Comma Sequences, Part II

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Joint work with Robert Dougherty-Bliss

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Comma Transform of Catalan Numbers

Catalan Sequence (A000108):

$$1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, \ldots$$

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Comma Transform:

$$12, 25,$$

Comma Transform of Catalan Numbers

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Comma Transform of Catalan Numbers

Catalan Sequence (A000108):

$$1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, \ldots$$

Comma Transform:

$$12, 25, 51,$$

Comma Transform of Catalan Numbers

Catalan Sequence (A000108):

$$1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, \ldots$$

Comma Transform:

$$12, 25, 51, 44, 21, 24, 91, 4, 21, 65, 62, 27 \ldots$$

Comma Transform of Catalan Numbers

Catalan Sequence (A000108):

$$1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, \ldots$$

Comma Transform:

$$12, 25, 51, 44, 21, 24, 91, 4, 21, 65, 62, 27 \ldots$$

Differences (A000245):

$$1, 3, 9, 28, 90, 297, 1001, 3432, 11934, 41990, 149226, 534888, \ldots$$

The Original Comma Sequence

Goal

Lexicographically earliest sequence where the comma transform is the difference.

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$$1,$$

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$$1,$$

Comma Transform:

$$1x$$

Next Term:

$$1 + 1x$$

The Original Comma Sequence

Goal

Lexicographically earliest sequence where the comma transform is the difference.

$$1,$$

Comma Transform:

$$11$$

Next Term:

$$1 + 11 = 12$$

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Goal

Lexicographically earliest sequence where the comma transform is the difference.

$$1, 12,$$

Comma Transform:

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Goal

Lexicographically earliest sequence where the comma transform is the difference.

$$1, 12,$$

Comma Transform:

$$11, 2x$$

Next Term:

$$12 + 2x$$

The Original Comma Sequence

Goal

Lexicographically earliest sequence where the comma transform is the difference.

$$1, 12,$$

Comma Transform:

$$11, 23$$

Next Term:

$$12 + 23 = 35$$

The Original Comma Sequence

Goal

Lexicographically earliest sequence where the comma transform is the difference.

$$1, 12, 35,$$

Comma Transform:

$$11, 23$$

Next Term:

$$12 + 23 = 35$$

The Original Comma Sequence

Goal

Lexicographically earliest sequence where the comma transform is the difference.

$$1, 12, 35,$$

Comma Transform:

$$11, 23, 5x$$

Next Term:

$$35 + 5x$$

The Original Comma Sequence

Goal

Lexicographically earliest sequence where the comma transform is the difference.

$$1, 12, 35,$$

Comma Transform:

$$11, 23, 58$$

Next Term:

$$35 + 58 = 93$$

The Original Comma Sequence

Goal

Lexicographically earliest sequence where the comma transform is the difference.

$$1, 12, 35,$$

Comma Transform:

$$11, 23, 59$$

Next Term:

$$35 + 59 = 94$$

The Original Comma Sequence

Goal

Lexicographically earliest sequence where the comma transform is the difference.

$$1, 12, 35, 94,$$

Comma Transform:

$$11, 23, 59$$

Next Term:

$$35 + 59 = 94$$

The Original Comma Sequence

Goal

Lexicographically earliest sequence where the comma transform is the difference.

Comma Sequence (A121805):

$1, 12, 35, 94, 135, 186, 248, 331, 344, 387, 461, 475, 530, 535, 590, 595, 651, 667, 744, 791, 809, 908, 997, 1068, 1149, 1240, 1241, 1252, 1273, 1304, 1345, 1396, 1457, 1528, 1609, 1700, 1701, 1712, 1733, 1764, \ldots$

Comma Transform/Differences (A366487):

$11, 23, 59, 41, 51, 62, 83, 13, 43, 74, 14, 55, 5, 55, 5, 56, 16, 77, 47, 18, 99, 89, 71, 81, 91, 1, 11, 21, 31, 41, 51, 61, 71, 81, 91, 1, 11, 21, 31, 41, \ldots$

The Original Comma Sequence

Comma Sequence (A121805):

(Submitted to the OEIS in 2006 by Eric Angelini)

$1, 12, 35, 94, 135, 186, 248, 331, 344, 387, 461, 475, 530, 535, 590, 595, 651, 667, 744, 791, 809, 908, 997, 1068, 1149, 1240, 1241, 1252, 1273, 1304, 1345, 1396, 1457, 1528, 1609, 1700, 1701, 1712, 1733, 1764, \ldots$

Question: Can we keep going forever?

