CHARACTERIZING TRANSCENDENCE IN COMBINATORICS

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I work with gratitude on the unceded Traditional Coast Salish Lands including the Tsleil-Waututh (səl̓ilw̓ətaʔɬ), Squamish (Sḵwx̱wú7mesh Úxwumixw) and Musqueam (xʷməθkʷəy̓əm) Nations.

Motivation

Classification

Rational

$$\frac{1}{1-t} = 1 + t + t^2 + \ldots$$

$$\frac{-1+\sqrt{1-4t^2}}{2t} = 1 + t + 2t^2 + \ldots$$

Algebraic

Differentiably Finite

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots$$

Differentially Algebraic

Complex

$\Gamma(t)$

D-finite : Satisfies a linear DE with polynomial coefficients. AKA Holonomic

D-Algebraic: Satisfies a polynomial DE.

D-Transcendental : NOT differentially algebraic

Combinatorial classes

A *combinatorial class* is a set equipped with a size function. *Ordinary Generating Functions (OGF)* encode enumerative data as integer coefficients of formal power series.

$$\mathscr{C} \implies C(t) := \sum_{n=0}^{\infty} \left|\mathscr{C}_n\right| t^n$$

TYPE OF CLASS	TYPICAL EXAMPLES	NATURE OF OGF
Finite class		Polynomial
Iterative grammar specification	Recognizable by a finite automaton Regular language, eg. Fibonacci	Rational function
Recursively grammar specification	Trees, Catalan classes, Maps	Algebraic function
?	Shuffles of Dyck Paths k-regular labelled graphs SYT of bounded height	D-finite
?	Families of decorated maps	D-algebraic

Rational

Unconstrained simple walks

Regular languages

$$\frac{1}{1-t} = 1 + t + t^2 + \dots$$

Fibonacci numbers

Algebraic

Walks in half plane

Excursions on Cayley graphs of free products of finite groups

Context free languages

$$\frac{-1+\sqrt{1-4t^2}}{2t} = 1 + t + 2t^2 + \dots$$

132- avoiding permutations

2-3 Trees

Catalan numbers

Differentiably Finite

Simple walks in quarter plane

Constrained regular languages

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots$$

Baxter permutations

K-regular graphs

Differentially Algebraic

Tree decorated maps

Bell numbers (EGF)

Complex

Simple walks in “transcendental” region

Excursions on Sierpinski gasket

Consecutive 1432-avoiding permutations**

Complete 2-3 Trees

Bell numbers (OGF)

Applications of classification

- **Theoretical Computer Science**
  The following language is not unambiguously context free:
  $\mathscr{C} = \{w \in \{a, b, c\}^* \mid |w|_a \neq |w|_b \text{ or } |w|_a \neq |w|_c\}$ because its generating function $C(t) = \sum_n c_n t^n$ is not algebraic. *(Flajolet 1988)*

- **Group Theory**
  Let G be a finitely generated amenable group that is not nilpotent-by-finite and let S be a finite symmetric generating set for G. The OGF for walks starting and ending at the origin on the Cayley Graph X(G;S) is not D-finite. *(Bell, M. 2021)*
  Gives a strategy to determine if Thompson's Group F is an amenable group. *(Elvey-Price, Guttmann 2019)*

Why the interest in D-finite series?

"Almost anything is non-holonomic unless it is holonomic by design."

- Flajolet, Gerhold & Salvy, 2005

- Closure properties mirror combinatorial actions
- The differential equation is a useful data structure for both reasoning and computation
- Clear proof strategies
- **Conjecture** *(Christol, 1990):* If a series with non-negative integer coefficients and a positive, finite, radius of convergence is furthermore D-finite, then it can be written as the diagonal of a multivariate rational function.

D-algebraic series are much more difficult to manipulate and characterize.

