Sorting probabilities for Young diagrams

Greta Panova (University of Southern California)

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Partially ordered set $P$ on $X$

e.g. $X = \{a, b, c, d, e\}$,
where $f \succ e$, $e \succ c$ etc
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eq

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where $f \succ e$, $e \succ c$ etc.

\[
\begin{array}{cccc}
    a & b & c & d \\
    \downarrow & \uparrow & \downarrow & \downarrow \\
    f & c & e & a \\
\end{array}
\]
**Partially ordered set** $P$ on $X$
eq \{a, b, c, d, e\}$, where $f \succ e, e \succ c$ etc

**Linear extension** of $P$:
completion to total order
$L : X \sim [|X|]$ s.t. $L(x) < L(y)$ if $x \prec y$
$\mathcal{E}(P) = \{L : \text{lin. ext. of } P\}$

$L(a) = 1, L(c) = 2, L(b) = 3, L(d) = 4, L(e) = 5, L(f) = 6$
Partially ordered set $P$ on $X$
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$L(a) = 1$, $L(c) = 2$, $L(b) = 3$, $L(d) = 4$, $L(e) = 5$, $L(f) = 6$
The **number of linear extensions** of $P$: $e(P) = |\mathcal{E}(P)|$
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Poset $P =$

Linear extensions $\mathcal{E}(P)$

\[
\begin{align*}
&\begin{array}{c}
1 \\
3 \\
\end{array} \quad \begin{array}{c}
2 \\
4 \\
\end{array} & \begin{array}{c}
1 \\
4 \\
\end{array} \quad \begin{array}{c}
2 \\
3 \\
\end{array} & \begin{array}{c}
2 \\
3 \\
\end{array} \quad \begin{array}{c}
1 \\
4 \\
\end{array} & \begin{array}{c}
3 \\
4 \\
\end{array} \quad \begin{array}{c}
1 \\
2 \\
\end{array}
\end{align*}
\]

- $acbd$
- $acdb$
- $cabd$
- $cadb$
- $cdab$
Sorting probability of $x, y \in P$:

$$
\Pr[x \prec y] = \frac{|\{L \in \mathcal{E}(P), \text{s.t. } L(x) < L(y)\}|}{e(P)}
$$
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\]

**Background motivation:**

[comparison] Sorting under partial information (completing a partial order to total order):

At each step compare \( x, y \in P \), s.t.

\[
1 - \epsilon \leq \Pr[x \prec y] \leq 1 + \epsilon
\]

Runtime is \( \Theta(\log e(P)) \).

\[
\Pr[a \prec c] = \frac{2}{3}
\]
**Sorting probability** of \( x, y \in P \):

\[
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\]

Background motivation:

[comparison] **Sorting under partial information** (completing a partial order to total order):

At each step compare \( x, y \in P \), s.t.

\[
\frac{1}{2} - \epsilon \leq \Pr[x \prec y] \leq \frac{1}{2} + \epsilon
\]

Runtime is \( \Theta(\log e(P)) \).
The $\frac{1}{3} - \frac{2}{3}$ Conjecture

Conjecture (Kislitsyn '68, Fredman '75, Linial '84)

*For every finite poset that is not a total order, there exist $x, y$:*

$$\frac{1}{3} \leq \Pr[x < y] \leq \frac{2}{3}.$$
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[Kahn-Saks’ 84]: For every finite poset, which is not totally ordered, there exist $x, y$, s.t.

$$\frac{3}{11} \leq \Pr[x < y] \leq \frac{8}{11}$$

(proof via mixed-volume inequalities)
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[Brightwell-Felsner-Trotter '95]:

$$\frac{5 - \sqrt{5}}{10} \leq \Pr[x < y] \leq \frac{5 + \sqrt{5}}{10}.$$
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[Linial’84]: conjecture holds for width 2 posets, improved by [Sah’18]. Other special cases: [Brightwell’89], [Zaguia’12,’19], [Trotter, Gehrlein, Fishburn], [Pouzet]. Improvement by [Saks’85]
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Improvement by [Saks’85]

[Olson-Sagan ’18]: conjecture holds if $P$ is a Young diagram.
Young diagrams and Standard Young Tableaux

**Integer partition** \( \lambda \vdash n \): \( \lambda_1 \geq \lambda_2 \geq \cdots \geq 0 \), \( \lambda_1 + \lambda_2 + \cdots = n \)
Young diagrams and Standard Young Tableaux

**Integer partition** $\lambda \vdash n$: $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$, $\lambda_1 + \lambda_2 + \cdots = n$

**Young diagram** of $\lambda$: for $\lambda = (5, 3, 2)$

**Standard Young Tableaux (SYT)** of shape $\lambda$:  
\[
\begin{array}{cccc}
1 & 3 & 5 & 6 & 9 \\
2 & 7 & 10 \\
4 & 8 \\
\end{array}
\]

The number of SYT of shape $\lambda$ is $f^\lambda$. 
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**Young diagram** of \( \lambda \): 

\[
\begin{array}{ccccc}
\_
\end{array}
\]

for \( \lambda = (5, 3, 2) \)

**Standard Young Tableaux (SYT)** of shape \( \lambda \):

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1 & 3 & 5 & 6 & 9 \\
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\end{array}
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The number of SYT of shape \( \lambda \) is \( f^\lambda \).

