Refinements and symmetries for volumes of flow polytopes

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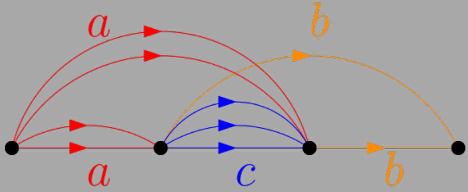
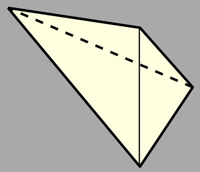
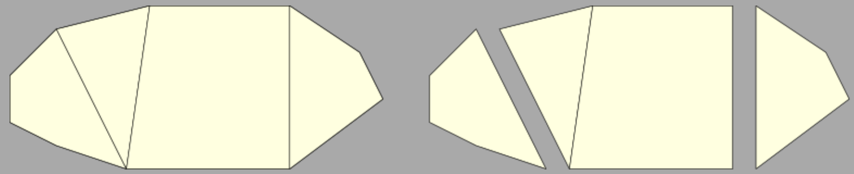




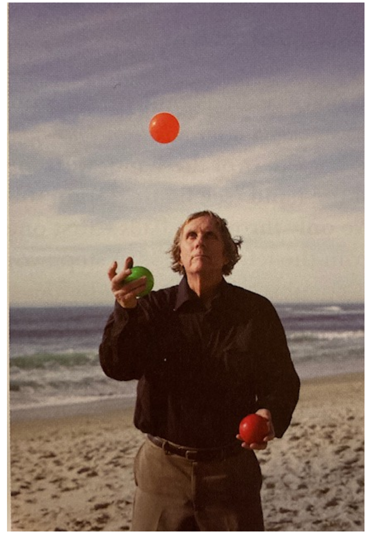


Rutgers Experimental Mathematics Seminar

March 25, 2021

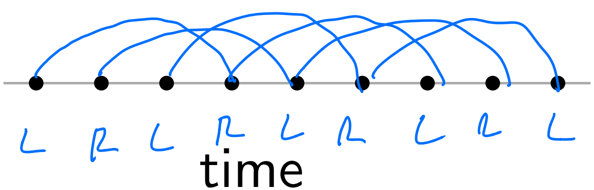
and with C. Benedetti, C. Hanusa, P. Harris, A. Simpson

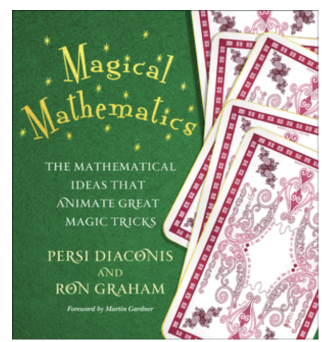
Juggling sequences



Ron Graham (1935-2020)

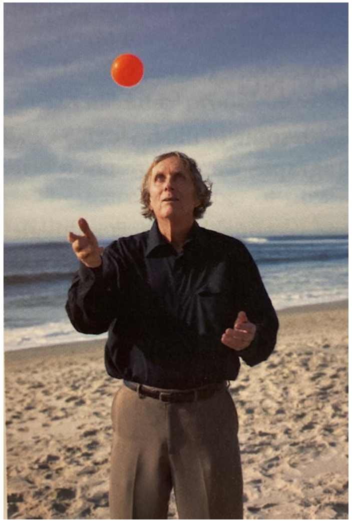
A *juggling sequence* is a sequence of arcs specifying amount of time a ball is thrown.



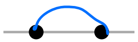


Mathematics of juggling

In how many ways can we juggle one ball with time n?

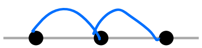


$n = 1$



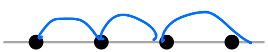
1

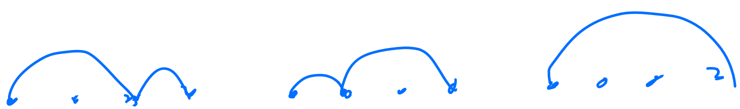
$n = 2$



2

$n = 3$





4

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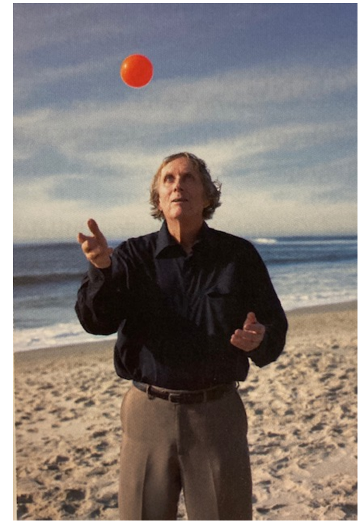
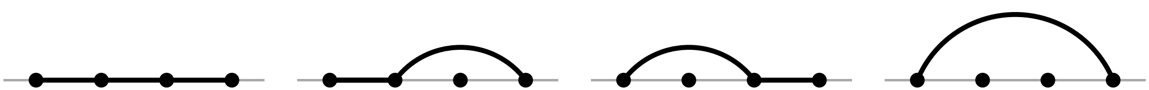


$n = 1$

$n = 2$

$n = 3$



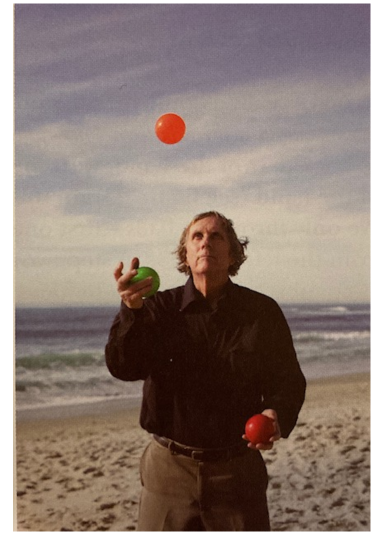
Answer: There are 2^{n-1} ways.

Mathematics of juggling

In how many ways $\mathrm{js}(n, t)$ can we juggle t balls with time n?

$n = 2$, $t = 2$

3



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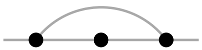
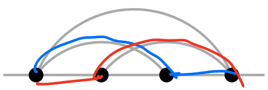
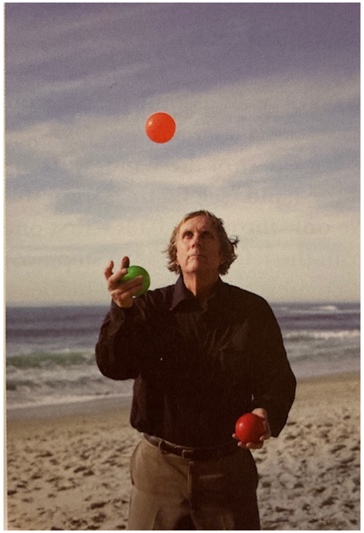


$n = 3,\ t = 2$

9 + 1 = 10





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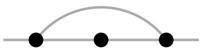

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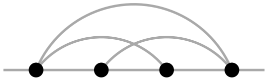

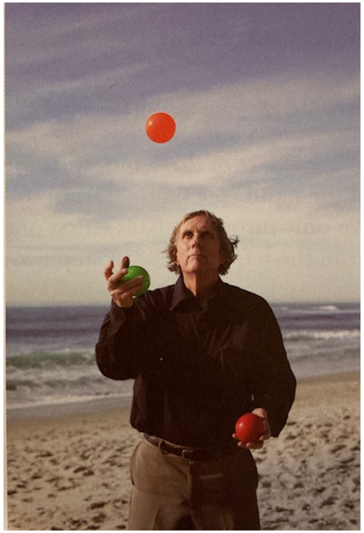

Answer: There is no nice known formula in general.

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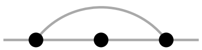

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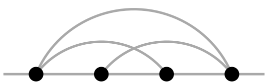
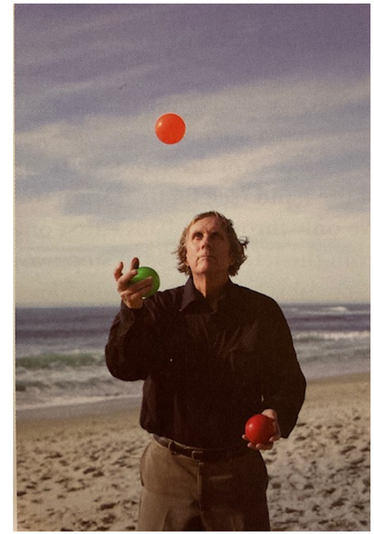


9 + 1



Answer: There is no nice ~~known formula in general.~~ satisfies recurrence with Par(t) ← # integer partitions

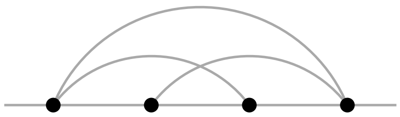
$$(t+1)^{n-1} \leq \mathrm{js}(n,t) \leq \prod_{1 \leq i \leq j \leq n} \frac{2t+i+j-1}{i+j-1}$$

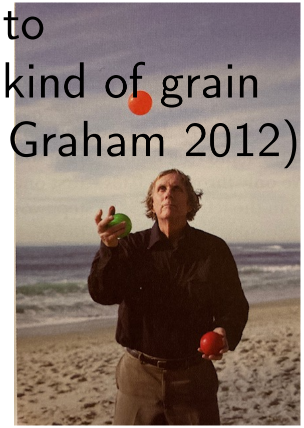
↑ tuples of sets   ↑ "alternating sign matrices"

Juggling with sand (ℝ-balls)

"To begin, it is necessary to have some reasonable objects to juggle!...Some jugglers fill tennis balls with *sand* or some kind of grain ..."

