, random sampling procedures, and more.

This database contains 23,845 permutation classes for which specifications have been automatically found. This includes many classes that have been previously enumerated by other means and many classes that have not been previously enumerated.

Some Notables Successes:

- 6 out of 7 of the principal classes of length 4
- all 56 symmetry classes avoiding two patterns of length 4
- all 317 symmetry classes avoiding three patterns of length 4
- the "domino set" used by Bevan, Brignall, Elvey Price, and Pantone to investigate Av(1324)
- the class Av(3412, 52341, 635241) of Alland and Richmond corresponding a type of Schubert variety
- the class Av(2341, 3421, 4231, 52143) equal to the (Av(12), Av(21))-staircase (see Albert, Pantone, and Vatter), which appears to be non-D-finite
- all of the permutation classes counted by the Schröder numbers conjectured by Eric Egge
- the class Av(34251, 35241, 45231), equal to the preimage of Av(321) under the West-stack-sorting operation (see Defant)

Section 2.4 of the article Combinatorial Exploration: An Algorithmic Framework for Enumeration gives a more comprehensive list of notable results.

The comb_spec_searcher github repository contains the open-source python framework for Combinatorial Exploration, and the tilings github repository contains the code needed to apply it to the field of permutation patterns.

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Av(1342)

View Raw Data

Generating Function

$$\frac{-8\sqrt{-8x+1}\,x - 8x^2 + \sqrt{-8x+1} + 20x + 1}{2(x+1)^3}$$

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Counting Sequence

1, 1, 2, 6, 23, 103, 512, 2740, 15485, 91245, 555662, 3475090, 22214707, 144640291, 956560748, ...

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Recurrence

$$a(0) = 1$$
$$a(1) = 1$$
$$a(n+2) = \frac{4(3+2n)a(n)}{n+2} + \frac{(-8+7n)a(n+1)}{n+2}, \quad n \geq 2$$

Copy to clipboard: latex Maple

Implicit Equation for the Generating Function

$$(x+1)^3 F(x)^2 + \left(8x^2 - 20x - 1\right) F(x) + 16x = 0$$

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Specification 1 Specification 2 Specification 3 Specification 4 Specification 5

This specification was found using the strategy pack "Point And Col Placements Tracked Fusion" and has 29 rules.

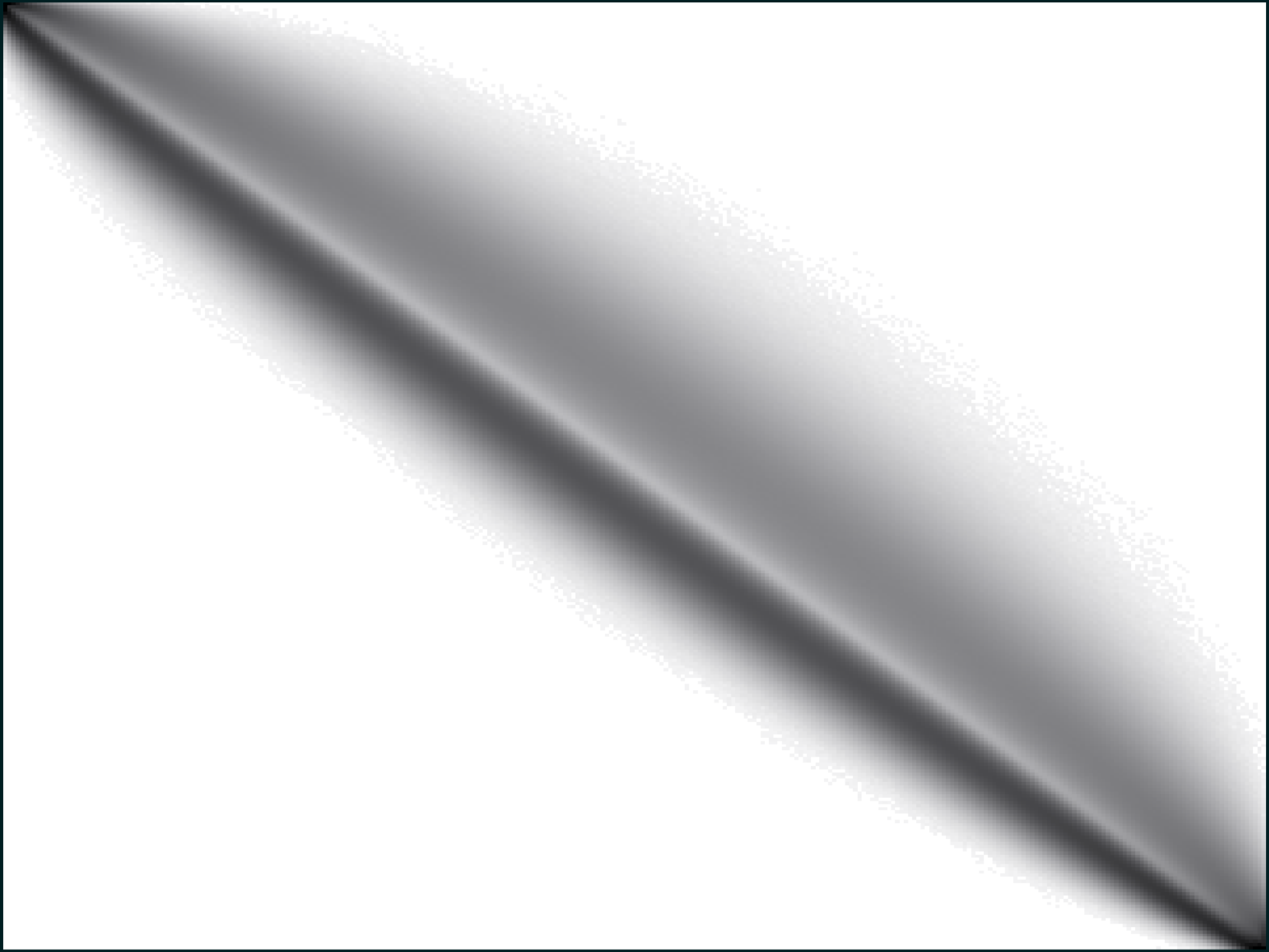
Found on May 26, 2021.