\( \iff \) poset \( P = [\lambda] \) and SYTs \( \leftrightarrow \) linear extensions, \( e(P) = f^\lambda \).

\( \lambda = (3, 3) \):

\[
\begin{array}{ccccc}
\_
\end{array}
\]

\[
\begin{array}{ccccc}
\_
\end{array}
\]

\[
\begin{array}{ccccc}
\_
\end{array}
\]

\[
\begin{array}{ccccc}
\_
\end{array}
\]

Skew SYT \( \lambda/\mu \):

\[
\begin{array}{cccc}
2 & 4 & 7 & 11 \\
1 & 5 & 9 \\
6 & 8 & 10 \\
3 & 12 \\
\end{array}
\]

for skew shape \( (7, 6, 4, 2)/(3, 2, 1) \).
Warm up: $\frac{1}{3} - \frac{2}{3}$ for Young diagrams
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$q_1 := \operatorname{Pr}[y_0 \prec x \prec y_1] = \frac{f^{\lambda/(2)}}{f^{\lambda}}$
Warm up: $\frac{1}{3} - \frac{2}{3}$ for Young diagrams

\[
\begin{array}{cccc}
  & y_0 & x & \cdots \\
 y_1 & \cdots & & \\
y_2 & & & \\
\vdots & & & \\
y_{\ell} & & & \\
\end{array}
\]

\[
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\]

\[
q_2 := \Pr[y_1 \prec x \prec y_2] = \frac{f_{\lambda/(2,1)}}{f_{\lambda}}
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$$q_k := \Pr[y_{k-1} \prec x \prec y_k] = \frac{f^{\lambda/(2,1^{k-1})}}{f^{\lambda}} \text{ for } 1 \leq k \leq \ell + 1$$
Warm up: $\frac{1}{3} - \frac{2}{3}$ for Young diagrams

\[
\begin{array}{c|c|c|c|c|c|c}
\hline
y_0 & x & \cdots & & & \\
\hline
y_1 & \cdots & & & & \\
\hline
y_2 & & & & & \\
\vdots & & & & & \\
\hline
y_\ell & & & & & \\
\hline
\end{array}
\]

\[
q_1 := \Pr[y_0 \prec x \prec y_1] = \frac{f_{\lambda}/(2)}{f_{\lambda}} \\
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\]

\[
q_1 \geq q_2 \geq \cdots \geq q_{\ell+1}, \quad q_1 + \cdots + q_{\ell+1} = 1
\]
Warm up: $\frac{1}{3} - \frac{2}{3}$ for Young diagrams

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c}
 & y_0 & x & \cdots & \\
\hline
y_0 & & & & \\
\hline
y_1 & & \cdots & & \\
\hline
y_2 & & & & \\
\hline
\vdots & & & & \\
\hline
y_\ell & & & & \\
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\]

Assume $q_1 \leq \frac{1}{2}$ (else transpose). If $q_1 \geq \frac{1}{3}$ then $\Pr[x \prec y_1] \in \left[\frac{1}{3}, \frac{2}{3}\right]$, done.
Warm up: $\frac{1}{3} - \frac{2}{3}$ for Young diagrams

\[
q_1 := \Pr[y_0 < x < y_1] = \frac{f^{\lambda/2}}{f^{\lambda}}
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q_2 := \Pr[y_1 < x < y_2] = \frac{f^{\lambda/2,1}}{f^{\lambda}}
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q_k := \Pr[y_{k-1} < x < y_k] = \frac{f^{\lambda/2,1^{k-1}}}{f^{\lambda}} \text{ for } 1 \leq k \leq \ell + 1
\]

\[q_1 \geq q_2 \geq \cdots \geq q_{\ell+1}, \quad q_1 + \cdots + q_{\ell+1} = 1\]

Assume \( q_1 \leq \frac{1}{2} \) (else transpose). If \( q_1 \geq \frac{1}{3} \) then \( \Pr[x < y_1] \in [\frac{1}{3}, \frac{2}{3}] \), done.