(Diaconis, Graham 2012)





Juggling with sand ($\mathbb{R}$-balls)

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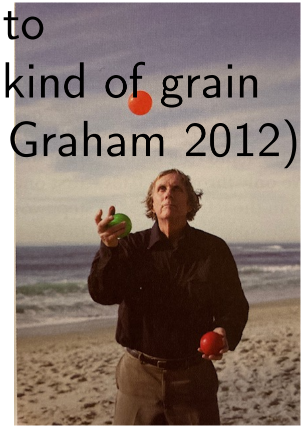

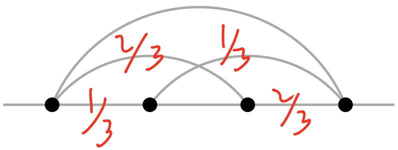

What is the set of $\mathbb{R}$-juggling sequences of one ball and time n?

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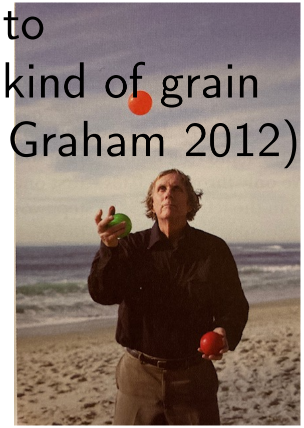

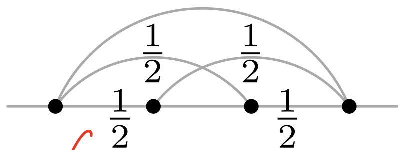

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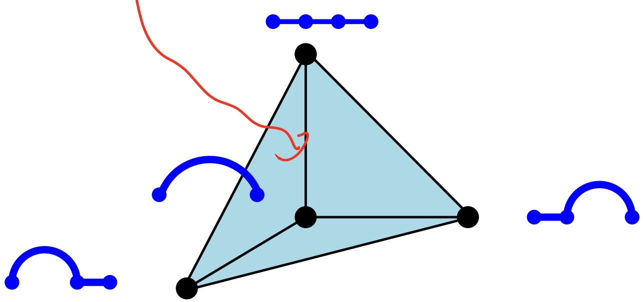

Answer: a *polytope*!

Integral polytopes

P a polytope in $\mathbb{R}^N$ with integral vertices:

P is the **convex hull** of finitely many vertices $\mathbf{v}$ in $\mathbb{Z}^N$

OR

P is the intersection of finitely many **half spaces**

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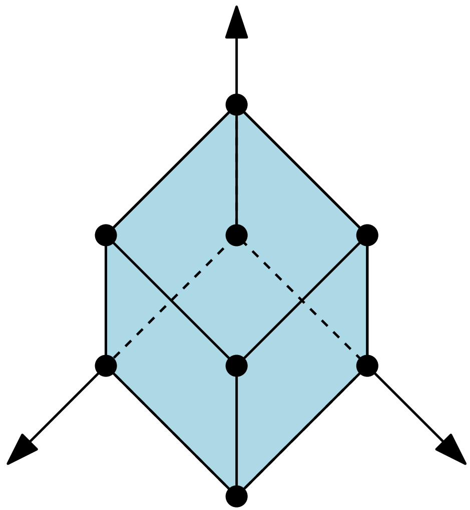
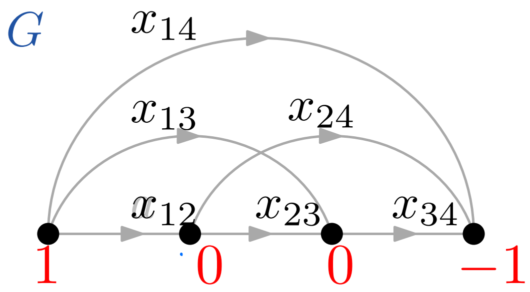


d-cube: convex hull of $\{0,1\}^d$

$$C_d = \left\{ (x_1, \ldots, x_d) \;\middle|\; 0 \le x_i \le 1, \quad i = 1, \ldots, d \right\}$$

$\mathbb{R}$-juggling to flow polytopes

Example



$$
\begin{cases}
x_{ij} \geq 0 \\
x_{12} + x_{13} + x_{14} = 1 \\
x_{23} + x_{24} - x_{12} = 0 \\
x_{34} - x_{13} - x_{23} = 0 \\
x_{14} + x_{24} + x_{34} = 1
\end{cases}
$$

$\mathbb{R}$-juggling to flow polytopes

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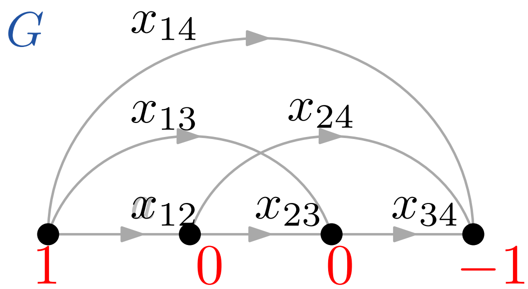

G graph $n+1$ vertices m edges

$\mathcal{F}_G(\mathbf{a}) = \{$flows $x(\epsilon) \in \mathbb{R}_{\geq 0}$, $\epsilon \in E(G) \mid$ netflow vertex $i = a_i\}$

$\mathbf{a} = (1, 0, \ldots, 0, -1)$

$\mathcal{F}_G(1, 0, \ldots, 0, -1)$:

$\mathbb{R}$-juggling to flow polytopes

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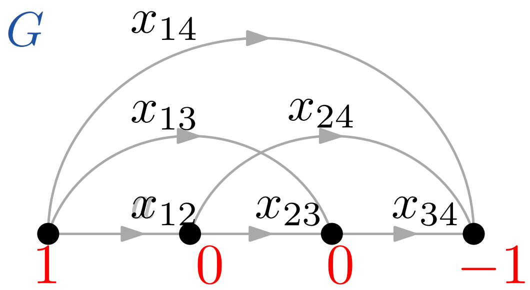

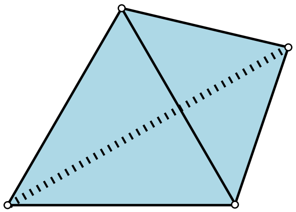

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$\mathcal{F}_G(1, 0, \ldots, 0, -1)$:

Connections to flow polytopes

flow polytopes have been related to:

- *Toric geometry* (Hille 2003)
- *Jeffrey–Kirwan residues* (Baldoni–Vergne 2009)
- *cluster algebras* (Danilov–Karzanov–Koshevoy 2012)
- *Root systems of Lie type* (Mészáros-M 11, Corteel-Kim-Mészáros 17)

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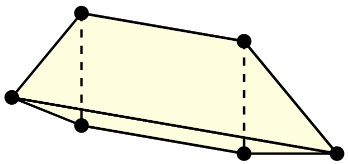
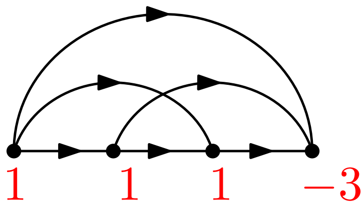




- *diagonal harmonics* (Mészáros-M-Rhoades 17, Liu-Mészáros-M 18)
- *generalized permutahedra* (Mészáros-St. Dizier 2017)
- *Schubert polynomials* (Escobar-Mészáros 2018)
  (Fink-Mészáros-St. Dizier 2018)
- *Gelfand-Tsetlin polytopes* (Liu-Mészáros-St. Dizier 2019)
- *associahedra, permutahedra*
  (von Bell, González D'Léon-Mayorga, Yip 20+, Barnard-McConville 20+)

Connections to flow polytopes

flow polytopes have been related to:

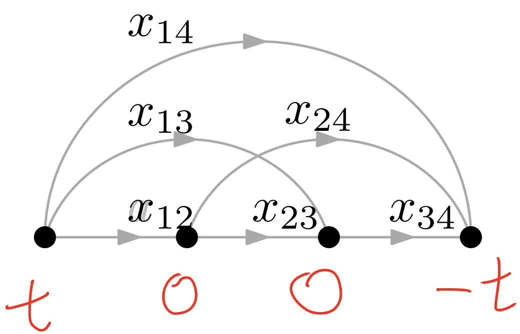
- *Resonance arrangement* (Gutekunst-Mészáros-Petersen 2019)
- *juggling sequences* (Benedetti-Hanusa-Harris-M-Simpson 2020)
- *rational Catalan combinatorics* (B-G-H-H-K-M-Y 2018, Yip 2019, Jang-Kim 2019)
- *Alternating sign matrices* (Mészáros-M-Striker 2019)

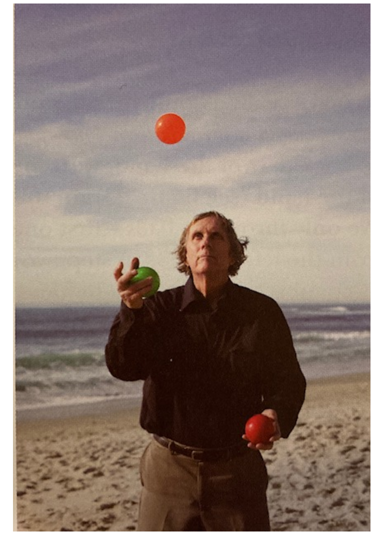
More balls and $\mathbb{R}$-balls revisited

In how many ways $\mathrm{js}(n,t)$ can we juggle t balls with time n?

