The method of brackets

A heuristic method for integration

Experimental Mathematics Seminar, February 2022.

Victor H. Moll
Tulane University
New Orleans
vhm@tulane.edu

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Collaborators

Ivan Gonzalez, chilean physicist

Igor Kondrashuk, another chilean physicist

Karen Kohl, former graduate student

Lin Jiu, another former graduate student

Zak Bradshaw, current graduate student

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A calculation

Consider the problem: evaluate

$$I(a) = \int_0^\infty e^{-ax}\,dx$$

Expand the integrand

$$I(a) = \int_0^\infty \sum_{k=0}^\infty \frac{(-1)^k}{k!} a^k x^k\,dx$$

exchange integral and sum

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A calculation. Continuation

$$I(a) = \int_0^\infty e^{-ax}\,dx = \sum_{k=0}^\infty \frac{(-1)^k}{k!} a^k \int_0^\infty x^k\,dx$$

introduce the notation

$$\phi_k = \frac{(-1)^k}{k!}$$

and

$$\langle b \rangle = \int_0^\infty x^{b-1}\,dx$$

so the integral now becomes

$$I(a) = \sum_{k=0}^\infty \phi_k a^k \langle k+1 \rangle$$

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$$I(a) = \sum_{k=0}^{\infty} \phi_k a^k \langle k+1 \rangle$$

solve the linear equation $k+1=0$ to get $k^* = -1$
give the sum the value

$$a^{k^*}\Gamma(-k^*)$$

dropping the ϕ_k and the symbol $\langle k+1 \rangle$
this gives

$$I(a) = \int_0^{\infty} e^{-ax}\,dx = a^{-1}\Gamma(1) = \frac{1}{a}$$

This is it. We are done.

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The general problem

Given a function

$$f : [a, b] \to \mathbb{R}$$

determine

$$I(f; a, b) := \int_a^b f(x)\, dx$$

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Wallis' formula

$$\int_0^\infty \frac{dx}{(x^2+1)^{m+1}} = \frac{\pi}{2^{2m+1}}\binom{2m}{m}$$

Clearly not stated in this form

A quartic analog

Theorem

For $m \in \mathbb{N}$ and $a > -1$

$$\int_0^\infty \frac{dx}{(x^4 + 2ax^2 + 1)^{m+1}} = \frac{\pi}{2} \frac{P_m(a)}{[2(a+1)]^{m+1/2}}$$

$$P_m(a) = \sum_{l=0}^{m} d_{l,m} a^l$$

$$d_{l,m} = 2^{-2m} \sum_{k=l}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} \binom{k}{l}$$

Coefficients $d_{l,m}$ have many interesting properties

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A related series

$$N_{0,4}(a;m) = \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}}$$

$$\sqrt{a+\sqrt{1+c}} = \sqrt{a+1} + \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} N_{0,4}(a,k-1)c^k$$

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Double square roots

In view of the relation of the quartic integral with double square roots

I looked for integrals having this function in the integrand

The famous table by Gradshteyn and Ryzhik has

$$\int_0^\infty \frac{dx}{(1+x^2)^{3/2}\left[\varphi(x)+\sqrt{\varphi(x)}\right]^{1/2}} = \frac{\pi}{2\sqrt{6}}$$

as entry 3.248.5 $\varphi(x)$ is a simple rational function

Beautiful

Unfortunately it is incorrect

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The correct question is

$$I_{true} := \int_0^{\infty} \frac{dx}{(1+x^2)^{3/2}\sqrt{\varphi(x)+\sqrt{\varphi(x)^{3}}}} = \frac{\pi}{2\sqrt{6}}$$

and the answer to the original question is

$$\int_0^{\infty} \frac{dx}{(1+x^2)^{3/2}\sqrt{\varphi(x)+\sqrt{\varphi(x)^{1}}}} =$$

$$\frac{\sqrt{3}-1}{\sqrt{2}}\Pi(\pi/2, n, 3^{-1/2}) - \frac{1}{\sqrt{2}}F(\alpha, 3^{-1/2}),$$

with $n = 2-\sqrt{3}$, $\alpha = \arcsin\sqrt{n}$, where $F(\varphi, k)$ and $\Pi(\varphi, n, k)$ are classical elliptic integrals. (Formulas not shown)

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Using integrals to define functions

$$\log x = \int_1^x t^{-1}\, dt$$

You have to make sure you have created a new function

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt$$

Then you can combine them. (Next page).