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Question: Can we keep going forever? If we start with 3 instead

$$3, 36$$

Next Term:

$$36 + 6x =$$

The Original Comma Sequence

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Next Term:

$$36 + 69 = 103$$

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Question: Can we keep going forever? If we start with 3 instead

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Next Term:

$$36 + 69 = 103 \text{ or } 36 + 61 = 97$$

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1, 12, 35, 94, 135, 186, 248, 331, 344, 387, 461, 475, 530, 535, 590, 595, 651, 667, 744, 791, 809, 908, 997, 1068, 1149, 1240, 1241, 1252, 1273, 1304, 1345, 1396, 1457, 1528, 1609, 1700, 1701, 1712, 1733, 1764, . . . , 99999945 (provided by W. Edwin Clark in 2006)

Next Term:

$$99999945 + 5x =$$

The Original Comma Sequence

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$1, 12, 35, 94, 135, 186, 248, 331, 344, 387, 461, 475, 530, 535, 590, 595, 651, 667, 744, 791, 809, 908, 997, 1068, 1149, 1240, 1241, 1252, 1273, 1304, 1345, 1396, 1457, 1528, 1609, 1700, 1701, 1712, 1733, 1764, \ldots, 99999945$ (provided by W. Edwin Clark in 2006)

Next Term:

$$99999945 + 59 = 100000004$$

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$1, 12, 35, 94, 135, 186, 248, 331, 344, 387, 461, 475, 530, 535, 590, 595, 651, 667, 744, 791, 809, 908, 997, 1068, 1149, 1240, 1241, 1252, 1273, 1304, 1345, 1396, 1457, 1528, 1609, 1700, 1701, 1712, 1733, 1764, \ldots, 99999945$ (provided by W. Edwin Clark in 2006)

Next Term:

$99999945 + 59 = 100000004$ or $99999945 + 51 = 99999996$

The Conjecture

Conjecture (Angelini et al, 2024)

For any positive initial value, the sequence will terminate.

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For any positive initial value, for any base greater than 2, the sequence will terminate.

From here, we will work in base b with $b \geq 3$.

Danger Zones

- For numbers from $d \cdot b^k$ to $(d+1) \cdot b^k - b^2$, comma numbers have a units digit of d (with $1 \leq d < b, k \geq 3$).
  eg 30045

Danger Zones

- For numbers from $d \cdot b^k$ to $(d+1) \cdot b^k - b^2$, comma numbers have a units digit of d (with $1 \leq d < b, k \geq 3$).
- From $d \cdot b^k - b^2$ to $d \cdot b^k - 1$ the units digit of a comma number is either $d - 1$ or d unless ...

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Danger Zone

For $k \geq 2$, the interval from $b^k - b^2$ to $b^k - 1$ is a **danger zone** and the intervals from $d \cdot b^k - b^2$ to $d \cdot b^k - 1$ for $1 < d < b$ is a **pseudo-danger zone**

Landmines

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$$(b-1)\cdots(b-1)xy$$

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where $x + y = b - 1$ and $0 < x, y < b-1$.

Next Term:

$$(b-1)\cdots(b-1)xy + y(b-1) \geq b^k$$
$$(b-1)\cdots(b-1)xy + y1 < b^k$$

Birthpoints

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A **birthpoint** is a number that has no comma predecessor.

- Birthpoints greater than b^2 have the form

$$d \cdot b^k$$

where $k \geq 2$ and $1 < d < b$.

Better Than Brute Force?

Brute Force

For every b and every number with no predecessor, calculate the comme sequence and see if it ends.

- Check for landmines
- Only need to check danger zones
- Easy to find comma successor outside of danger and pseudo danger zones

Fast Computation

How to compute quickly:

1. Start with $d \cdot b^k - u$

An example with $b = 10$:

1. Start with $5992 = 6 \cdot 10^3 - 8$

Fast Computation

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1. Start with $d \cdot b^k - u$
2. Compute the comma numbers

An example with $b = 10$:

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2. Comma numbers: $26, 86, 46, 06, 66,$

Fast Computation

How to compute quickly:

1. Start with $d \cdot b^k - u$
2. Compute the cycle of comma numbers

An example with $b = 10$:

1. Start with $5992 = 6 \cdot 10^3 - 8$
2. Comma numbers: $26, 86, 46, 06, 66,$ repeat!

Fast Computation

How to compute quickly:

1. Start with $d \cdot b^k - u$
2. Compute the cycle of comma numbers
3. Add the cycle as many times as possible (ie find b^k mod the sum of the cycle)

An example with $b = 10$:

1. Start with $5992 = 6 \cdot 10^3 - 8$
2. Comma numbers: $26, 86, 46, 06, 66,$ repeat!
3. Sum of the cycle: 230 10^3 mod 230 is 80, so we get $7 \cdot 10^3 - 88 = 6912$

Fast Computation

How to compute quickly:

1. Start with $d \cdot b^k - u$
2. Compute the cycle of comma numbers
3. Add the cycle as many times as possible (ie find b^k mod the sum of the cycle)
4. Try to fill in as many more comma numbers as possible

An example with $b = 10$:

1. Start with $5992 = 6 \cdot 10^3 - 8$
2. Comma numbers: $26, 86, 46, 06, 66,$ repeat!
3. Sum of the cycle: 230
   10^3 mod 230 is 80, so we get $7 \cdot 10^3 - 88 = 6912$
4. $6912 + 26 = 6938$. The new d is 7 and the new u is 62.