$n = 3$, t

integer flows





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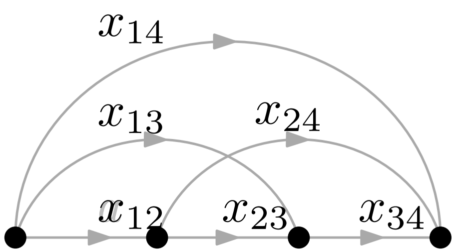
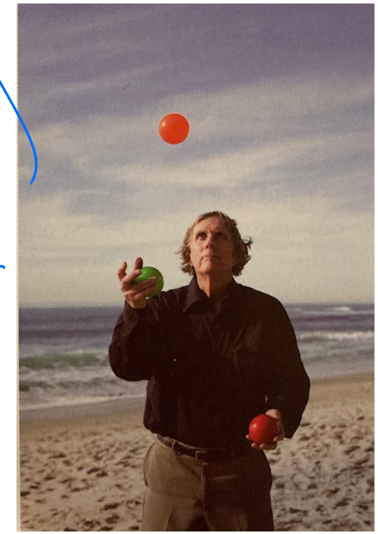


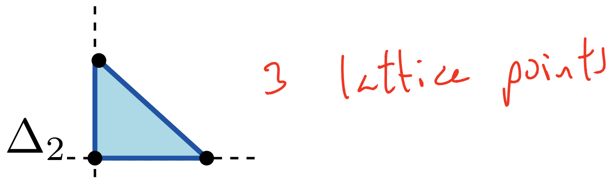


Answer: # of *lattice points* of $\mathcal{F}_G(t, 0, \ldots, 0, -t)$

Lattice points of polytopes

- $\#P \cap \mathbb{Z}^N$ number of lattice points (discrete volume)

Example:



standard simplex $\Delta_d = \{(x_1, \ldots, x_d) \mid \sum_{i=1}^{d} x_i \leq 1,\ x_i \geq 0\}$

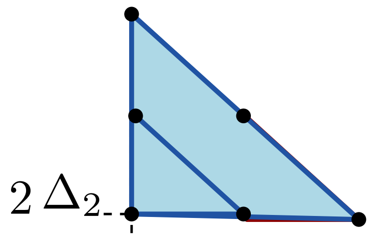
Lattice points of polytopes

- $\#P \cap \mathbb{Z}^N$ number of lattice points (discrete volume)

$L_P(t) := \#(tP \cap \mathbb{Z}^N)$ counts lattice points in t-**dilation** of P.

Ehrhart polynomial (Thm)

Example:



t

standard simplex $\Delta_d = \{(x_1, \ldots, x_d) \mid \sum_{i=1}^{d} x_i \leq 1,\ x_i \geq 0\}$

$t\Delta_d \cap \mathbb{Z}^d = \binom{t+d}{d}$ ← polynomial in t

$t\,\text{cube}_d \cap \mathbb{Z}^d = (t+1)^d$

Lattice points: Kostant's partition function

lattice points of $\mathcal{F}_G(\mathbf{a})$ are integral flows on G with netflow $\mathbf{a}$

let $K_G(\mathbf{a}) := \#(\mathcal{F}_G(\mathbf{a}) \cap \mathbb{Z}^m) = L_{\mathcal{F}_G(\mathbf{a})}(1)$

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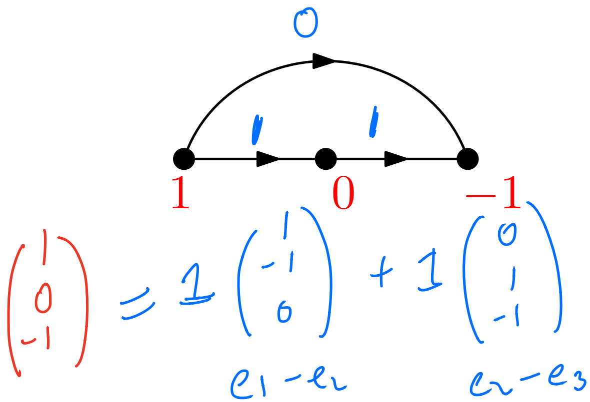

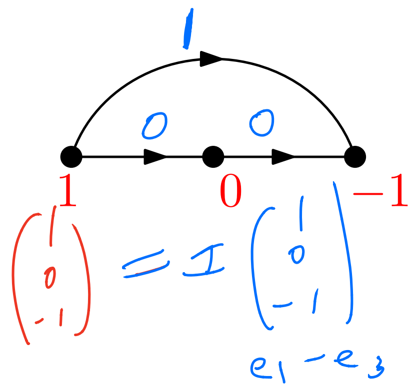

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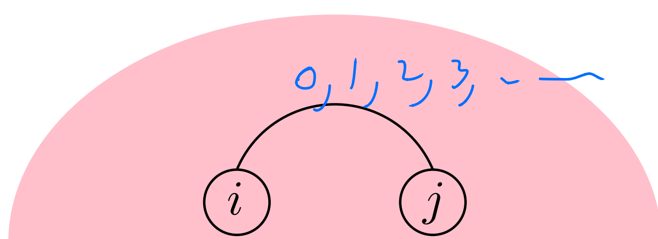
Generating function for $K_{k_{n+1}}(\mathbf{a})$:

$$\sum_{\mathbf{a}} K_G(\mathbf{a})\mathbf{x}^{\mathbf{a}} = \prod_{(i,j)\in E(G)} \frac{1}{(1 - x_i x_j^{-1})}.$$

geometric series



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- there are formulas for **weight multiplicities** and **tensor product multiplicities** of *type A semisimple Lie algebras* in terms of $K_{k_{n+1}}(\mathbf{a})$.

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- there are formulas for **weight multiplicities** and **tensor product multiplicities** of *type A semisimple Lie algebras* in terms of $K_{k_{n+1}}(\mathbf{a})$.
- Recall $\mathrm{js}(n,t) = K_{k_{n+1}}(t,0,\ldots,0,-t)$ is difficult

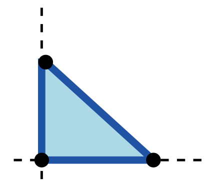
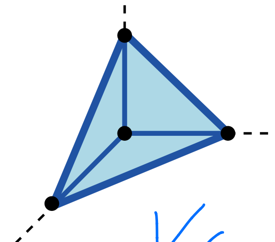
Volume of polytopes

normalized volume of $P := \dim(P)! \cdot$ (euclidean volume of P)

$\in \mathbb{N}$ ☺

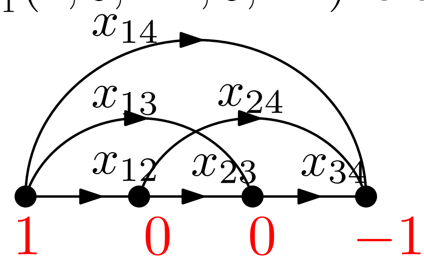
Example:

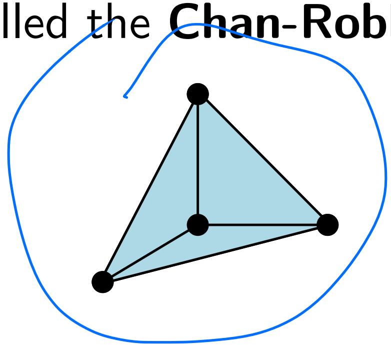
standard simplex $\Delta_n = \{(x_1, \ldots, x_n) \mid \sum x_i \leq 1,\ x_i \geq 0\}$

	Δ_1	Δ_2	Δ_3
			
euclidean volume	1	1/2	1/6
(normalized) volume	1	1	1

The CRY polytope and its volume

$\mathcal{F}_{k_{n+1}}(1,0,\ldots,0,-1)$ is called the **Chan-Robbins-Yuen** (CRY_n) polytope