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Powers of loggamma $L_n = \int_0^1 \log^n \Gamma(q)\, dq$

Theorem
(L. Euler)

$$L_1 := \int_0^1 \log \Gamma(q)\, dq = \ln \sqrt{2\pi}$$

Theorem
(O. Espinosa, V.M., 2002)

$$\begin{aligned} L_2 &:= \int_0^1 \log^2 \Gamma(q)\, dq \\ &= \frac{\gamma^2}{12} + \frac{\pi^2}{48} + \frac{1}{3}\gamma L_1 + \frac{4}{3} L_1^2 - (\gamma + 2L_1)\frac{\zeta'(2)}{\pi^2} + \frac{\zeta''(2)}{2\pi^2} \end{aligned}$$

This came as a corollary of our work on Hurwitz zeta function.

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Problem

We asked in 2002: what is

$$L_3 = \int_0^1 \log^3 \Gamma(q)\, dq \ ?$$

Borwein-Bailey-Crandall finally got it in 2013

Theorem

$$\begin{aligned} L_3 &= \frac{3}{4}\left(\frac{\zeta(3)}{\pi^2}+\frac{L_1}{3}\right)A^2 \\ &- \frac{3}{2}\left(\frac{\zeta'(2,1)}{\pi^2}+2L_1\frac{\zeta'(2)}{\pi^2}\right)A+\frac{3}{2}L_1\frac{\zeta''(2)}{\pi^2} \\ &+ L_1^3+\frac{\pi^2}{16}L_1+\frac{3}{16}\zeta(3) \\ &- \frac{3}{8\pi^2}\left(\omega_{1,1,0}(1,1,1)-2\omega_{1,0,1}(1,1,1)\right) \end{aligned}$$

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$$\omega(r,s,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{n^r m^s (n+m)^t}$$

Tornhein-Witten-zeta functions

$$\omega(r,s,t) = \frac{1}{\Gamma(t)}\int_0^1 \mathrm{Li}_r(\sigma)\mathrm{Li}_s(\sigma)(-\log\sigma)^{t-1}\,dt$$

$$\mathrm{Li}_s(x) = \sum_{n=1}^{\infty}\frac{x^n}{n^s} \text{ for } |x|<1 \text{ and } \mathrm{Re}\, s > 1$$

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Problems

Prove that

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A second problem

Invent a new function

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}\,dx}{e^x - 1}$$

Riemann did it.

Evaluate the integral

$$\int_0^\infty \frac{(1-12t^2)}{(1+4t^2)^3} \int_{1/2}^\infty \log|\zeta(\sigma + it)|\,d\sigma\,dt = \frac{\pi(3-\gamma)}{32}$$

This might be hard

Is equivalent to the Riemann hypothesis.

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The method of brackets

$$\langle a \rangle := \int_0^\infty x^{a-1}\, dx \quad \text{for } a \in \mathbb{R}$$

this is the *bracket* corresponding to $a \in \mathbb{R}$.
Given the expansion

$$f(x) = \sum_{n=0}^{\infty} c_n x^{\alpha n + \beta - 1} \quad c_n,\ \alpha,\ \beta \in \mathbb{C}$$

integrating term by term gives the value of the integral

$$\int_0^\infty f(x)\, dx = \sum_n c_n \langle \alpha n + \beta \rangle$$

as a bracket series.

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The method of brackets. What is left to do?

The integrals are divergent. Regularize.

We need to decide rules of evaluation for *bracket series*

We need to be able to produce bracket series in an efficient manner.

What about rigor?
Some people still care.

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We need to be able to produce bracket series in an efficient manner.

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Some people still care.

The method of brackets. What is left to do?

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Rules to generate bracket series

Rule 1:

$$(a_1 + a_2 + \cdots + a_r)^{\alpha} \to$$

$$\sum_{n_1, n_2, \cdots, n_r} \phi_{1,2,\cdots,r} a_1^{n_1} \cdots a_r^{n_r} \frac{\langle -\alpha + n_1 + n_2 + \cdots + n_r \rangle}{\Gamma(-\alpha)}$$

$$\phi_n = \frac{(-1)^n}{\Gamma(n+1)}$$

$$\phi_{1,2} = \phi_{n_1} \phi_{n_2}$$

Rules for brackets. Continuation

Rule 2. Evaluation of a bracket series:

$$\sum_n \phi_n f(n)\langle \alpha n + \beta \rangle \to \frac{1}{|\alpha|} f(n^*)\Gamma(-n^*)$$

where n^* solves the equation $\alpha n + \beta = 0$.