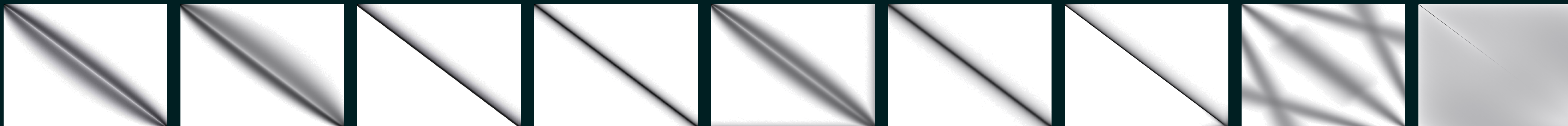
Finding the specification took 1720 seconds.

System of Equations

Copy 29 equations to clipboard: latex Maple sympy

$$
\begin{aligned}
F_0(x) &= F_1(x) + F_2(x)\\
F_1(x) &= 1\\
F_2(x) &= F_3(x)\\
F_3(x) &= F_4(x)F_5(x)\\
F_4(x) &= x\\
F_5(x) &= F_6(x,1)\\
F_6(x,y) &= F_0(x) + F_7(x,y)\\
F_7(x,y) &= F_8(x,y)\\
F_8(x,y) &= F_{14}(x,y)F_9(x,y)\\
F_9(x,y) &= F_{10}(x,y) + F_{15}(x,y)\\
F_{10}(x,y) &= F_{11}(x,y)F_6(x,y)\\
F_{11}(x,y) &= F_1(x) + F_{12}(x,y)\\
F_{12}(x,y) &= F_{13}(x,y)\\
F_{13}(x,y) &= F_{11}(x,y)^2F_{14}(x,y)\\
F_{14}(x,y) &= yx\\
F_{15}(x,y) &= F_{16}(x,y)\\
F_{16}(x,y) &= F_{17}(x,y)F_4(x)F_6(x,y)\\
F_{18}(x,y) &= F_0(x)F_{17}(x,y)F_4(x)\\
F_{18}(x,y) &= F_{19}(x,y)\\
F_{20}(x,y) &= F_{19}(x,y) + F_{28}(x,y)\\
F_{20}(x,y) &= F_{21}(x,y) + F_6(x,y)\\
F_{21}(x,y) &= F_{22}(x,y)\\
F_{22}(x,y) &= F_{23}(x,y)F_4(x)\\
F_{23}(x,y) &= \frac{yF_{24}(x,y) - F_{24}(x,1)}{-1+y}\\
F_{24}(x,y) &= F_{25}(x,y) + F_{26}(x,y)\\
F_{25}(x,y) &= F_{11}(x,y)F_5(x)\\
F_{26}(x,y) &= F_{27}(x,y)\\
F_{27}(x,y) &= F_{17}(x,y)F_4(x)F_5(x)\\
F_{28}(x,y) &= F_0(x)F_{11}(x,y)
\end{aligned}
$$





Av(1234, 1243) Av(1234, 1342) Av(1234, 1432) Av(1234, 2143) Av(1234, 2341) Av(1234, 2413) Av(1234, 2431) Av(1234, 3412) Av(1234, 3421)

Av(1234, 4231) Av(1243, 1324) Av(1243, 1342) Av(1243, 2134) Av(1243, 2143) Av(1243, 2314) Av(1243, 2341) Av(1243, 2413) Av(1243, 2431)

Av(1243, 3214) Av(1243, 3241) Av(1243, 3412) Av(1243, 3421) Av(1243, 4231) Av(1324, 1342) Av(1324, 1432) Av(1324, 2143) Av(1324, 2341)

Av(1324, 2413) Av(1324, 2431) Av(1324, 3412) Av(1324, 4231) Av(1342, 1423) Av(1342, 1432) Av(1342, 2143) Av(1342, 2314) Av(1342, 2341)

