Unification of set partitions

Orli Herscovici

School of Mathematics
Georgia Tech

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Some facts about Touchard polynomials, $T_n(x)$

g.f. $$e^{x(e^t-1)} = \sum_{n\geq 0} T_n(x) \frac{t^n}{n!}$$

recurrence relation

$$T_{n+1}(x) = x\left(1 + \frac{d}{dx}\right) T_n(x)$$

$$T_{n+1}(x) = x \sum_{k=0}^{n} \binom{n}{k} T_k(x)$$

explicit form $$T_n(x) = \sum_{k=1}^{n} S(n,k)\, x^k$$

$$S(n,k) = S(n-1,k-1) + kS(n-1,k)$$

where $S(n,k)$ - Stirling numbers of the second kind

Generalization of Touchard polynomials
q-calculus

Difference operator (Jackson, 1910)

$$D_q f = \frac{f(qx) - f(x)}{qx - x} \quad , \quad q \neq 1$$

q-integer $\quad [n]_q = \frac{q^n - 1}{q - 1} = 1 + q + q^2 + \dots + q^{n-1}$

$$D_q x^n = [n]_q x^{n-1} \quad , \quad \lim_{q \to 1} D_q f = \frac{df}{dx}, \quad \lim_{q \to 1} [n]_q = n$$

q-exponent

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} \quad , \quad [n]_q! = [1]_q [2]_q \dots [n]_q$$

$T_n(x) = \sum S(n,k)\, x^k$, $S(n,k) = S(n-1,k-1) + k\,S(n-1,k)$

$x^n = \sum S(n,k)\, x_{(k)}$, $x_{(k)} = x(x-1)\dots(x-k+1)$

Carlitz (1933, 1948)

$$[x]_q^n = \sum_{k=0}^{n} q^{\frac{1}{2}k(k-1)} S_q(n,k)\,[x]_{q,(k)}$$

$$S_q(n,k) = S_q(n-1,k-1) + [k]_q S_q(n-1,k)$$

more natural

Milne (1978)

$$[x]_q^n = \sum_{k=0}^{n} \hat{S}_q(n,k)\,[x]_{q,(k)}$$

$$\hat{S}_q(n,k) = q^{k-1}\hat{S}_q(n-1,k-1) + [k]_q \hat{S}_q(n-1,k)$$

$$e^{x(e^t-1)} = \sum T_n(x)\frac{t^n}{n!}$$

q-analogue ?

(Non-Extensive) Statistical Mechanics

Tsallis, 1988 $\qquad e_q(x) = \left(1+(1-q)x\right)^{\frac{1}{1-q}}$

Borges, 1998 $\qquad e_q(x) = 1 + \sum_{n=1}^{\infty} Q_{n-1}(q)\frac{x^n}{n!}$

where $Q_n(q) = 1\cdot q(2q-1)\cdots(nq-(n-1))$

$\Downarrow$

$\tilde{D}_q e_q(x) = e_q(x) \quad \Rightarrow \quad \tilde{D}_q f(x) = \left[1+(1-q)x\right]\frac{df(x)}{dx}$

Carlitz, 1956 $\qquad \lim_{q\to 1} e_q(x) = e^x$

$(1+\lambda x)^{\mu}$, $\lambda\mu = 1$

p,q- Touchard polynomials and Stirling numbers

e^x $\qquad\qquad$ $e_q(x) = (1+(1-q)x)^{\frac{1}{1-q}}$

$e^{x(e^t-1)} = \sum T_n(x)\frac{t^n}{n!}$ $\qquad\qquad$ $e_p\big(x(e_q(t)-1)\big) = \sum T_n(x;p,q)\frac{t^n}{n!}$

$T_n(x) = \sum S(n,k)x^k$ $\qquad\qquad$ $T_n(x;p,q) = \sum S_{p,q}(n,k)x^k$

$$S(n,k) = S(n-1,k-1) + kS(n-1,k)$$

Prop. (M., Mansour 2017)

$$S_{p,q}(n,k) = \big(1-(1-p)(k-1)\big)S_{p,q}(n-1,k-1) + \big(k-(1-q)n\big)S_{p,q}(n-1,k)$$