Faster Computation

How to compute even faster:

1. Start with $1 \cdot b^k - u$
2. Run $b - 1$ fast computations
3. Check if you ended at a landmine

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1. Start with $1 \cdot b^k - u$
2. Run $b - 1$ fast computations
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Just need to check every

$$2 \leq d < b$$
$$2 < k$$

Making the Computation Finite

Key Idea

For any mod, b^k is eventually periodic.

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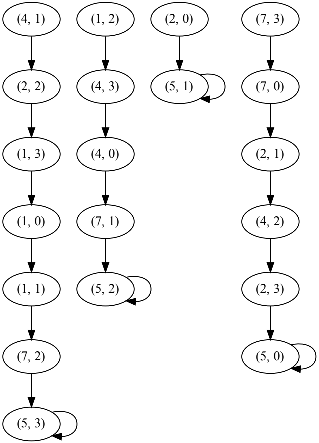
Just need to check every

$$2 \leq d < b$$
$$C < k \leq L + C$$

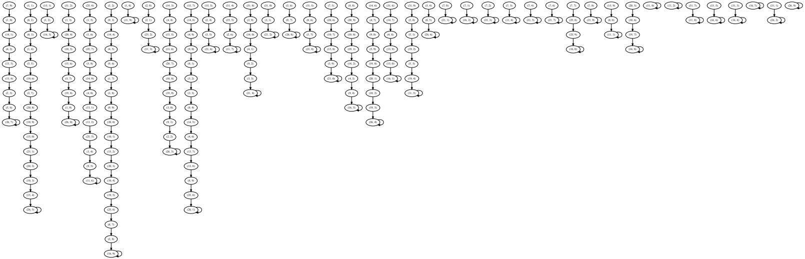
where L is the lcm of the periods of b^k mod the cycle sum.

$b = 3 (L = 4)$



- Actually depict only $d = 1$
- Labeled by (u, k)
- Do not depict birthpoints
- Landmines have loops

$b = 6$



b=?

Theorem (Dougherty-Bliss & TS)

For all initial values, comma sequences are finite in bases 3 through 19, as well as 22 and 23.

- Base 3 was known by Angelini et al
- Code generates paths and checks if all the vertices are there
- In theory, one could run the code for any base

Uniformity of u

Conjecture

For a given value of u, the distribution of u' is (approximately) uniform over all (valid) u.

- Experimentally true
- Any cycle must have length cL for some c
  e.g. $b = 10 : L = 924$, average length of path: 6.75, longest path: 45
- Should be unlikely to survive that long, but no independence

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- If initial value lands on a path P, average number of danger zones is $\binom{|P|}{2}$
- So total average is

$$\frac{\sum_P \binom{|P|}{2}}{\sum_P |P|}$$

Complex Model

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- If initial value lands on a path P, average number of danger zones is $\binom{|P|}{2}$
- So total average is

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- Note that this only works if we know there are no cycles

Simple vs Complex Model: $b = 10$

- Simple Model: Out of the 100 numbers before 10^k, 12 die, so expect to see $100/12 \approx 8.33$ danger zones

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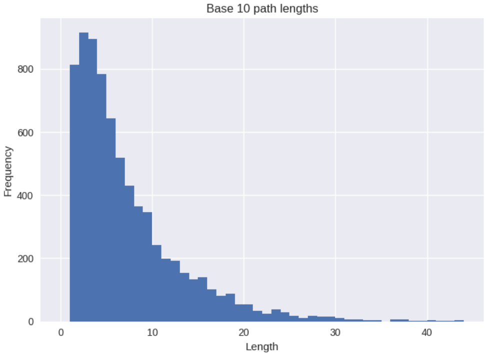
Simple vs Complex Model: $b = 10$

- Simple Model: Out of the 100 numbers before 10^k, 12 die, so expect to see $100/12 \approx 8.33$ danger zones
- Complex Model: Summing up over all paths, expect to see ≈ 5.42
- Empirical Value: (using A330129) ≈ 5.28

Complex Model Approximated: $b = 10$

- Really want to approximate distribution of path lengths
- In base 10,



Thank You!

Any questions?