Av(1342, 2413) Av(1342, 2431) Av(1342, 3124) Av(1342, 3142) Av(1342, 3214) Av(1342, 3241) Av(1342, 3412) Av(1342, 4123) Av(1342, 4213)

Av(1432, 2143) Av(1432, 2341) Av(1432, 2413) Av(1432, 3214) Av(1432, 3412) Av(2143, 2413) Av(2143, 3412) Av(2413, 3142)

Computational Difficulties

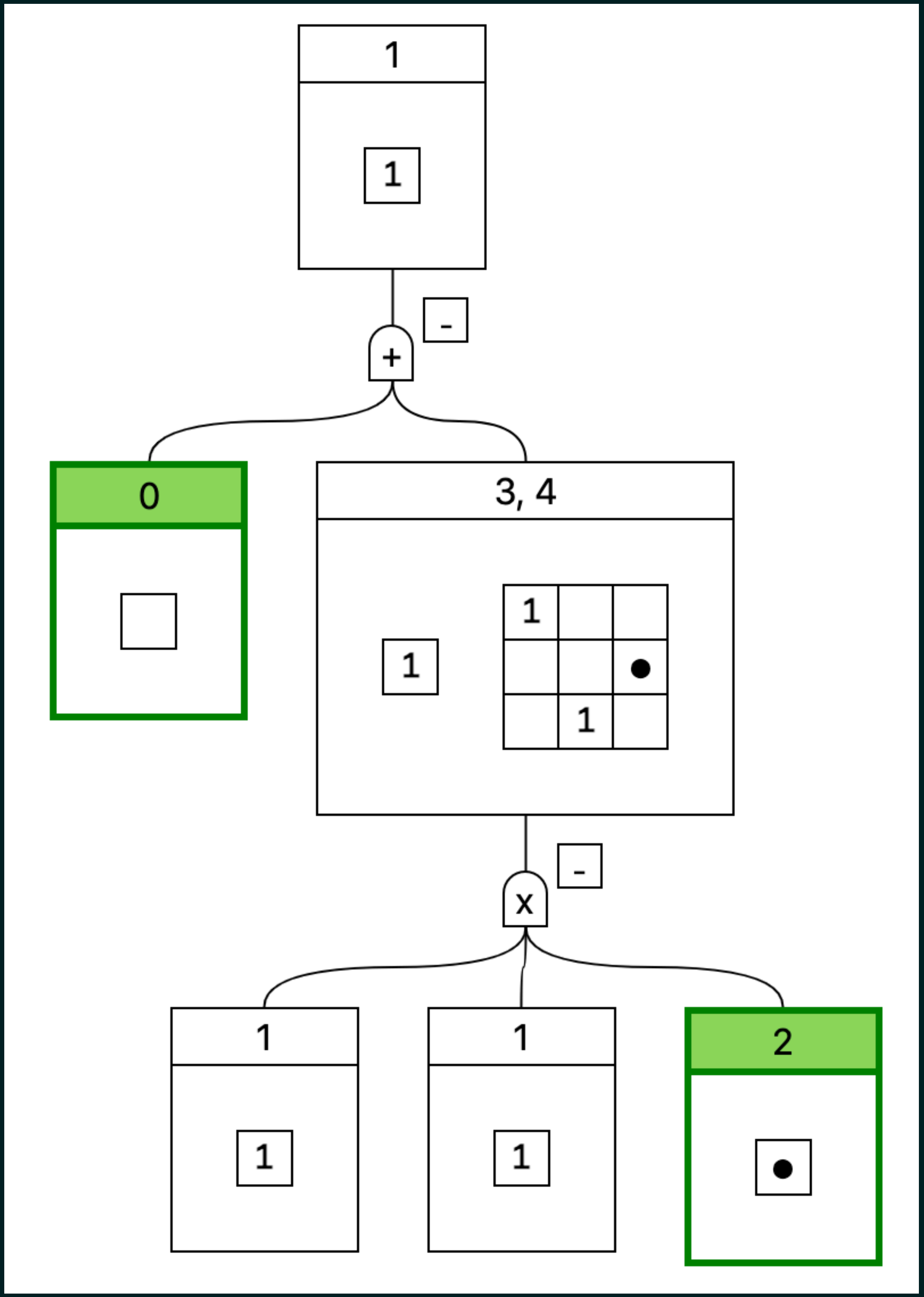
Permutations avoiding 132:

