Sorting probabilities for Young diagrams

Greta Panova (University of Southern California)

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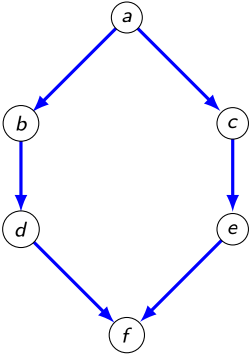
Partially ordered set P on X
e.g. $X = \{a, b, c, d, e\}$,
where $f \succ e, e \succ c$ etc



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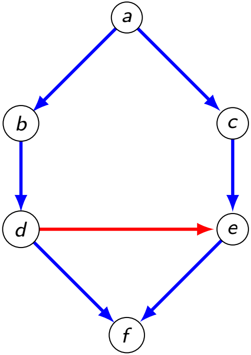

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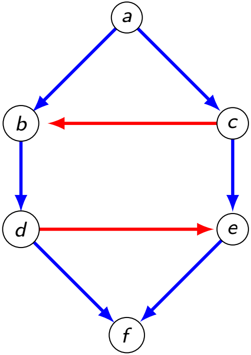
Linear extension of P:
completion to total order
$L : X \xrightarrow{\sim} [|X|]$ s.t. $L(x) < L(y)$ if $x \prec y$
$\mathcal{E}(P) = \{L : \text{lin. ext. of } P\}$



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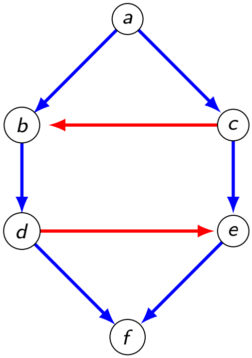

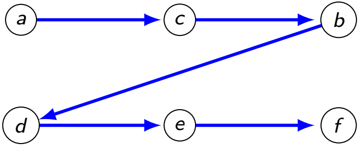

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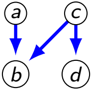
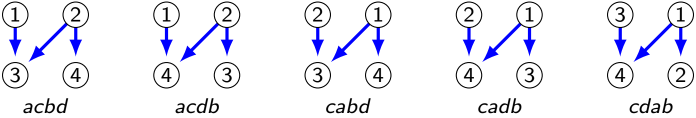
Poset $P =$

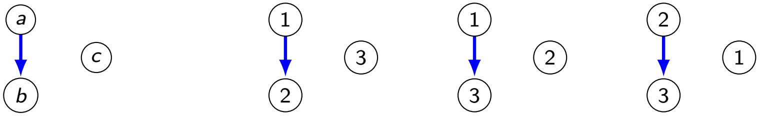


Linear extensions $\mathcal{E}(P)$



Sorting probability of $x, y \in P$:

$$\Pr[x \prec y] = \frac{|\{L \in \mathcal{E}(P), \text{ s.t. } L(x) < L(y)\}|}{e(P)}$$



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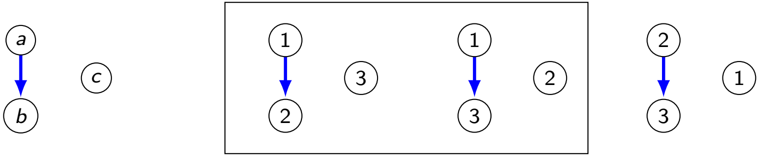

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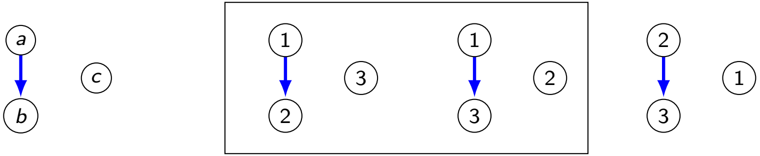

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Background motivation:
[comparison] Sorting under partial information (completing a partial order to total order):
At each step compare $x, y \in P$, s.t.

$$\frac{1}{2} - \epsilon \leq \Pr[x \prec y] \leq \frac{1}{2} + \epsilon$$

Runtime is $\Theta(\log e(P))$.

The $\frac{1}{3} - \frac{2}{3}$ Conjecture

Conjecture (Kislitsyn '68, Fredman '75, Linial '84)

For every finite poset that is not a total order, there exist x, y*:*

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(proof via mixed-volume inequalities)

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[Olson-Sagan '18]: conjecture holds if P is a Young diagram.

Young diagrams and Standard Young Tableaux

Integer partition $\lambda \vdash n$: $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$, $\lambda_1 + \lambda_2 + \cdots = n$

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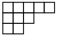
Young diagram of λ:  for $\lambda = (5, 3, 2)$

Standard Young Tableaux (SYT) of shape λ:

1	3	5	6	9
2	7	10		
4	8			

The number of SYT of shape λ is f^{λ}.