This is Ramanujan master theorem.

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Rules for brackets. Continuation

Rule 3:

$$\sum_{n_1,n_2} \phi_{n_1,n_2} f(n_1,n_2) \langle a_{11}n_1 + a_{12}n_2 + c_1 \rangle \langle a_{21}n_1 + a_{22}n_2 + c_2 \rangle$$

$$\rightarrow \frac{1}{|a_{11}a_{22} - a_{12}a_{21}|} f(n_1^*, n_2^*) \Gamma(-n_1^*) \Gamma(-n_2^*)$$

where (n_1^*, n_2^*) solves the linear system obtained by the vanishing of brackets

We do not assign a value if the determinant vanishes.

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Rules for brackets. Continuation

Rule 4: If the system is not square, consider separately all square subsystems.

Rule 5: Divergent answers should be discarded.

May be.

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Wallis' formula

$$J_m := \int_0^\infty \frac{dx}{(x^2+1)^{m+1}} = \int_0^\infty (1+x^2)^{-m-1}\, dx$$

$$(1+x^2)^{-m-1} \mapsto \sum_{n_1,n_2} \phi_{1,2} \frac{\langle m+1+n_1+n_2\rangle}{\Gamma(m+1)} 1^{n_1} x^{2n_2}$$

$$J_m \mapsto \sum_{n_1,n_2} \phi_{1,2} \frac{1}{\Gamma(m+1)} \langle m+1+n_1+n_2\rangle \langle 2n_2+1\rangle$$

System: $m+1+n_1+n_2 = 0, \quad 2n_2+1=0$

Solution: $n_1^* = -(m+1/2), \quad n_2^* = -1/2.$

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Wallis' formula. Continuation

Rule 3 implies

$$J_m = \frac{1}{2}\frac{\Gamma(-n_1^*)\,\Gamma(-n_2^*)}{\Gamma(m+1)} = \frac{\Gamma(m+1/2)\,\Gamma(1/2)}{2\Gamma(m)}$$

The values

$$\Gamma(m) = m! \text{ and } \Gamma(m+1/2) = \frac{\sqrt{\pi}}{2^{2m}}\frac{(2m)!}{m!}$$

produce

$$\int_0^\infty \frac{dx}{(x^2+1)^{m+1}} = \frac{\pi}{2}\frac{\binom{2m}{m}}{2^{2m}}$$

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A Bessel integral

$$I := \int_0^\infty J_0(ax)\sin(bx)dx$$

$$J_0(ax) = \sum_{m=0}^{\infty} \phi_m \frac{a^{2m}}{\Gamma(m+1)2^{2m}} x^{2m}$$

$$\sin(bx) = \sum_{n=0}^{\infty} \phi_n \frac{\Gamma(n+1)}{\Gamma(2n+2)} b^{2n+1} x^{2n+1}$$

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A multi-dimensional example: 4.638.3 in GR

$$I_n(s) = \int_0^\infty \int_0^\infty \cdots \int_0^\infty \frac{x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1} \quad dx_1 dx_2 \cdots dx_n}{\left(1 + (r_1 x_1)^{q_1} + \cdots + (r_n x_n)^{q_n}\right)^s},$$

$$\text{denominator} \mapsto \sum_{k_0, k_1, \cdots, k_n} \phi_{0,\cdots,n} \prod_{j=1}^{n} (r_j x_j)^{q_j k_j} \frac{\langle s + k_0 + \cdots + k_n \rangle}{\Gamma(s)}.$$

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The solutions are $k_0 = -s + \sum_{j=1}^{n} \frac{p_j}{q_j}$ and $k_j = -\frac{p_j}{q_j}$ for $1 \leq j \leq n$.

$$I_n = \frac{1}{\Gamma(s)} \Gamma\left(s - \sum_{j=1}^{n} \frac{p_j}{q_j}\right) \prod_{j=1}^{n} \frac{\Gamma\left(\frac{p_j}{q_j}\right)}{q_j r_j^{p_j}}.$$

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Ising integrals

$$C_{n,k} := \frac{4}{n!} \int_0^\infty \int_0^\infty \cdots \int_0^\infty \left(\sum_{j=1}^{n} (u_j + 1/u_j)^2 \right)^{-k-1} \frac{du_1}{u_1} \frac{du_2}{u_2} \cdots \frac{du_n}{u_n}$$

reduces to (Jon Borwein-David Bradley)

$$C_{n,k} = \frac{2^{n-k+1}}{n!\,k!} \int_0^\infty t^k K_0^n(t)\,dt$$

where

$$K_0(x) = \lim_{\nu \to 0} \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin \pi\nu}$$

is the modified Bessel function.