$v_n :=$ volume(CRY_n)

n	2	3	4	5	6	7
v_n	1	1	2	10	140	5880
v_n / v_{n-1}		1	2	5	14	42

✓

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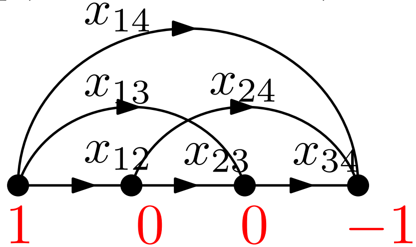

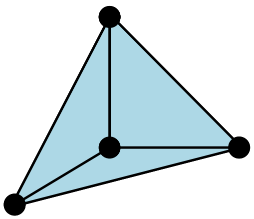

$v_n := \text{volume}(CRY_n)$

n	2	3	4	5	6	7
v_n	1	1	2	10	140	5880

- $v_n = C_{n-2}C_{n-3}\cdots C_1$ (Zeilberger 99)

$C_n := \frac{1}{n+1}\binom{2n}{n}$ are the **Catalan numbers**

Volume of the CRY_n polytope

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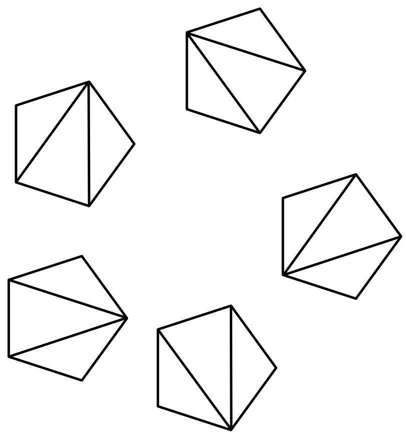
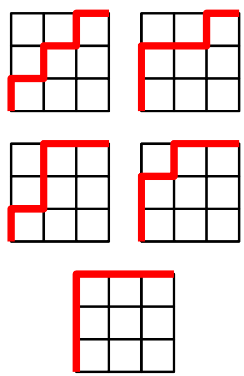
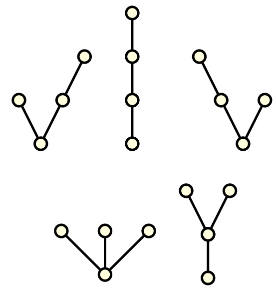
Catalan numbers $(1, 1, 2, 5, 14, 42, \ldots)$ count more than **200 different objects**







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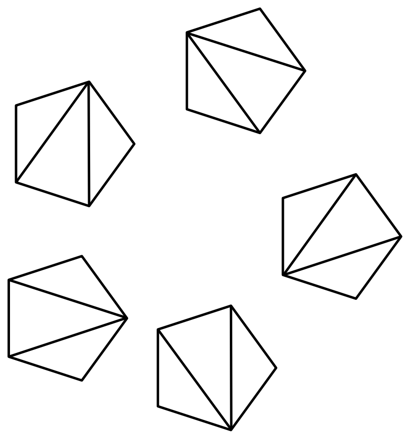

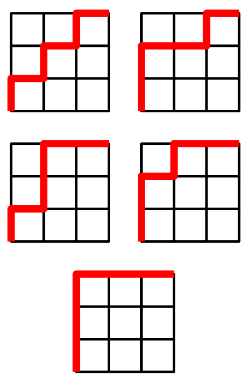

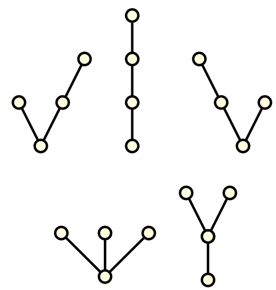

. . . however, there is no combinatorial proof of formula for v_n

Outline

- $\mathcal{F}_G(\mathbf{a})$ flow polytope of a graph G netflow $\mathbf{a}$
- volumes and lattice points important in representation theory
- volume $\mathcal{F}_{k_{n+1}}(1, 0, \ldots, 0, -1) = C_{n-2} C_{n-3} \cdots C_1$
- about the proof

Fundamental theorem volume of flow polytopes

let $K_G(\mathbf{a}) := \#(\mathcal{F}_G(\mathbf{a}) \cap \mathbb{Z}^m)$

Theorem (Stanley-Postnikov 99, Baldoni-Vergne 08)

$$\text{volume}\mathcal{F}_G(1, 0, \ldots, 0, -1) = K_G(0, i_2, i_3, \ldots, -\textstyle\sum_k i_k),$$

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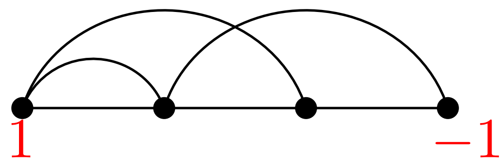
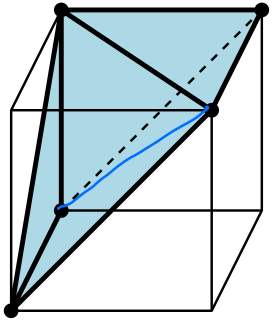
Example





volume = 2

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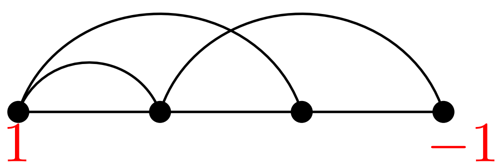

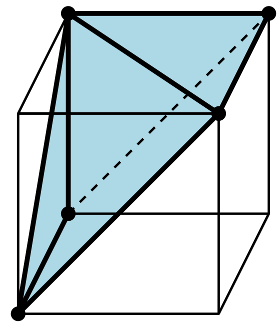

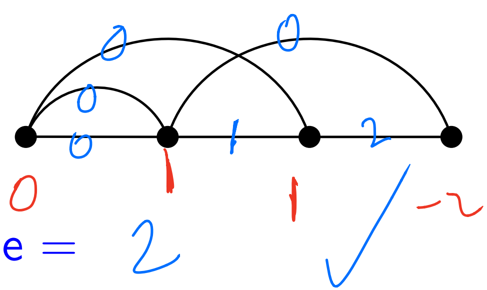

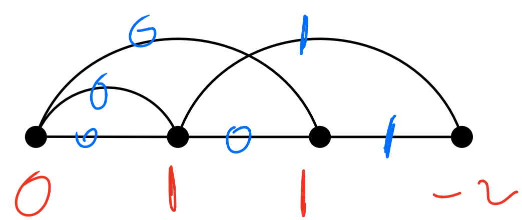

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Corollary

$$\text{volume}(CRY_n) = K_{k_{n+1}}(0,0,1,2,3,\ldots,n-2,-\tbinom{n-1}{2}))$$

Chan-Robbins-Yuen

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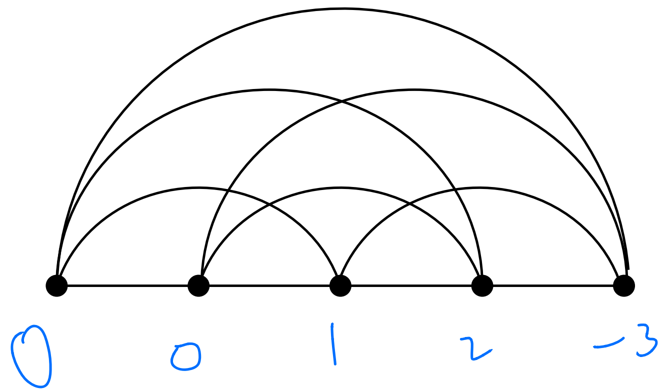

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$$M_n(a,b,c) := \text{constant term of } \prod_{i=1}^{n} x_i^{-a+1}(1-x_i)^{-b} \prod_{1\leq i<j\leq n} (x_i-x_j)^{-c}$$

$\Gamma(n+1) = n!$

$$= \prod_{j=0}^{n-1} \frac{\Gamma(1+\frac{c}{2})\Gamma(a+b-1+(n+j-1)\frac{c}{2})}{\Gamma(1+(j+1)\frac{c}{2})\Gamma(a+j\frac{c}{2})\Gamma(b+j\frac{c}{2})}.$$

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at $a=b=c=1$ gives $C_{n-2}C_{n-3}\cdots C_1$.

Zeilberger's entire paper

PROOF OF A CONJECTURE OF CHAN, ROBBINS, AND YUEN

Doron ZEILBERGER [1]

Abstract: Using the celebrated Morris Constant Term Identity, we deduce a recent conjecture of Chan, Robbins, and Yuen (math.CO/9810154), that asserts that the volume of a certain $n(n-1)/2$-dimensional polytope is given in terms of the product of the first $n-1$ Catalan numbers.