Connection to set partitions

a set $[n]=\{1,2,\dots,n\}$, partition into k non-empty disjoint blocks $B_1/\dots/B_k$

standard: $\pi = 139/2/457/68$

u^2

non-standard $\pi = 2/86/913/475$

v^3

$$S(n,k,u,v) = \sum_{\pi} u^{nsb(\pi)} v^{nse(\pi)}$$

Prop. (H., Mansour 2017)

$$S(n,k,u,v) = (1+u(k-1))S(n-1,k-1,u,v) + (k+vn)S(n,k,u,v)$$

$$S_{p,q}(n,k) = S(n,k,p-1,q-1)$$

Theorem (H., 2019)

$$[u^i v^j]\, S(n,k,u,v) = |s(n,n-j)|\, S(n-j,k)\, |s(k,k-i)|$$

$S(n,k)$ - Stirling numbers of the second kind
$s(n,k)$ - Stirling numbers of the first kind

Theorem (H, 2019)

$$T_n(x;p,q) = \sum_{k=0}^{n} \sum_{m=0}^{k-1} \sum_{\ell=0}^{n-k} \sum_{i=m}^{k-1} \sum_{j=\ell}^{n-k} \binom{i}{m}\binom{j}{\ell} (-1)^{m+\ell} s(n,n-j)\, S(n-j,k)\, s(k,k-i)\, p^m q^\ell x^k$$

$$e_p\left(x(e_q(t)-1)\right) = \sum_{n\geq 0} T_n(x;p,q)\frac{t^n}{n!}$$

in notation of Motzkin, 1971

Sets of sets A008277 $e^{x(e^t-1)} = \sum_{n\geq 0} T_n(x;1,1)\frac{t^n}{n!}$

Sets of lists A105278 $e^{\frac{xt}{1-t}} = \sum T_n(x;1,2)\frac{t^n}{n!}$

Lists of sets A019538 $\frac{1}{1-x(e^t-1)} = \sum T_n(x;2,1)\frac{t^n}{n!}$

Lists of lists A156992 $\frac{1-t}{1-t-tx} = \sum T_n(x;2,2)\frac{t^n}{n!}$

Cyclically ordered blocks/elements ?

Cyclically ordered elements

unordered blocks $-\log\left(1 - x(e^t - 1)\right)$

ordered blocks $-\log\left(1 - \frac{xt}{1-t}\right)$

cyclically ordered blocks $-\log\left(1 + x\log(1-t)\right)$

Cyclically ordered blocks

unordered elements $(1-t)^{-x}$

ordered elements $\frac{1}{1 + x\log(1-t)}$

Back to q-calculus

deformed exp. function $\quad e_q(t) = \left(1+(1-q)t\right)^{\frac{1}{1-q}}$, $q \neq 1$

combinatorics $\Rightarrow$ $e_\beta(t) = (1-\beta t)^{-\frac{1}{\beta}}$ $\quad \beta \neq 0$

deformed log function $\quad \log_\beta(t) = \dfrac{t^{-\beta}-1}{-\beta}$

$-\log(1-t) \rightsquigarrow -\log_\beta(1-t) = \dfrac{(1-t)^{-\beta}-1}{-\beta}$

$$\Rightarrow \quad -\log_\beta(1-t) = \frac{1}{\beta}\, e_{\frac{1}{\beta}}(\beta t) - \frac{1}{\beta}$$

Unified generating function

$$e_{\frac{\beta}{\alpha}}(\alpha t) = \left(1 - \frac{\beta}{\alpha}\cdot \alpha t\right)^{-\frac{\alpha}{\beta}}$$

s.t. $\alpha\cdot\beta = 0 \Rightarrow \begin{cases} \alpha = 0, \ \beta \neq 0 \\ \alpha \neq 0, \ \beta = 0 \end{cases}$ OR

"$\alpha = 0$" $= \lim\limits_{\alpha\to 0}$, "$\beta = 0$" $= \lim\limits_{\beta \to 0}$

$$F(x,t;\alpha,\beta,\lambda,\mu) = \frac{1}{\alpha} e_{\frac{\beta}{\alpha}}\left[\alpha x\left(\frac{1}{\lambda} e_{\frac{\mu}{\lambda}}(\lambda t) - \frac{1}{\lambda}\right)\right] - \frac{1}{\alpha} + \frac{\delta_{\alpha,1}}{\alpha}$$