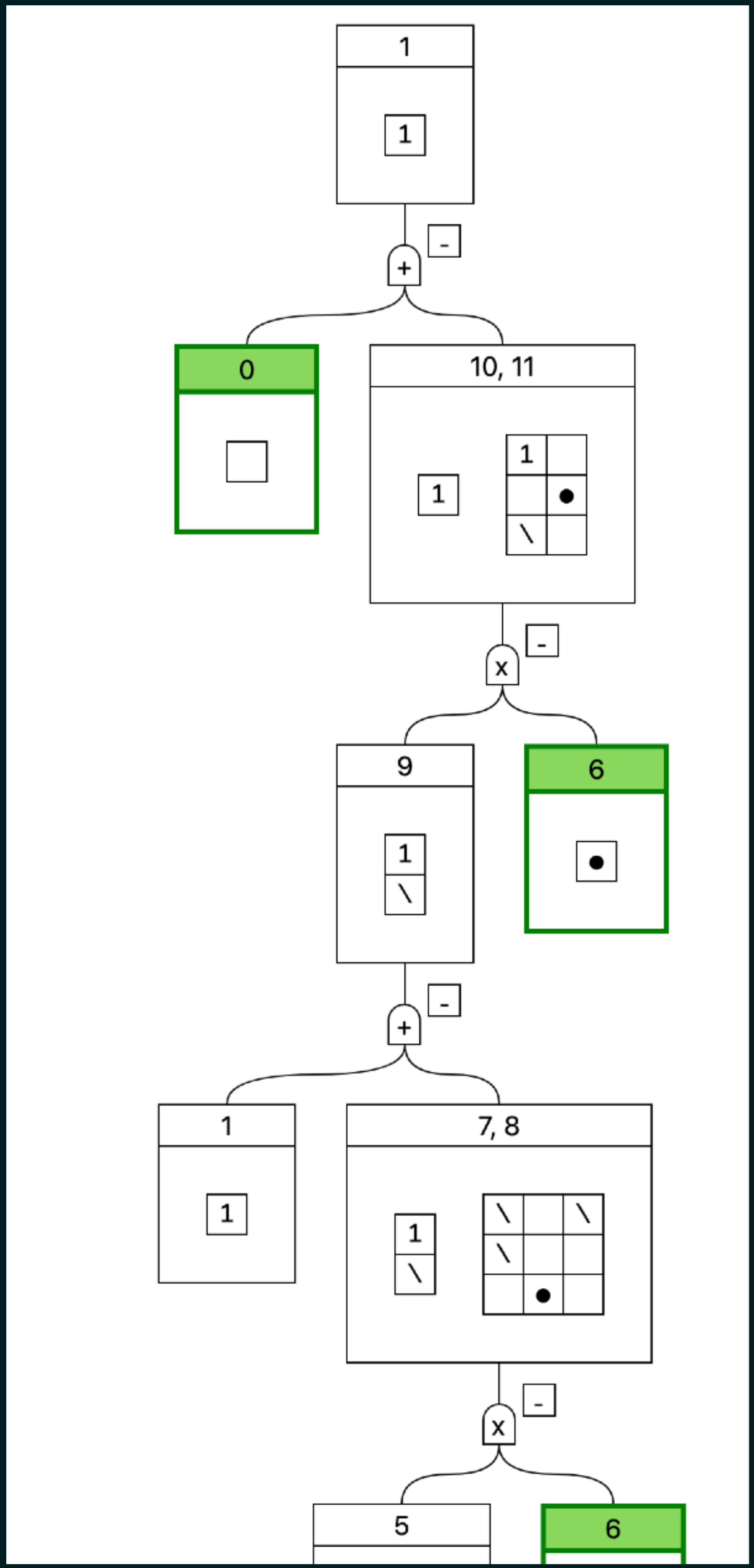
$$F_0(x) = F_1(x) + F_2(x)$$
$$F_1(x) = F_0(x)^2 \cdot F_3(x)$$
$$F_2(x) = 1$$
$$F_3(x) = x$$

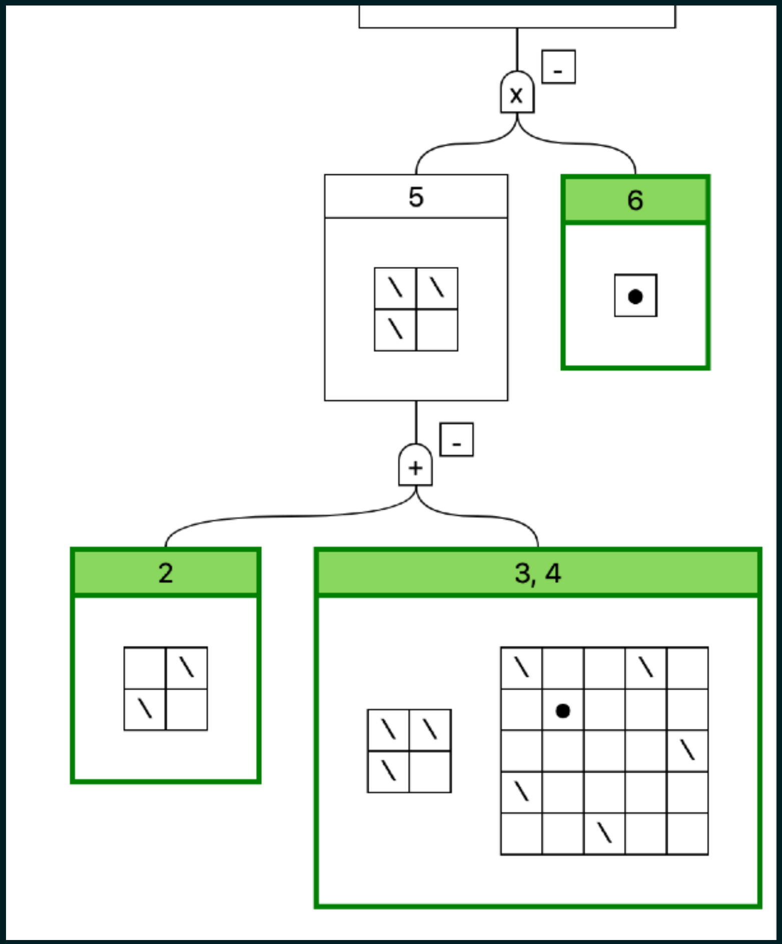
Permutations avoiding 1432 and 2143:

$$F_0(x) = F_{547}(x) + F_{373}(x)$$
$$F_1(x) = F_0(x) - F_{118}(x)$$
$$\cdots$$
$$F_{549}(x) = 0$$

550 equations $\longrightarrow$ guess-and-check


1
1
-
+
0
3, 4
1
1
1
-
x
1
1
1
1
2


1
1
+
-
0
10, 11
1
1
x
-
9
1
6
+
-
1
1
7, 8
1
x
-
5
6


x
-
5
6
+
-
2
3, 4

Computational Difficulties — with 1 catalytic variable!

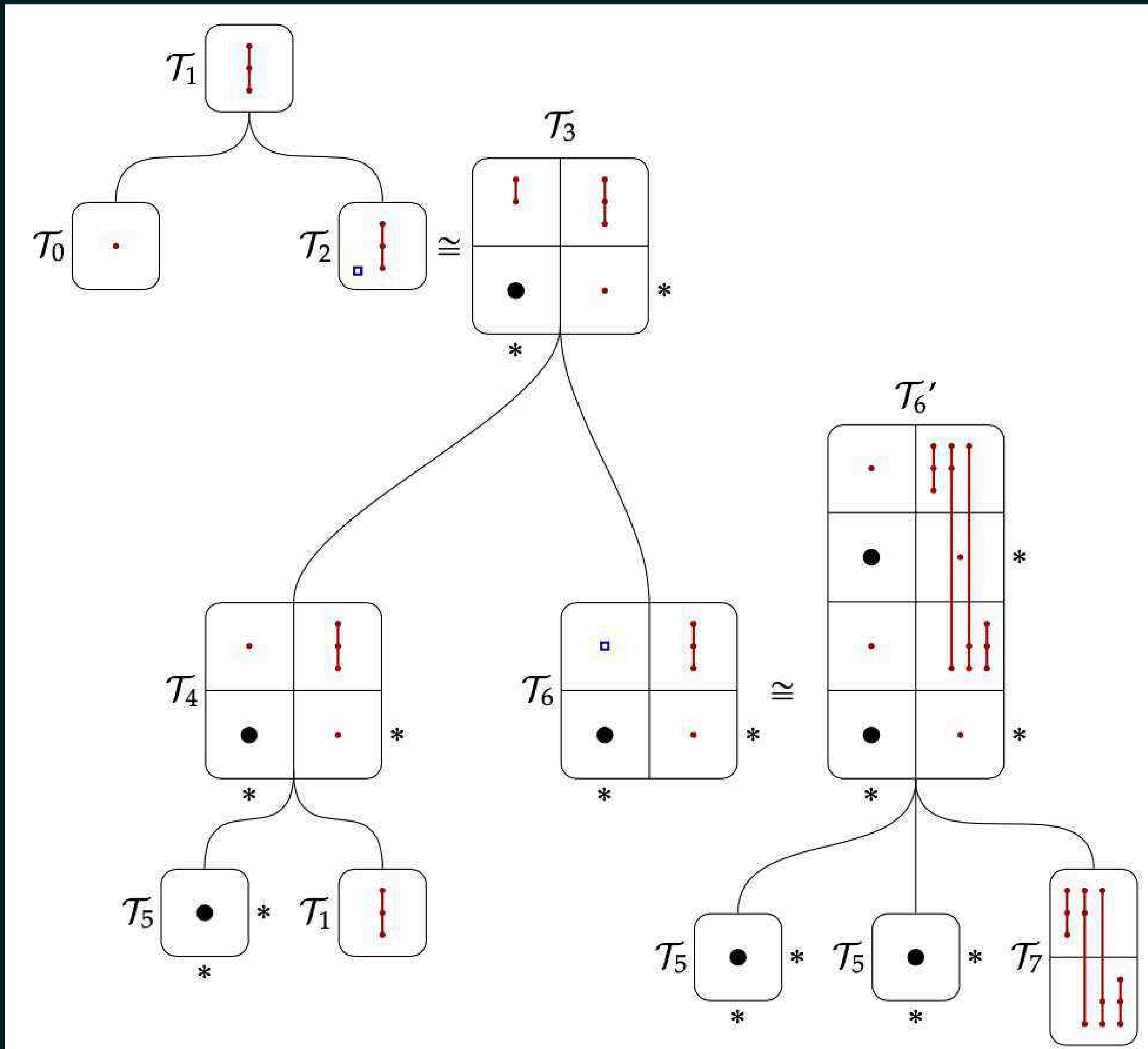
Permutations avoiding 123:

$$
\begin{aligned}
F_0(x) &= F_{11}(x) + F_6(x) \\
F_1(x) &= F_{12}(x) \cdot F_2(x) \\
&\boxed{F_2(x) = F_3(x,1)} \\
F_3(x,y) &= F_7(x,y) + F_8(x,y) \\
F_4(x,y) &= F_{12}(x) \cdot F_5(x,y) \cdot F_8(x,y) \\
&\boxed{F_5(x,y) = \frac{yF_3(x,y) - F_3(x,1)}{y-1}} \\
F_6(x) &= F_1(x) \\
F_7(x,y) &= F_4(x,y) \\
F_8(x,y) &= F_{10}(x,y) + F_{11}(x) \\
F_9(x,y) &= F_{13}(x,y) \cdot F_8(x,y) \\
F_{10}(x,y) &= F_9(x,y) \\
F_{11}(x) &= 1 \\
F_{12}(x) &= x \\
F_{13}(x,y) &= xy
\end{aligned}
$$

Computational Difficulties — with 2+ catalytic variables!

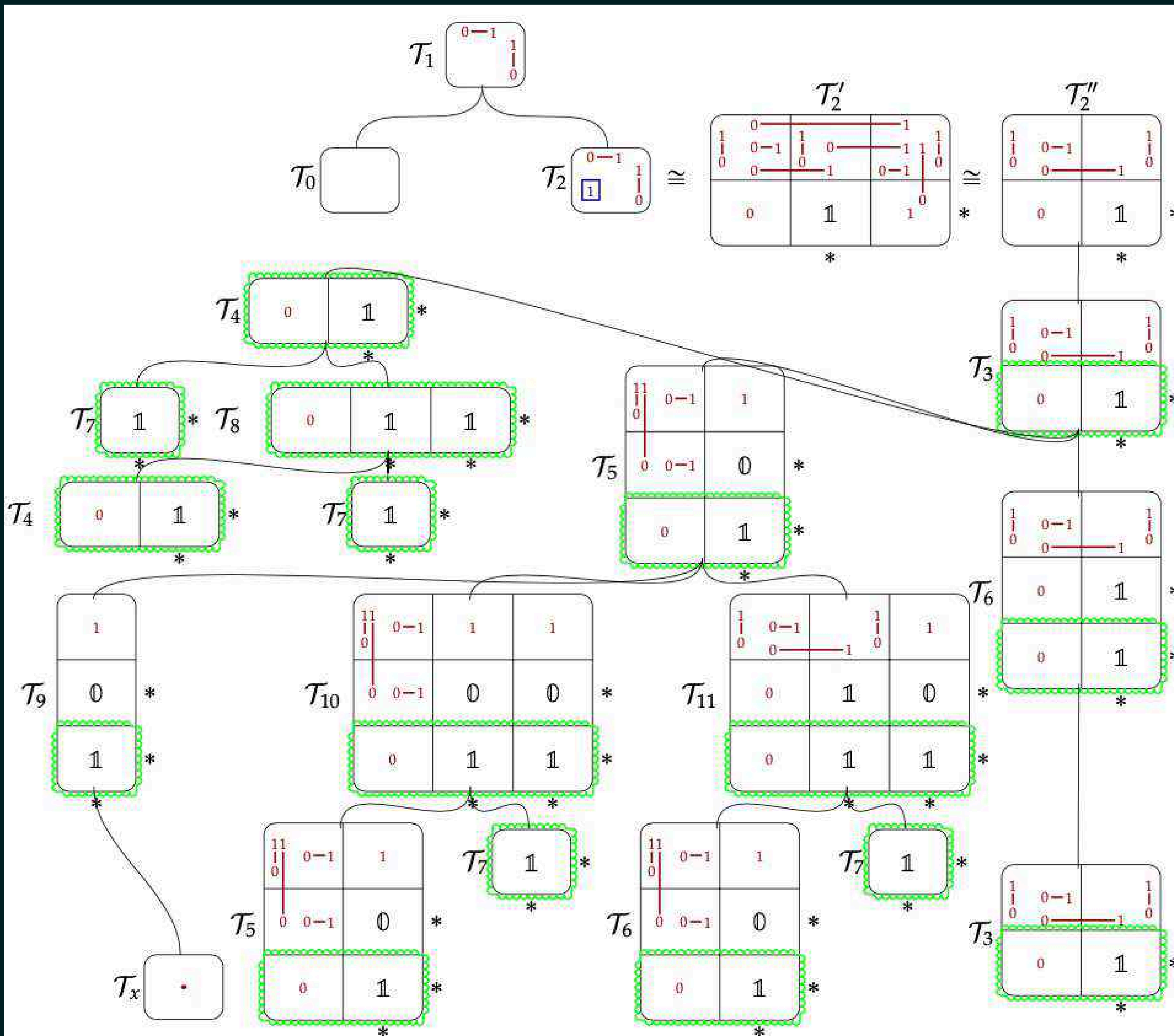
$$F_0(x) = F_1(x) + F_{15}(x)$$
$$F_1(x) = F_{16}(x) \cdot F_2(x)$$
$$F_2(x) = F_3(x,1)$$
$$F_3(x,y) = F_{12}(x,y) + F_{15}(x) + F_4(x,y)$$
$$F_4(x,y) = F_{17}(x,y) \cdot F_5(x,y)$$
$$\boxed{F_5(x,y) = F_{14}(x,1,y)}$$
$$F_6(x,y,z) = F_{11}(x,y,z) + F_{15}(x) + F_7(x,y,z) + F_9(x,y,z)$$
$$F_7(x,y,z) = F_{17}(x,z) \cdot F_8(x,y,z)$$
$$\boxed{F_8(x,y,z) = \frac{yF_{14}\left(x, \frac{y}{z}, z\right) - z \cdot F_{14}(x,1,z)}{y-z}}$$
$$F_9(x,y,z) = F_{10}(x,y,z) \cdot F_{16}(x)$$