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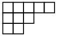
Young diagram of λ:  for $\lambda = (5, 3, 2)$

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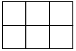
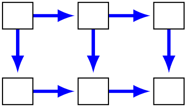
$\Longleftrightarrow$ poset $P = [\lambda]$ and SYTs $\leftrightarrow$ linear extensions, $e(P) = f^\lambda$.

$\lambda = (3, 3)$:





1	3	4
2	5	6

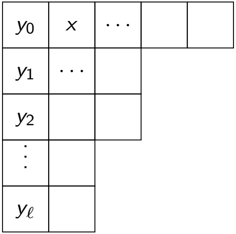
$e(P) = f^{(3,3)} = C_3 = 5$

Skew SYT λ/μ:

			2	4	7	11
		1	5	9		
	6	8	10			
3	12					

for skew shape $(7, 6, 4, 2)/(3, 2, 1)$.

Warm up: $\frac{1}{3} - \frac{2}{3}$ for Young diagrams



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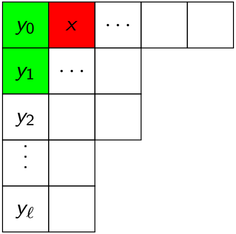

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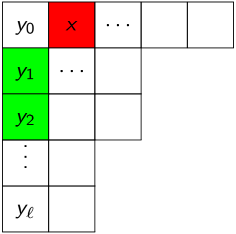

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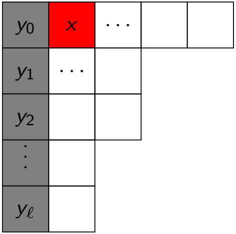

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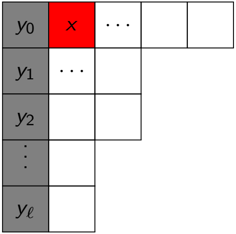

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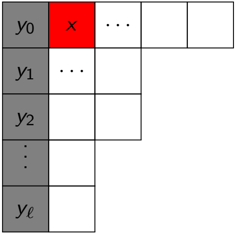

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Assume $q_1 \leq \frac{1}{2}$ (else transpose). If $q_1 \geq \frac{1}{3}$ then $\Pr[x \prec y_1] \in [\frac{1}{3}, \frac{2}{3}]$, done.

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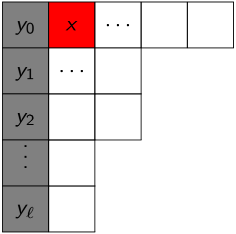

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Else, $q_1 < \frac{1}{3}$ and pick k–min, s.t. $\Pr[x \prec y_k] = q_1 + \cdots + q_k \geq \frac{1}{3}$. Then $\Pr[x \prec y_k] \leq \Pr[x \prec y_{k-1}] + q_k \leq \frac{1}{3} + \frac{1}{3}$. Done.

Sorting probability for posets

Sorting probability of P:

$$\delta(P) = \min_{x,y \in P} |\Pr[x \prec y] - \Pr[y \prec x]|$$

Conjecture (Kahn-Saks'84)

For every finite poset P on n elements

$$\delta(P) \to 0 \qquad \textit{as width}(P) \to \infty.$$

($width(P)$ = size of max antichain)
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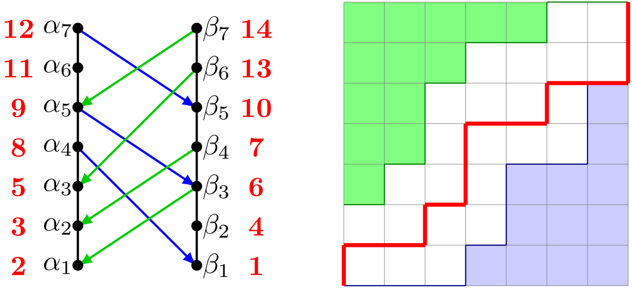
Let $\lambda = (n, n)$, so P_λ is the "Catalan poset". Then

$$\delta(P_\lambda) \leq \frac{C}{n^{\frac{5}{4}}}.$$

Linear extension ↔ lattice paths



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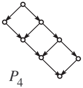
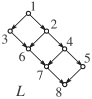


P_4



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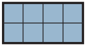

λ

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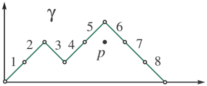
A