Chan, Robbins, and Yuen[CRY] conjectured that the cardinality of a certain set of triangular arrays $\mathcal{A}_n$ defined in pp. 6-7 of [CRY] equals the product of the first $n-1$ Catalan numbers. It is easy to see that their conjecture is equivalent to the following *constant term identity* (for any rational function $f(z)$ of a variable z, $CT_z f(z)$ is the coeff. of z^0 in the formal Laurent expansion of $f(z)$ (that always exists)):

$$CT_{x_n} \dots CT_{x_1} \prod_{i=1}^{n} (1-x_i)^{-2} \prod_{1 \le i < j \le n} (x_j - x_i)^{-1} = \prod_{i=1}^{n} \frac{1}{i+1} \binom{2i}{i} \quad . \qquad (CRY)$$

But this is just the special case $a=2$, $b=0$, $c=1/2$, of the *Morris Identity*[M] (where we made some trivial changes of discrete variables, and 'shadowed' it)

$$CT_{x_n} \dots CT_{x_1} \prod_{i=1}^{n} (1-x_i)^{-a} \prod_{i=1}^{n} x_i^{-b} \prod_{1 \le i < j \le n} (x_j - x_i)^{-2c} = \frac{1}{n!} \prod_{j=0}^{n-1} \frac{\Gamma(a+b+(n-1+j)c)\Gamma(c)}{\Gamma(a+jc)\Gamma(c+jc)\Gamma(b+jc+1)} \, . \qquad (Chip)$$

To show that the right side of $(Chip)$ reduces to the right side of (CRY) upon the specialization $a=2, b=0, c=1/2$, do the plugging in the former and call it M_n. Then manipulate the products to simplify M_n/M_{n-1}, and *then* use *Legendre's duplication formula* $\Gamma(z)\Gamma(z+1/2) = \Gamma(2z)\Gamma(1/2)/2^{2z-1}$ three times, and *voilà*, up pops the Catalan number $\binom{2n}{n}/(n+1)$. $\square$

Remarks: 1. By converting the left side of $(Chip)$ into a contour integral, we get the same integrand as in the Selberg integral (with $a \to -a$, $b \to -b-1$, $c \to -c$). Aomoto's proof of the Selberg integral (SIAM J. Math. Anal. **18**(1987), 545-549) goes verbatim. **2.** Conjecture 2 in [CRY] follows in the same way, from (the obvious contour-integral analog of) Aomoto's extension of Selberg's integral. Introduce a new variable t, stick $CT_t t^{-k}$ in front of (CRY), and replace $(1-x_i)^{-2}$ by $(1-x_i)^{-1}(t + x_i/(1-x_i))$. **3.** Conjecture 3 follows in the same way from another specialization of $(Chip)$.

References

[CRY] Clara S. Chan, David P. Robbins, and David S. Yuen, *On the volume of a certain polytope*, `math.CO/9810154`.

[M] Walter Morris, *"Constant term identities for finite and affine root systems, conjectures and theorems"*, Ph.D. thesis, University of Wisconsin, Madison, Wisconsin, 1982.

[1] Department of Mathematics, Temple University, Philadelphia, PA 19122, USA. `zeilberg@math.temple.edu` `http://www.math.temple.edu/~zeilberg/` . Nov. 17, 1998. Supported in part by the NSF.

Refining the product of Catalans

Theorem (Zeilberger 99)

volume$(CRY_{n+1}) = \#$ $= C_{n-1}C_{n-2}\cdots C_1$

$0\,0\,1\,2 \;\cdots\; n-1-\binom{n}{2}$

Refining the product of Catalans

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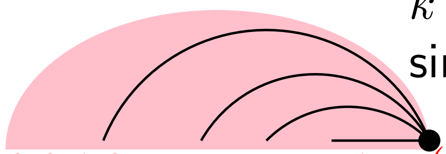
Theorem (Zeilberger 99)

$$\#$$



$$0\,0\,1\,2 \quad \cdots \quad n-1-\binom{n-1}{2}$$

k zero flow sink edges

$$= N(n-1,k)C_{n-2}\cdots C_1$$

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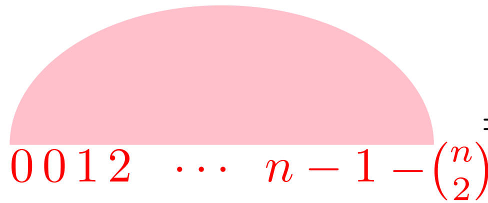

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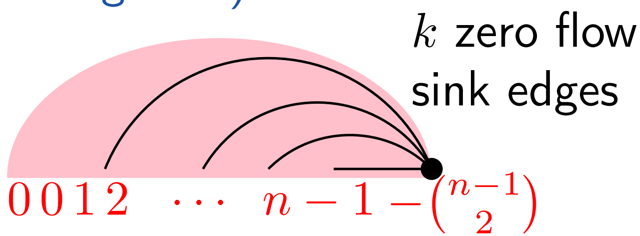
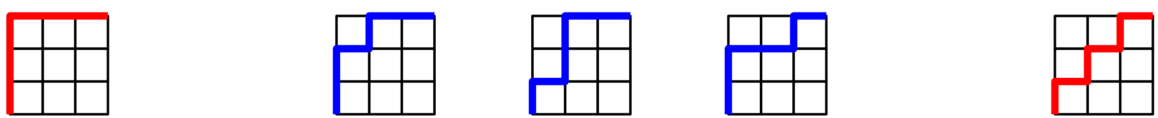


$$= N(n-1,k)C_{n-2}\cdots C_1$$

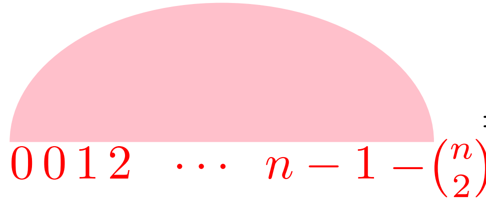
The **Narayana numbers** $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ refine the Catalan numbers, i.e. count Dyck paths by # peaks.



Refining the product of Catalans

Theorem (Zeilberger 99)

$$\mathrm{volume}(CRY_{n+1}) = \#$$



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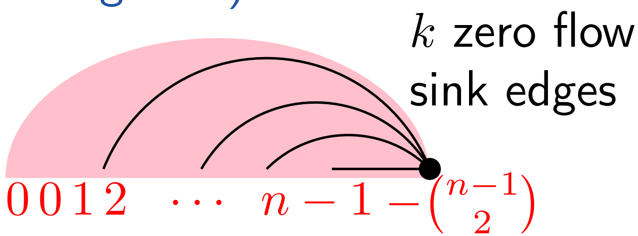
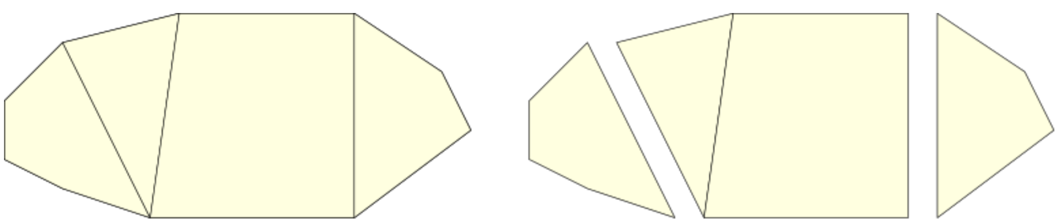


$$= N(n-1,k)C_{n-2}\cdots C_1$$

- Mészáros (2011) gave a collection of interior disjoint polytopes with volumes that sum to $N(n-1,k)C_{n-2}\cdots C_2C_1$



Outline

- $\mathcal{F}_G(\mathbf{a})$ flow polytope of a graph G netflow $\mathbf{a}$
- $K_G(\mathbf{a}) := \#(\mathcal{F}_G(\mathbf{a}) \cap \mathbb{Z}^m)$
- volume $\mathcal{F}_{k_{n+1}}(1, 0, \ldots, 0, -1) = C_{n-2} C_{n-1} \cdots C_1$
- about the proof:
  $\text{volume}\mathcal{F}_G(1, 0, \ldots, 0, -1) = K_G(0, i_2, i_3, \ldots, -\sum_k i_k)$.

Flow polytopes with volume $M_n(a,b,c)$

$$M_n(a,b,c) := \text{ constant term of } \prod_{i=1}^{n} x_i^{-a+1}(1-x_i)^{-b} \prod_{1\le i<j\le n} (x_i-x_j)^{-c}$$

$$= \prod_{j=0}^{n-1} \frac{\Gamma(1+\frac{c}{2})\Gamma(a+b-1+(n+j-1)\frac{c}{2})}{\Gamma(1+(j+1)\frac{c}{2})\Gamma(a+j\frac{c}{2})\Gamma(b+j\frac{c}{2})}.$$

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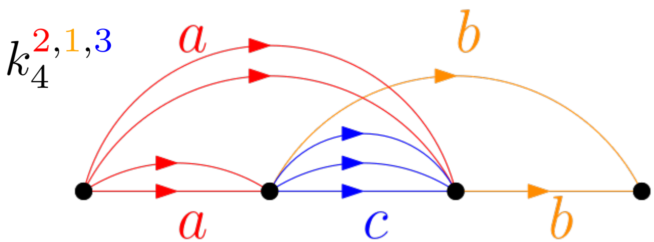
Let $k_{n+2}^{a,b,c}$ be graph with vertices $\{0,\ldots,n+1\}$ and for $i \in [n]$, edge $(0,i)$ a times, $(i,n+1)$ b times, and (i,j) c times.