$$F_{10}(x,y,z) = \frac{zF_6(x,y,z) - F_6(x,y,1)}{z-1}$$
$$F_{11}(x,y,z) = F_{17}(x,y) \cdot F_6(x,y,z)$$
$$F_{12}(x,y) = F_{13}(x,y) \cdot F_{16}(x)$$
$$F_{13}(x,y) = \frac{yF_3(x,y) - F_3(x,1)}{y-1}$$
$$\boxed{F_{14}(x,y,z) = F_6(x,yz,z)}$$
$$F_{15}(x) = 1$$
$$F_{16}(x) = x$$
$$F_{17}(x,y) = xy$$



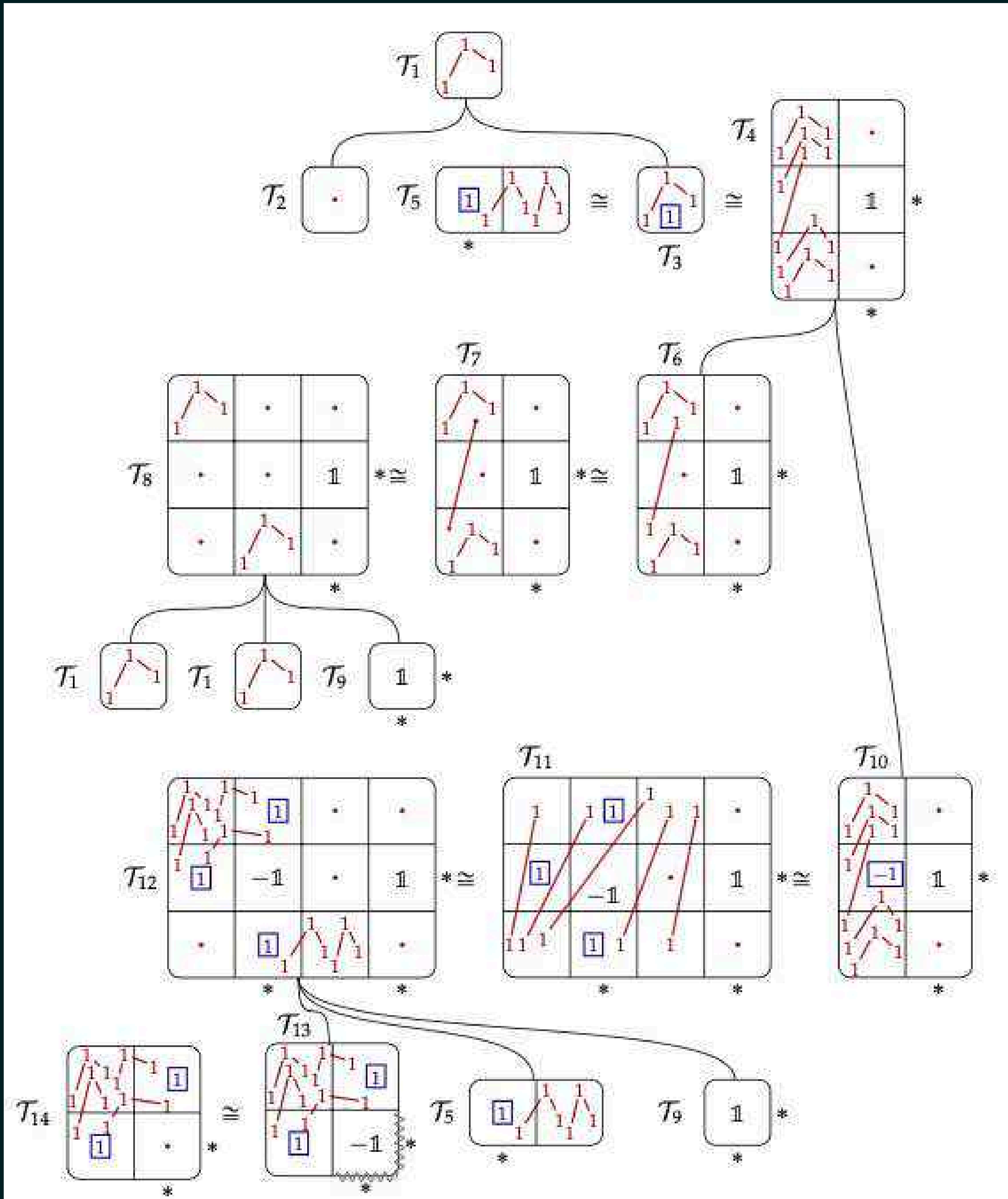
$$T_1(x) = 1 + T_2(x)$$
$$T_2(x) = T_4(x) + T_6(x)$$
$$T_4(x) = T_1(x) \cdot T_5(x)$$
$$T_5(x) = x$$
$$T_6(x) = T_5(x)^2 \cdot T_7(x)$$
$$T_7(x) = \frac{d}{dx}(x \cdot T_1(x))$$