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Ising integrals. Continuation.

The only known values

$$C_n := C_{n,1}$$

$$C_1 = 2, \qquad C_2 = 1,$$

$$C_3 = L_{-3}(2) = \sum_{n=0}^{\infty} \left(\frac{1}{(3n+1)^2} - \frac{1}{(3n+2)^2} \right),$$

$$C_4 = \frac{7}{12}\zeta(3)$$

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Difficulties with the method

$$I_2 = \frac{1}{2} C_{2,1} := \int_0^\infty \int_0^\infty \frac{dx\, dy}{xy(x + y + 1/x + 1/y)^2}$$

$$\int_0^\infty \int_0^\infty \frac{dx\, dy}{xy} \sum_{n_1,n_2,n_3,n_4} \phi_{1,2,3,4} x^{n_1-n_3} y^{n_2-n_4} \frac{\langle 2+n_1+n_2+n_3+n_4 \rangle}{\Gamma(2)}$$

$$\sum_{n_1,n_2,n_3,n_4} \phi_{1,2,3,4} \langle n_1 - n_3 \rangle \langle n_2 - n_4 \rangle \langle 2 + n_1 + n_2 + n_3 + n_4 \rangle$$

4 indices and 3 brackets

One free variable

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$$\int_0^\infty \int_0^\infty \frac{dx\, dy}{xy} \sum_{n_1,n_2,n_3,n_4} \phi_{1,2,3,4} x^{n_1 - n_3} y^{n_2 - n_4} \frac{\langle 2 + n_1 + n_2 + n_3 + n_4 \rangle}{\Gamma(2)}$$

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4 indices and 3 brackets

One free variable

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Difficulties. Continuation

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n_1 free: $n_2^* = -n_1 - 1$, $n_3^* = n_1$, $n_4^* = -n_1 - 1$

$$\begin{aligned} I_2 &= \sum_{n_1} \frac{1}{\det(A)} \phi_1 \Gamma(-n_2^*)\Gamma(-n_3^*)\Gamma(-n_4^*) \\ &= \frac{1}{2}\sum_{n_1}(-1)^{n_1}\Gamma(n_1+1)\Gamma(-n_1). \end{aligned}$$

Divergent the method gives no information

The same occurs with the other three choices of free variable.

Difficulties. Continuation

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$$I_2 := \int_0^\infty \int_0^\infty \frac{dx\,dy}{xy(x+y+1/x+1/y)^2} = \int_0^\infty \int_0^\infty \frac{xy\,dx\,dy}{(x^2y+xy^2+x+y)^2}$$

Every form produces divergent integrals
The same occur with

$$I_2 = \int_0^\infty \int_0^\infty \frac{xy\,dx\,dy}{(xy(x+y)+x+y)^2}$$

$$\text{But} \quad I_2 = \int_0^\infty \int_0^\infty \frac{xy\,dx\,dy}{(xy(x+y)+(x+y))^2}$$

Finally gives $I_2 = \frac{1}{2}$

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Integrals coming from Feynman diagrams

The figure shows interaction of three particles

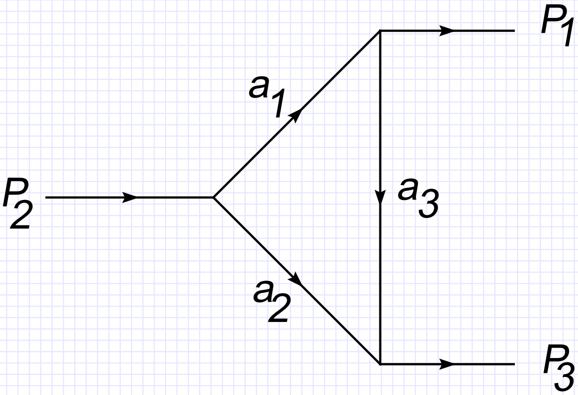

Figure: The triangle

Integrals coming from Feynman diagrams. Continuation.