Unified generating function

$$F(x,t;\alpha,\beta,\lambda,\mu) = \frac{1}{\alpha} e_{\frac{\beta}{\alpha}}\left[\alpha x\left(\frac{1}{\lambda} e_{\frac{\mu}{\lambda}}(\lambda t) - \frac{1}{\lambda}\right)\right] - \frac{1}{\alpha} + \frac{\delta_{\alpha,1}}{\alpha}$$

let $\bar{v} := (\alpha, \beta, \lambda, \mu)$

$$F(x,t;\bar{v}) = \sum_{n\geq 0} u_n(x;\bar{v}) \frac{t^n}{n!}$$

Unified polynomials

$$U_0(x;\bar{v}) = \frac{\delta_{\alpha,1}}{\alpha}$$

$$U_1(x;\bar{v}) = x$$

$$U_2(x;\bar{v}) = (\alpha+\beta)x^2 + (\lambda+\mu)x$$

$$U_3(x;\bar{v}) = (\alpha+\beta)(\alpha+2\beta)x^3 + 3(\alpha+\beta)(\lambda+\mu)x^2 + (\lambda+\mu)(\lambda+2\mu)x$$

$\forall n$, $U_n(x;\bar{v})$ is a polynomial in $\alpha, \beta, \lambda, \mu, x$

Some generalizations

$$T_{n+1}(x) = x\left(1 + \frac{d}{dx}\right) T_n(x)$$

Thm (H., 2022)

$$U_{n+1}(x;\bar{v}) = (\alpha x + \mu n) U_n(x;\bar{v}) + x(\beta x + \lambda) \frac{\partial}{\partial x} U_n(x|\bar{v})$$

$$U_1(x;\bar{v}) = x$$

Some generalizations

$$T_n(x) = \sum_{k=1}^{n} S(n,k)\, x^k \quad , \quad S(n+1,k) = S(n,k-1) + k\,S(n,k)$$

$$U_n(x;\bar{v}) = \sum_{k=1}^{n} S(n,k;\bar{v})\, x^k$$

Prop. (H., 2022) $\quad S(0,0;\bar{v}) = \dfrac{\delta_{\alpha,1}}{\alpha}, \quad S(1,1;\bar{v}) = 1$

$$S(n+1,k;\bar{v}) = \begin{cases} (\lambda + n\mu)\, S(n,1;\bar{v}) \;, & k = 1 \\ (\alpha + (k-1)\beta)\, S(n,k-1;\bar{v}) + (k\lambda + n\mu)\, S(n,k;\bar{v}) \;, & 2 \le k \le n \\ (\alpha + n\beta)\, S(n,n;\bar{v}) & k = n+1 \\ 0 \;, & \text{otherwise} \end{cases}$$

Combinatorics of set partitions

nsB - non standard block
nse - non standard element

$\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$

σ_j - right-to-left minimum if $\sigma_j < \sigma_k \quad \forall k > j$

Prop (H., 2019) Let $RLM(\sigma) = \{\sigma_j \mid \sigma_j < \sigma_k \ \forall k > j\}$,
$NSE(\sigma) = \{\sigma_j \mid \exists k > j \ \sigma_k < \sigma_j\}$,
$rlm(\sigma) = |RLM(\sigma)|$, $nse(\sigma) = |NSE(\sigma)|$

$\Rightarrow NSE(\sigma) \cap RLM(\sigma) = \emptyset$, $nse(\sigma) + rlm(\sigma) = n$

Combinatorics of set partitions

$$\pi = B_{1,1} \cdots B_{1,j_1} / B_{2,1} \cdots B_{2,j_2} / \cdots / B_{k,1} \cdots B_{k,j_k}$$