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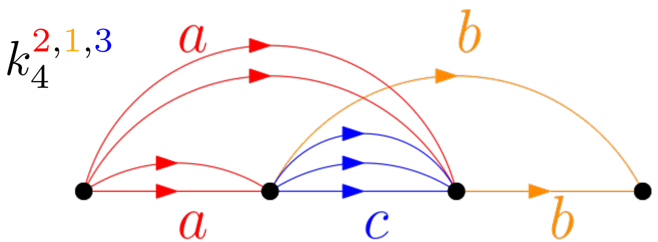

Theorem (Corteel-Kim-Mészáros 2017)

$$\text{volume}\mathcal{F}_{k_{n+2}^{a,b,c}}(1,0,\ldots,0,-1) = K_{k_{n+2}^{a,b,c}}(0,s_1,\ldots,s_n,-\sum s_i)$$
$$= M_n(a,b,c).$$

where $s_i = a-1+c(i-1)$.

Outline

- $\mathcal{F}_G(\mathbf{a})$ flow polytope of a graph G netflow $\mathbf{a}$
- $\mathcal{F}_G = \mathcal{F}_G(1, 0, \ldots, 0, -1)$

$$\text{volume}\mathcal{F}_{k_{n+2}} = M_n(1,1,1) = C_{n-1}C_{n-2}\cdots C_1$$

$$\text{volume}\mathcal{F}_{k^{a,b,c}_{n+2}} = M_n(a, b, c)$$

refinement $N(n-1,k)C_{n-2}\cdots C_1$

- lattice point interpretation
- geometric interpretation

Baldoni-Vergne's refinement

Let $\Phi_n(k, a, b, c) := \mathrm{CT}_x \; e_k \prod_{i=1}^{n} (1 - x_i)^{-b} x_i^{-a+1} \prod_{1 \le i < j \le n} (x_j - x_i)^{-c}.$

Baldoni-Vergne's refinement

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Question (Corteel 2016)

Can you use geometric interpretation of $M_n(a, b, c)$ to combinatorialize these recurrences.

Outline

- $\mathcal{F}_G(\mathbf{a})$ flow polytope of a graph G netflow $\mathbf{a}$
- $\mathcal{F}_G = \mathcal{F}_G(1, 0, \ldots, 0, -1)$

$$\text{volume}\mathcal{F}_{k_{n+2}} = M_n(1,1,1) = C_{n-1}C_{n-2}\cdots C_1$$

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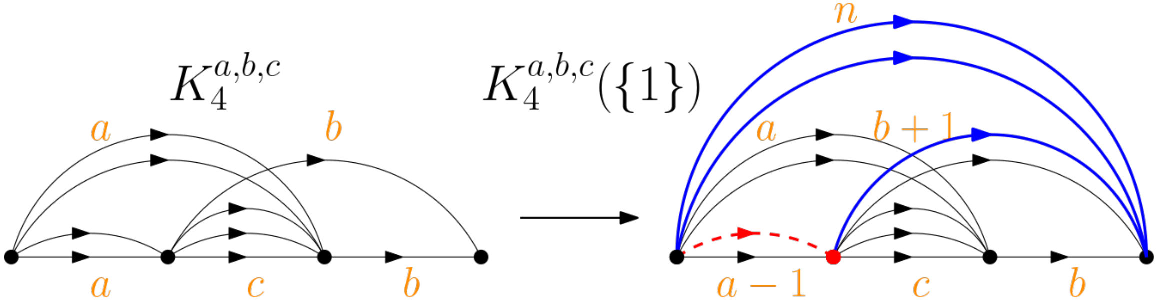
For $S \subseteq [n]$, the graph $k_{n+2}^{a,b,c}(S)$ takes $k_{n+2}^{a,b,c}$, adds n edges $(0, n+1)$, and for each $i \in S$, deletes an edge $(0, i)$ and adds an edge $(i, n+1)$.



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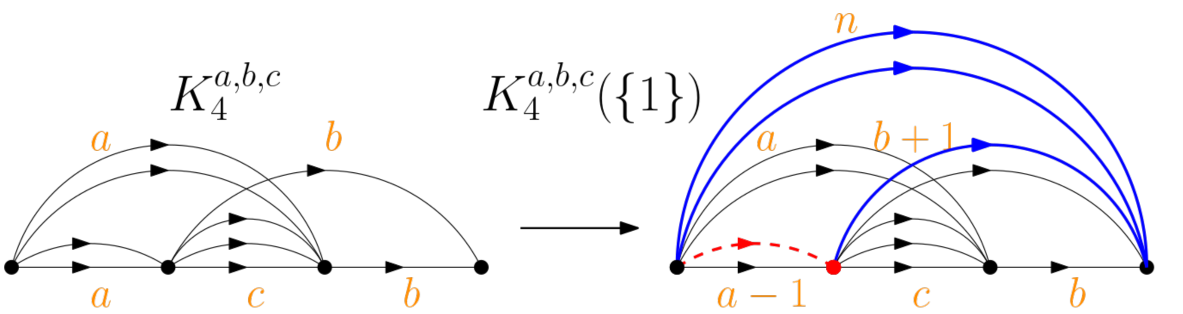

Theorem (Morales-Shi 21)
For $n, a, b \in \mathbb{Z}^+$ and $c, k \in \mathbb{Z}_{\geq 0}$ with $k \leq n$,

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- at $a = b = c = 1$ gives Mészáros' geometric interpretation.

Example geometric interpretation of $\Psi_n(k,a,b,c)$

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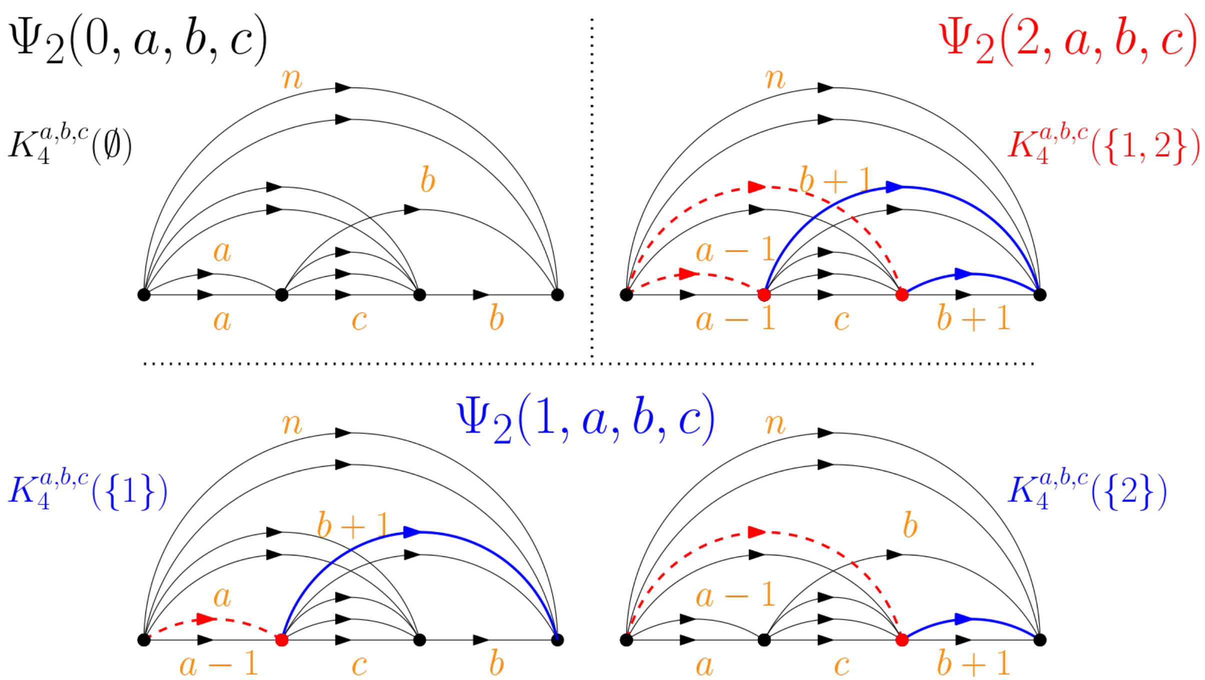
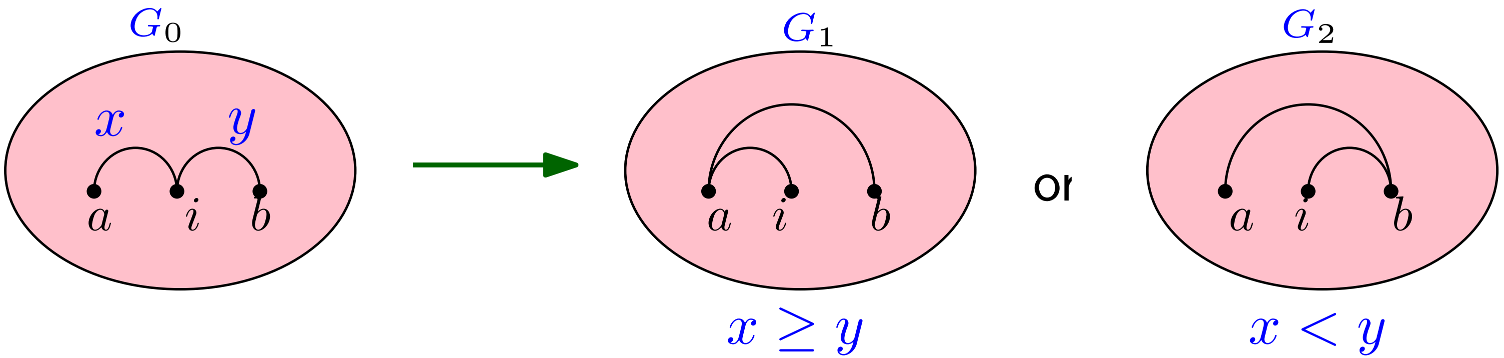


About proof of geometric interpretation

- The **subdivision lemma** (Postnikov-Stanley) gives a map that reduces a flow polytope to two interior disjoint polytopes.