Schwinger parametrization gives

$$\begin{aligned} G &= \frac{(-1)^{-D/2}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \int_0^\infty \int_0^\infty \int_0^\infty \frac{x_1^{a_1-1} x_2^{a_2-1} x_3^{a_3-1}}{(x_1+x_2+x_3)^{D/2}} \\ &\times \exp(x_1 m_1^2 + x_2 m_2^2 + x_3 m_3^2) \\ &\times \exp\left(-\frac{C_{11}P_1^2 + 2C_{12}P_1\cdot P_2 + C_{22}P_2^2}{x_1+x_2+x_3}\right) dx_1 dx_2 dx_3. \end{aligned}$$

The coefficients C_{ij} are given by

$$C_{11} = x_1(x_2+x_3), \quad C_{12} = x_1 x_3, \quad C_{22} = x_3(x_1+x_2).$$

Integrals coming from Feynman diagrams. Continuation.

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Integrals coming from Feynman diagrams. Continuation.

Conservation of momentum gives $P_3 = P_1 + P_2$ and then

$$
\begin{aligned}
G \quad = \quad & \frac{(-1)^{-D/2}}{\prod_{j=1}^{3}\Gamma(a_j)} \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} x^{a_1-1} x^{a_2-1} x^{a_3-1} \times \\
\times \quad & \exp\left(x_1 m_1^2 + x_2 m_2^2 + x_3 m_3^2\right) \\
\times \quad & \frac{\exp\left(-\frac{x_1 x_2 P_1^2 + x_2 x_3 P_2^2 + x_3 x_1 P_3^2}{x_1 + x_2 + x_3}\right)}{(x_1 + x_2 + x_3)^{D/2}} dx_1 dx_2 dx_3 .
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Integrals coming from Feynman diagrams. Continuation.

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Integrals coming from Feynman diagrams. Continuation.

Problem. Evaluate $G = G(P_1, P_2, m_i, D, a_i)$.

Special case $m_1 = m_2 = m_3 = 0$ and $P_1^2 = P_2^2 = 0$.

$$
\begin{aligned}
G &= \frac{(-1)^{-D/2}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \\
&\times \int_{\mathbb{R}_+^3} x_1^{a_1-1} x_2^{a_2-1} x_3^{a_3-1} \frac{\exp\left(-\frac{x_1 x_3}{x_1+x_2+x_3} P_3^2\right)}{(x_1+x_2+x_3)^{D/2}} \, dx_1 \, dx_2 \, dx_3.
\end{aligned}
$$

The method of brackets gives

$$
G = \frac{(-1)^{-D/2}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \sum_{n_1} \sum_{n_2} \sum_{n_3} \sum_{n_4} \phi_{1234} (P_3^2)^{n_1} \frac{\Delta_1 \Delta_2 \Delta_3 \Delta_4}{\Gamma(D/2+n_1)},
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Integrals coming from Feynman diagrams. Continuation.

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Integrals coming from Feynman diagrams. Continuation.

The brackets Δ_j are

$$\begin{aligned}
\Delta_1 &= \langle D/2 + n_1 + n_2 + n_3 + n_4 \rangle, \\
\Delta_2 &= \langle a_1 + n_1 + n_2 \rangle, \\
\Delta_3 &= \langle a_2 + n_3 \rangle, \\
\Delta_4 &= \langle a_3 + n_1 + n_4 \rangle.
\end{aligned}$$

This problem has no free indices.

$$n_1^* = \frac{D}{2} - a_1 - a_2 - a_3, \; n_2^* = -\frac{D}{2} + a_2 + a_3, \; n_3^* = -a_2, \; n_4^* = -\frac{D}{2} + a_1 + a_2.$$

Integrals coming from Feynman diagrams. Continuation.

This gives

$$\begin{aligned} G &= \frac{(-1)^{-D/2}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}(P_3^2)^{D/2-a_1-a_2-a_3} \times \\ &\times \frac{\Gamma(a_1+a_2+a_3-\frac{D}{2})\Gamma(\frac{D}{2}-a_2-a_3)\Gamma(a_2)\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}-a_1-a_2)}{\Gamma(D-a_1-a_2-a_3)}. \end{aligned}$$

Up to now

Every integral for which the method gives an answer, it gives the correct answer.

Thanks for your attention.