$B_m = \min\{B_{m,1}, \ldots, B_{m,j_m}\}$ – opener of the m^{th} block

$$nsb(\pi) = |\{j \mid \exists m > j \quad B_m < B_j\}|$$

$$nse(\pi) = \sum_{\ell} |\{j \mid \exists m > j \quad B_{\ell,m} < B_{\ell,j}\}|$$

$$rlb(\pi) = |\{j \mid \forall m > j \quad B_m > B_j \text{ AND } B_j > 1\}|$$

$$rle(\pi) = \sum_{\ell} |\{j \mid \forall m > j \quad B_{\ell,m} > B_{\ell,j} \text{ AND } B_{\ell,j} > B_\ell\}|$$

$$\bar{S}(n,k;\bar{v}) = \sum \alpha^{rlb(\pi)} \beta^{nsb(\pi)} \lambda^{rle(\pi)} \mu^{nse(\pi)}$$

Insertion element n, while the set [n-1] is ordered

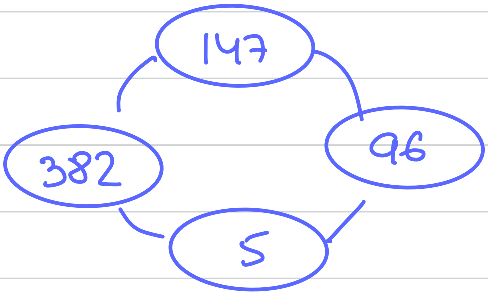
$\underbrace{\square\,\square\,\cdots\,\square}_{k-1\text{ blocks}}$ OR $\underbrace{\square\,\square\,\cdots\,\square}_{k\text{ blocks}}$

$\underbrace{\square\,\square\,\cdots\,\square}_{k-1\text{ blocks}}$		OR	$\underbrace{\square\,\square\,\cdots\,\square}_{k\text{ blocks}}$
$\Downarrow$			$\Downarrow$
insert as a block			insert into one of blocks

as last block	otherwise	as last element	otherwise
↗rlb	↗nsb	↗rle	↑nse
α.	$(k-1)\beta$	$k\lambda$.	$(n-1)\mu$.
$\bar{S}(n-1,k-1;\bar{v})$	$\bar{S}(n-1,k-1;\bar{v})$	$\bar{S}(n-1,k;\bar{v})$	$\bar{S}(n-1,k;\bar{v})$

$$\bar{S}(n,k;\bar{v}) = (\alpha+(k-1)\beta)\bar{S}(n-1,k-1;\bar{v}) + (k\lambda+(n-1)\mu)\bar{S}(n-1,k;\bar{v})$$

Example



147/96/5/382
96/5/382/147
5/382/147/96
382/147/96/5

382/147/96/5 nse = 3 , rle = 2

openers 2 1 6 5 $\alpha^1 \beta^2 \lambda^2 \mu^3$

nsb = 2 rlb = 1

Thm (H., 2022)

$$\left[\alpha^{k-1-i}\beta^{i}\lambda^{n-k-j}\mu^{j}\right]\bar{S}(n,k;\bar{v}) = |s(n,n-j)|\,S(n-j,k)\,|s(k,k-i)|$$

$$\frac{1}{\alpha}e_{\frac{\beta}{\alpha}}\left[\alpha x\left(\frac{1}{\lambda}e_{\frac{\mu}{\lambda}}(\lambda t)-\frac{1}{\lambda}\right)\right]-\frac{1}{\alpha}+\frac{\delta_{\alpha,1}}{\alpha} =$$

$$= \sum_{n=0}^{\infty}\sum_{k=1}^{n}\sum_{i=0}^{k-1}\sum_{j=0}^{n-k}|s(n,n-j)|\,S(n-j,k)\,|s(k,k-i)|\,\alpha^{k-1-i}\beta^{i}\lambda^{n-k-j}\mu^{j}x^{k}\frac{t^{n}}{n!}$$

α	β	λ	μ	
1	β	1	μ	$e_\beta[x(e_\mu(t)-1)]$
α	1	λ	1	$-\log_\alpha[1+x\log_\lambda(1-t)] + \frac{\delta_{\alpha,1}}{\alpha}$
0	1	0	1	$-\log[1+x\log(1-t)]$
0	1	1	1	$-\log\left[1-\frac{xt}{1-t}\right]$
1	1	0	1	$\frac{1}{1+x\log(1-t)}$
0	1	1	0	$-\log[1-x(e^t-1)]$
1	0	0	1	$(1-t)^{-x}$

Thank you!