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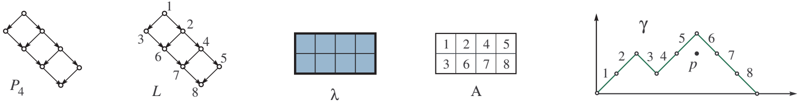

Dyck paths $\gamma : (0,0) \to (2n,0)$:

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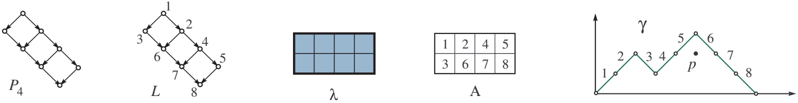

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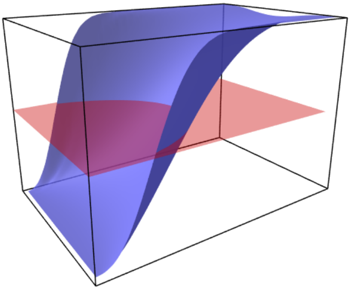
Limit shape of $R_n(\frac{h}{n}, \frac{z}{\sqrt{2n}})$



Catalan posets II:

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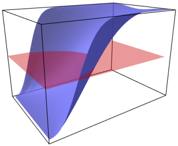
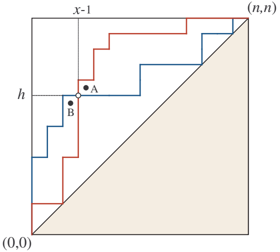


$$R_n(h,h-x) - R_n(h+1,h-x) = \frac{\#\gamma_1 - \#\gamma_2}{C_n}$$



$$R_n(h, \frac{\sqrt{n}}{10}) \le \frac{1}{4} \qquad R_n(h, 10\sqrt{n}) \ge \frac{3}{4}$$

$$\frac{c_1}{\sqrt{n}} \le R_n(h,z+1) - R_n(h,z) \le \frac{c_2}{\sqrt{n}}$$

$$R_n(h,z) = R_n(n+z-h,z)$$

$$c_3 \frac{n-2h+z}{n^2} \le R_n(h,z) - R_n(h+1,z) \le c_4 \frac{n-2h+z}{n^2}$$

Young diagrams of fixed length

Theorem (Chan-Pak-Panova'21+)

Let $\lambda \vdash n$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq \epsilon n$. Let P_λ be the poset of the Young diagram of λ. Then there is a constants $C > 0$ depending on d, ϵ, s.t.

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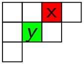

$$\{L \in \mathcal{E}(P) : L(x) < L(y)\} = \left\{ \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & \\ 8 & & & \end{array}, \quad \begin{array}{cccc} 1 & 3 & 4 & 5 \\ 2 & 7 & 8 & \\ 6 & & & \end{array} \ldots \right\}$$

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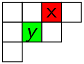

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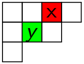

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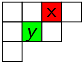

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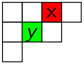

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$$\begin{array}{|c|c|c|c|} \hline \ & \ & x & \ \\ \hline \ & y & \ \\ \cline{1-3} \ \\ \cline{1-1} \end{array}$$

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- Apply Naruse Hook-Length Formula[1] for asymptotic bounds.

Skew SYTs

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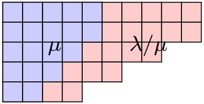

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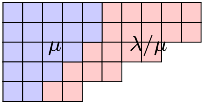

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$$f^{\lambda/\mu} = |\lambda/\mu|! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{u \in [\lambda] \setminus D} \frac{1}{h(u)},$$

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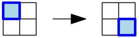
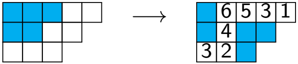
$\mathcal{E}(\lambda/\mu) = \{D \subset \lambda :$ obtained from μ via  $\}$



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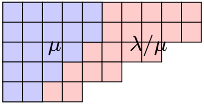

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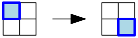
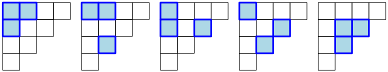
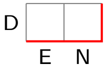
$\mathcal{E}(\lambda/\mu) = \{D \subset \lambda :$ obtained from μ via  $\}$



$$f^{(4321/21)} = 7!\left(\frac{1}{1^4 \cdot 3^3} + \frac{1}{1^3 \cdot 3^3 \cdot 5} + \frac{1}{1^3 \cdot 3^3 \cdot 5} + \frac{1}{1^2 \cdot 3^3 \cdot 5^2} + \frac{1}{1^2 \cdot 3^2 \cdot 5^2 \cdot 7}\right) = 61$$

Thank you!

The 

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