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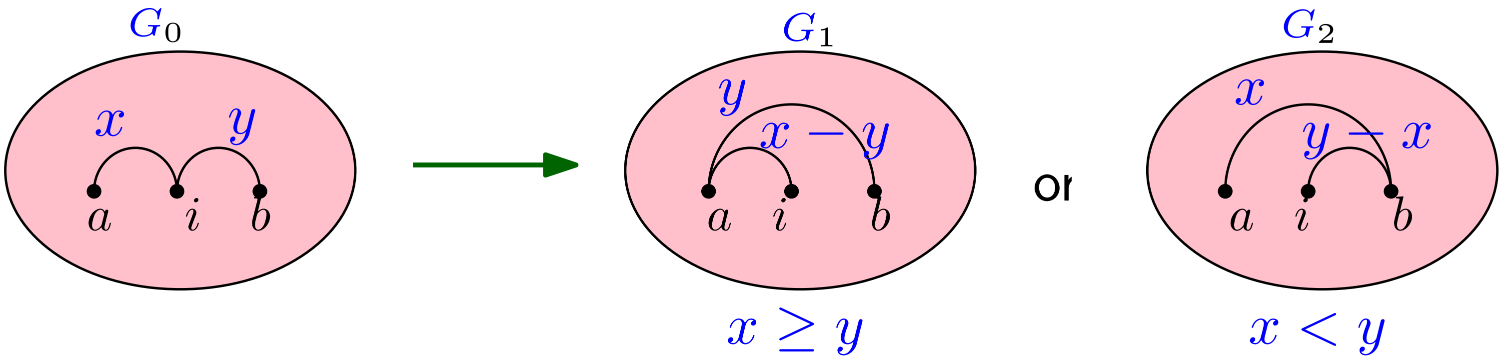

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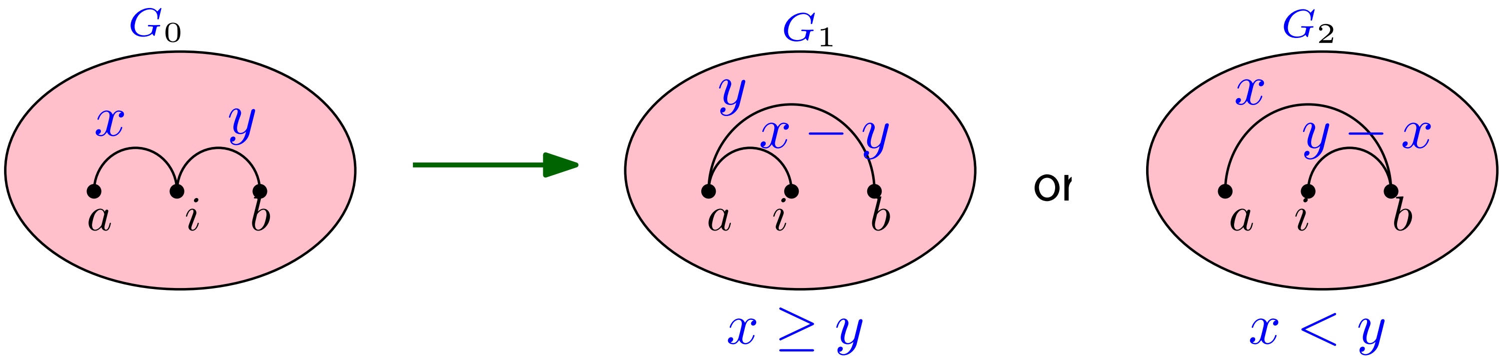
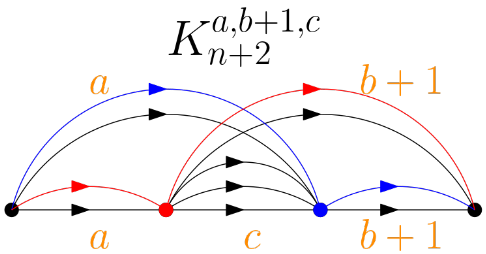
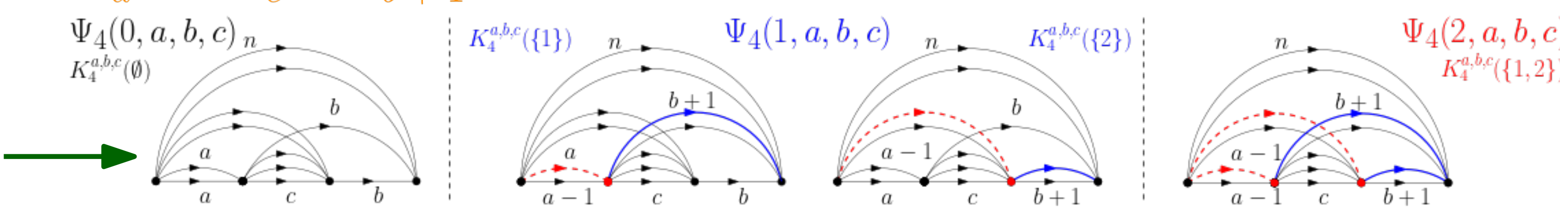


- We apply this to $\mathcal{F}_{k_{n+2}^{a,b+1,c}}$ once to each internal vertex.



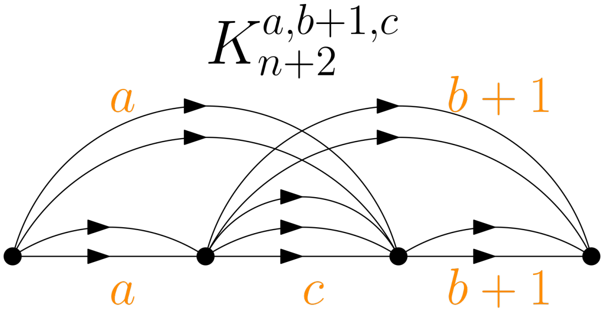


Refining the Morris identity

Corollary (Morales-Shi 21)

For $n, a, b \in \mathbb{Z}^+$ and $c, k \in \mathbb{Z}_{\geq 0}$ with $k \leq n$,

$$M_n(a, b+1, c) = \sum_{k=0}^{n} \Psi_n(k, a, b, c).$$



Outline

$$\text{volume}\mathcal{F}_{k_{n+2}} = K_{k_{n+2}}(0,0,1,2,\ldots)$$
$$= M_n(1,1,1)$$
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Combinatorial proof?

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Fundamental theorem + symmetry

Theorem

$$K_G(0, i_2, i_3, \ldots, -\sum_k i_k) = K_{G^r}(0, i'_2, i'_3, \ldots, -\sum_k i'_k)$$

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Proof

- $\text{volume}\mathcal{F}_G(1, 0, \ldots, 0, -1) = K_G(0, i_2, i_3, \ldots, -\sum_k i_k)$,

Both sides are the volume of $\mathcal{F}_G(10 \cdots 0 - 1) \equiv \mathcal{F}_{G^r}(10 \cdots 0 - 1)$. □

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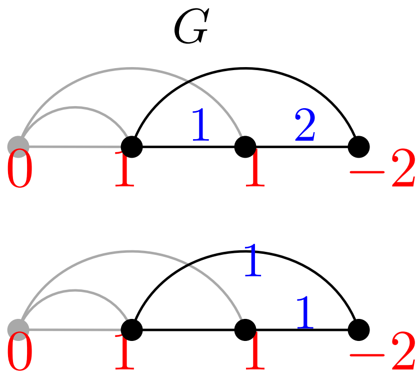

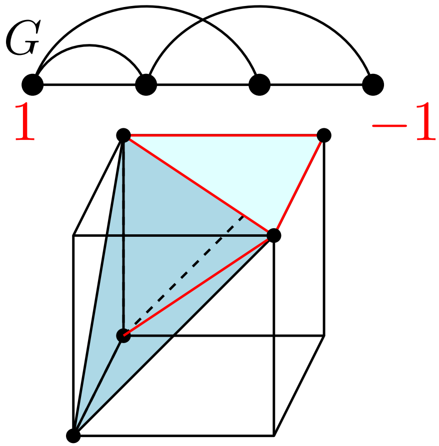

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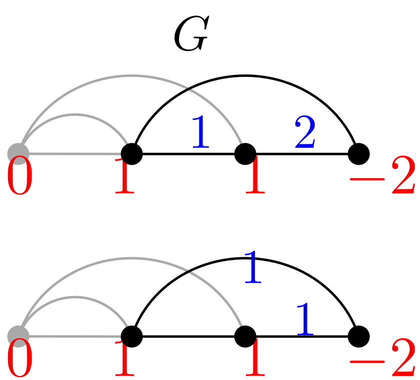