Else, \( q_1 < \frac{1}{3} \) and pick \( k\)–min, s.t. \( \Pr[x < y_k] = q_1 + \cdots + q_k \geq \frac{1}{3} \). Then \( \Pr[x < y_k] \leq \Pr[x < y_{k-1}] + q_k \leq \frac{1}{3} + \frac{1}{3} \). Done.
Sorting probability for posets

**Sorting probability of** \( P \):

\[
\delta(P) = \min_{x,y \in P} |\Pr[x \prec y] - \Pr[y \prec x]| 
\]

**Conjecture (Kahn-Saks'84)**
For every finite poset \( P \) on \( n \) elements

\[
\delta(P) \to 0 \quad \text{as width}(P) \to \infty. 
\]

(\text{width}(P) = \text{size of max antichain})

[Komlós '90] Proved for posets with \( \Omega\left(\frac{1}{\log \log n}\right) \) minimal elements.
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**Theorem (Chan-Pak-Panova'21+)**

Let \( \lambda \vdash n \) with \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq \epsilon n \). Let \( P_\lambda \) be the poset of the Young diagram of \( \lambda \). Then there is a constants \( C > 0 \) depending on \( d, \epsilon, \) s.t.

\[
\delta(P_\lambda) < \frac{C}{\sqrt{n}}. 
\]
Sorting probability for posets

**Sorting probability of** $P$

$$\delta(P) = \min_{x,y \in P} |\Pr[x < y] - \Pr[y < x]|$$

**Conjecture (Kahn-Saks’84)**

For every finite poset $P$ on $n$ elements

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**Theorem (Chan-Pak-Panova’21+)**

Let $\lambda \vdash n$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq \epsilon n$. Let $P_\lambda$ be the poset of the Young diagram of $\lambda$. Then there is a constants $C > 0$ depending on $d, \epsilon$, s.t.

$$\delta(P_\lambda) < \frac{C}{\sqrt{n}}.$$  

**Theorem (Chan-Pak-Panova’21+)**

Let $\lambda = (n, n)$, so $P_\lambda$ is the “Catalan poset”. Then

$$\delta(P_\lambda) \leq \frac{C}{n^4}.$$
Linear extension ↔ lattice paths

### Diagram

**Left Side**
- Vertices labeled with $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7$
- Vertices labeled with $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7$
- Edges connecting $\alpha_i$ to $\beta_i$

**Right Side**
- Lattice path diagram
- Yellow squares represent $\alpha_i$
- Green squares represent $\beta_i$
- Blue squares represent $\alpha_i, \beta_i$
- Red squares represent $\alpha_i, \beta_i$

**Numerical Values**
- $\alpha_1: 2$
- $\alpha_2: 3$
- $\alpha_3: 5$
- $\alpha_4: 8$
- $\alpha_5: 9$
- $\alpha_6: 11$
- $\alpha_7: 12$
- $\beta_1: 1$
- $\beta_2: 4$
- $\beta_3: 6$
- $\beta_4: 7$
- $\beta_5: 10$
- $\beta_6: 13$
- $\beta_7: 14$
Sorting probabilities for Catalan posets

\[ P_4 \]

\[ L \]

\[ \lambda \]

\[ A \]

\[ \gamma \]

Limit shape of \[ R_n(\mathbf{h}, \mathbf{z}) := \text{Pr}[L(2, h - z) < L(1, h)] \]
Sorting probabilities for Catalan posets

Dyck paths $γ : (0, 0) \rightarrow (2n, 0)$:

$$Pr[L(1, a) < L(2, b)] = Pr[γ \text{ passes above } p = (a + b - 1, a - b)]$$
Sorting probabilities for Catalan posets

Dyck paths $\gamma : (0, 0) \rightarrow (2n, 0)$:

$$Pr[L(1, a) < L(2, b)] = Pr[\gamma \text{ passes above } p = (a + b - 1, a - b)]$$

$$R_n(h, z) := Pr[L(2, h - z) < L(1, h)]$$

Limit shape of $R_n\left(\frac{h}{n}, \frac{z}{\sqrt{2n}}\right)$
Catalan posets II:

\[ R_n(h, z) := \Pr[L(2, h - z) < L(1, h)] \]

Limit shape of \( R_n\left(\frac{h}{n}, \frac{z}{\sqrt{2}n}\right) \)

\[ R_n(h, \frac{\sqrt{n}}{10}) \leq \frac{1}{4} \quad R_n(h, 10\sqrt{n}) \geq \frac{3}{4} \]

\[ \frac{c_1}{\sqrt{n}} \leq R_n(h, z + 1) - R_n(h, z) \leq \frac{c_2}{\sqrt{n}} \]

\[ R_n(h, z) = R_n(n + z - h, z) \]

\[ \frac{n - 2h + z}{n^2} \leq R_n(h, z) - R_n(h+1, z) \leq \frac{n - 2h + z}{n^2} \]

\[ R_n(h, h-x) - R_n(h+1, h-x) = \frac{\#\gamma_1 - \#\gamma_2}{C_n} \]
Young diagrams of fixed length