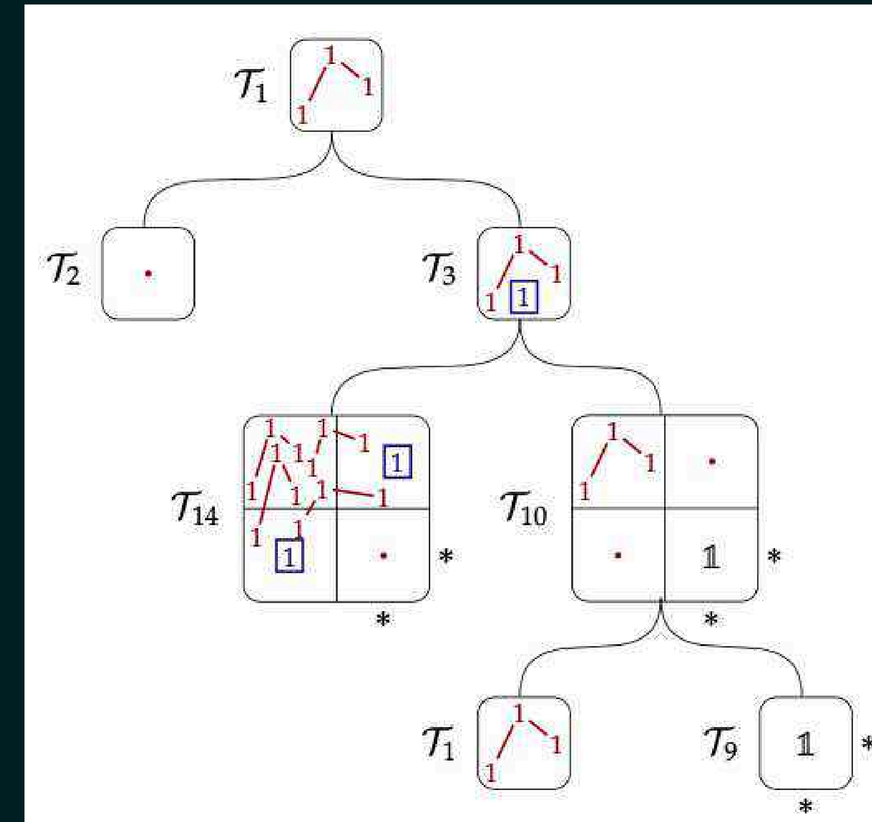
$$T_1(x) = 1 + (x + x^2)T_1(x) + x^3 \frac{d}{dx} T_1(x)$$



$$T_1(x) = 1 + T_2(x)$$
$$T_2(x) = T_3(x,1)$$
$$T_3(x,y) = T_4(x,y) + T_5(x,y) + T_6(x,y)$$
$$T_4(x,y) = T_7(x,y) + T_8(x,y)$$
$$T_5(x,y) = T_9(x,y) + T_{10}(x,y) + T_{11}(x,y)$$
$$T_6(x,y) = T_3(x,xy)$$
$$T_7(x,y) = xy$$
$$T_8(x,y) = T_4(x,y) \cdot T_7(x,y)$$
$$T_9(x,y) = 0$$
$$T_{10}(x,y) = T_5(x,y) \cdot T_7(x,y)$$
$$T_{11}(x,y) = T_6(x,y) \cdot T_7(x,y)$$

$$T_1(x) = \prod_{i=1}^{\infty} \frac{1}{1 - x^i}$$





$$T_1(x) = \frac{3 - x - \sqrt{1 - 6x + x^2}}{2}$$

Thank you!

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