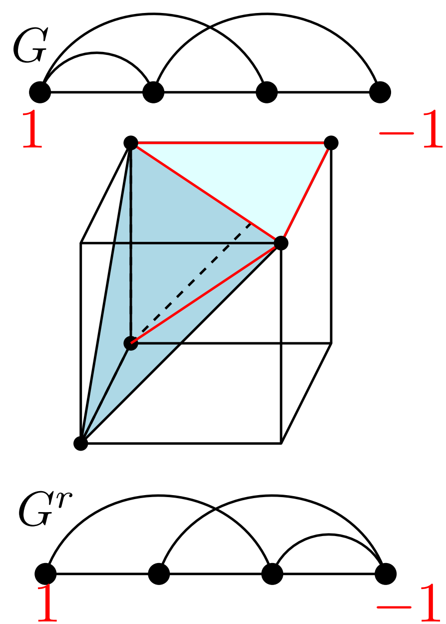

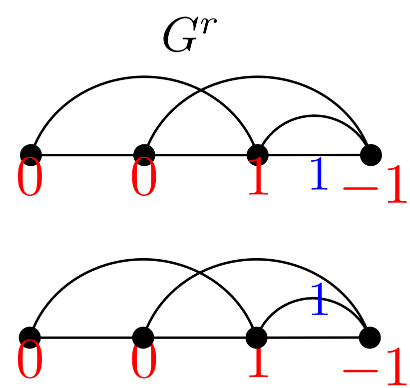

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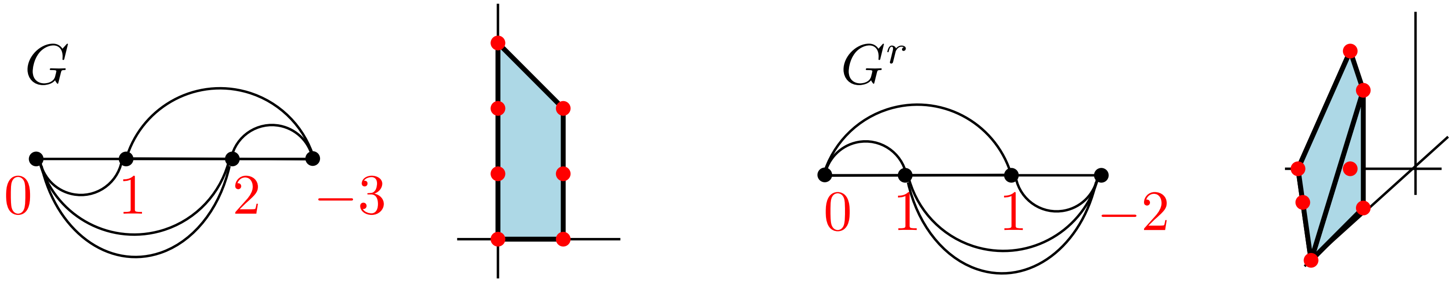

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Bijection between lattice points K_G and K_{G^r}

- for a route P with vertex v, Pv and vP are subpaths ending and starting at v.

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$$\Phi : \mathcal{C} \mapsto fl,$$
$$fl((u,v)) = \# \text{ times } (u,v) \text{ appears in } \{Pv \mid P \in C\} - 1$$

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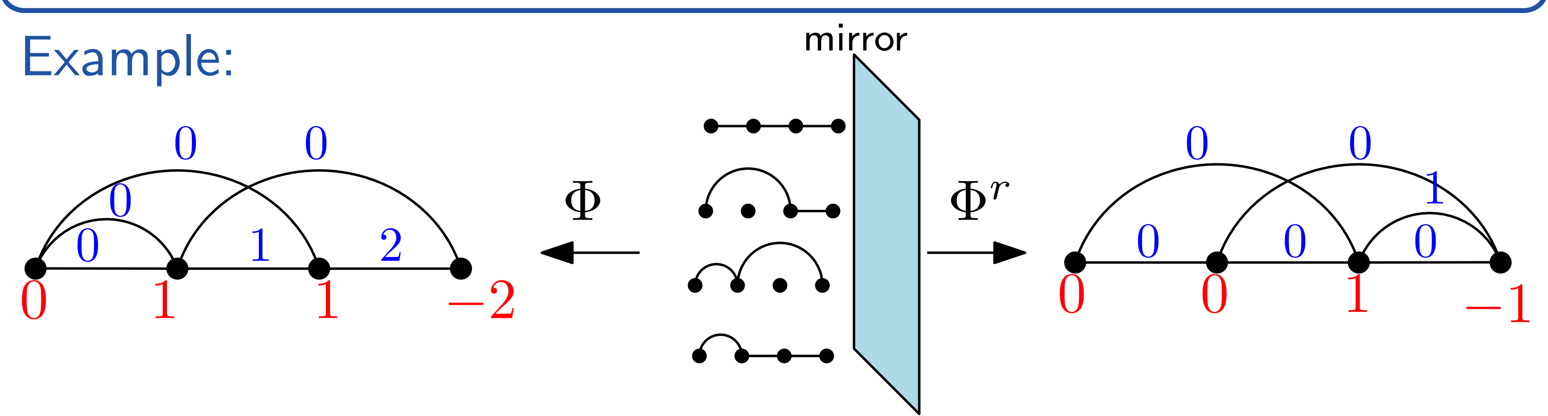
Example:



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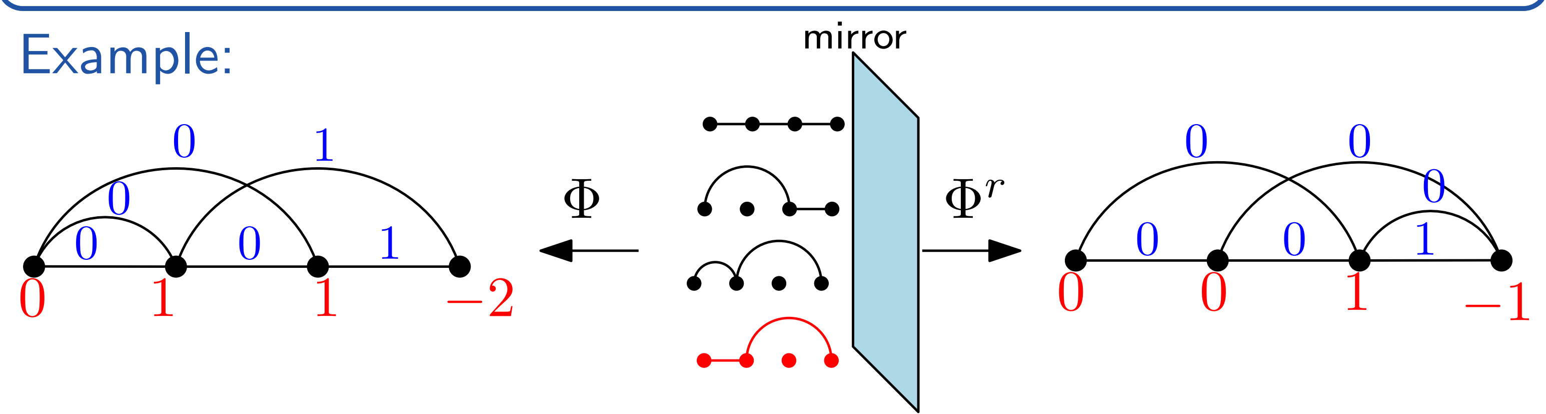

Outline

$$\mathrm{volume}\mathcal{F}_{k_{n+2}} = K_{k_{n+2}}(0,0,1,2,\ldots)$$
$$= M_n(1,1,1)$$
$$= C_{n-1}C_{n-2}\cdots C_1$$

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$$\mathrm{volume}\mathcal{F}_{k_{n+2}^{a,b,c}} = K_{k_{n+2}^{a,b,c}}(\ldots)$$
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(Corteel-Kim-Mészáros 17)

refinement $N(n-1,k)C_{n-2}\cdots C_1$

- lattice point interpretation (Zeilberger 99)
- geometric interpretation (Mészáros 11)

new refinement $\Psi_n(k,a,b,c)$ of $M_n(a,b,c)$

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Bijective proof

$M_n(a,bc) = M_n(b,a,c)$

- (Morales, Shi) Refinements and Symmetries of the Morris identity for volumes of flow polytopes arxiv:2102.05825

- (Benedetti, Hanusa, Harris, Morales, Simpson) Kostant's partition function and magic multiplex juggling sequences, arxiv:2001.03219

Thank you Gracias