Theorem (Chan-Pak-Panova’21+)

Let $\lambda \vdash n$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq \epsilon n$. Let $P_\lambda$ be the poset of the Young diagram of $\lambda$. Then there is a constant $C > 0$ depending on $d$, $\epsilon$, s.t.

$$\delta(P_\lambda) < \frac{C}{\sqrt{n}}.$$

\[
\begin{align*}
\{L \in \mathcal{E}(P) : L(x) < L(y)\} &= \left\{ \begin{array}{c} 1 & 2 & 3 & 4, \\ 5 & 6 & 7 \\ 8 \\ 6 \end{array}, \begin{array}{c} 1 & 3 & 4 & 5 \\ 2 & 7 & 8 \\ 6 \end{array} \right\} \\
\{L \in \mathcal{E}(P) : L(x) > L(y)\} &= \left\{ \begin{array}{c} 1 & 2 & 5 & 6, \\ 3 & 4 & 8 \\ 7 \\ 5 \end{array}, \begin{array}{c} 1 & 3 & 6 & 7, \\ 2 & 4 & 8 \\ 5 \end{array} \right\}
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\[
\begin{align*}
\{L \in \mathcal{E}(P) : L(x) < L(y)\} &= \left\{ \begin{array}{c}
1234, \\
567, \\
8, \\
\end{array} \right. \\
\{L \in \mathcal{E}(P) : L(x) > L(y)\} &= \left\{ \begin{array}{c}
1256, \\
348, \\
7, \\
\end{array} \right.
\end{align*}
\]

Proof ingredients:
- Intuition: Linear extensions $\leftrightarrow$ lattice paths in a simplex in $\mathbb{Z}^d$. 

---

**Note:**
- See Morales, Pak, Panova, "Hook formulas for skew shapes"
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Proof ingredients:

- **Intuition**: Linear extensions $\leftrightarrow$ lattice paths in a simplex in $\mathbb{Z}^d$.
- **Let $x$ – first row, $y$ –second row of $\lambda$.** $\Pr[x < y]$ = Probability the lattice path passes on “one side” of a codim 2 hyperplane.
- **Reduction via Hoeffding’s inequality.**
Young diagrams of fixed length

Theorem (Chan-Pak-Panova’21+)

Let $\lambda \vdash n$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq \epsilon n$. Let $P_\lambda$ be the poset of the Young diagram of $\lambda$. Then there is a constants $C > 0$ depending on $d, \epsilon$, s.t.

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- Reduction via Hoeffding’s inequality.
- Estimate

$$\delta(P_\lambda) \leq \sum_{\mu \approx \frac{1}{2} \lambda + c \sqrt{n}} \frac{f_\mu f_\lambda/\mu}{f_\lambda}$$
Young diagrams of fixed length

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Let $\lambda \vdash n$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq \epsilon n$. Let $P_\lambda$ be the poset of the Young diagram of $\lambda$. Then there is a constants $C > 0$ depending on $d, \epsilon$, s.t.

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• Let $x$ – first row, $y$ –second row of $\lambda$. $\Pr[x < y] = \text{Probability the lattice path passes on “one side” of a codim 2 hyperplane}$.

• Reduction via Hoeffding’s inequality.

• Estimate

$$\delta(P_\lambda) \leq \sum_{\mu \approx \frac{1}{2} \lambda + c \sqrt{n}} \frac{f_\mu f_\lambda/\mu}{f_\lambda}$$

• Apply Naruse Hook-Length Formula$^1$ for asymptotic bounds.
Skew SYTs

\[ \delta(P_\lambda) \leq \sum_{\mu \approx \frac{1}{2} \lambda + c \sqrt{n}} \frac{f^\mu f^{\lambda/\mu}}{f^\lambda} \]

where $E(\lambda/\mu)$ is the set of excited diagrams of $\lambda/\mu$. Excited diagrams:

$E(\lambda/\mu) = \{ D \subset \lambda : obtained from \mu via \}$
Skew SYTs

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**Theorem (Naruse, Morales-Pak-Panova)**

\[ f^{\lambda/\mu} = |\lambda/\mu|! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{u \in [\lambda] \setminus D} \frac{1}{h(u)}, \]

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\[ f(4321/21) = 7! \left( \frac{1}{1^4 \cdot 3^3} + \frac{1}{1^3 \cdot 3^3 \cdot 5} + \frac{1}{1^3 \cdot 3^3 \cdot 5} + \frac{1}{1^2 \cdot 3^3 \cdot 5^2} + \frac{1}{1^2 \cdot 3^2 \cdot 5^2 \cdot 7} \right) = 61 \]
Thank you!

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