“Classic” Strategies

To show a series is D-finite:

- Build it from other D-finite series
- Show the coefficients satisfy a linear recurrence
- Write it as the constant term (with respect to auxiliary variables) of a multivariable D-finite series (essentially, a Cauchy integral)

To show a series is NOT D-finite

- Show asymptotic growth of the coefficients is not of the correct form
- Show that it comes from a function with an infinite number of singularities
- *It is sufficient to show it is D-Transcendendal*

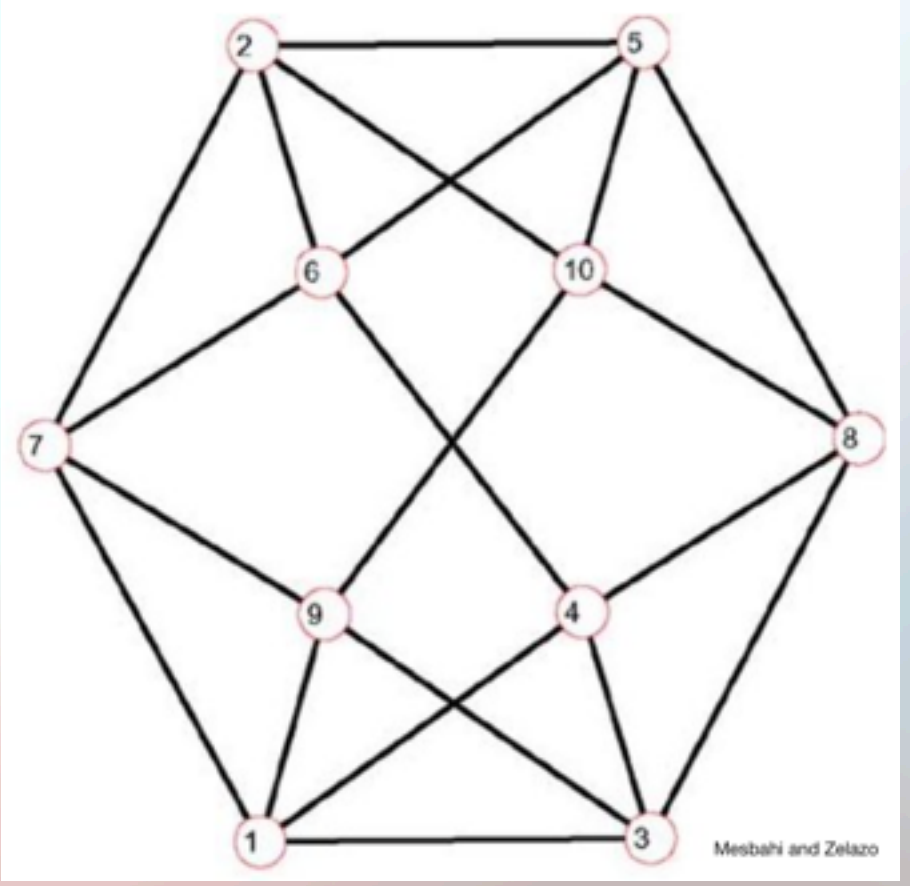
D-finite series in combinatorics

- Richard Stanley's 1980 article plants several seeds, many which were considered by Gessel (1990):
  - Baxter permutations
  - Young Tableaux of bounded height
  - *k*-regular graphs
- Cited by > 500
  > 12 000 hits to {Holonomic | D-finite }+combinatorics
- Most D-finite classes are in some bijection with a class of *lattice walks*

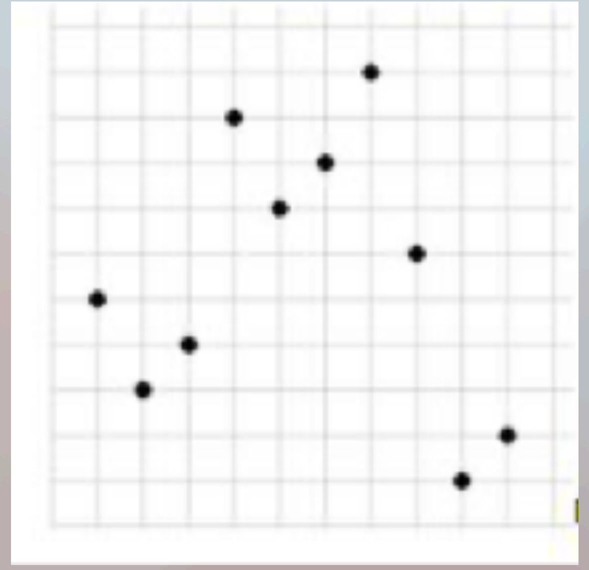
Differentiably Finite Power Series

R. P. STANLEY*

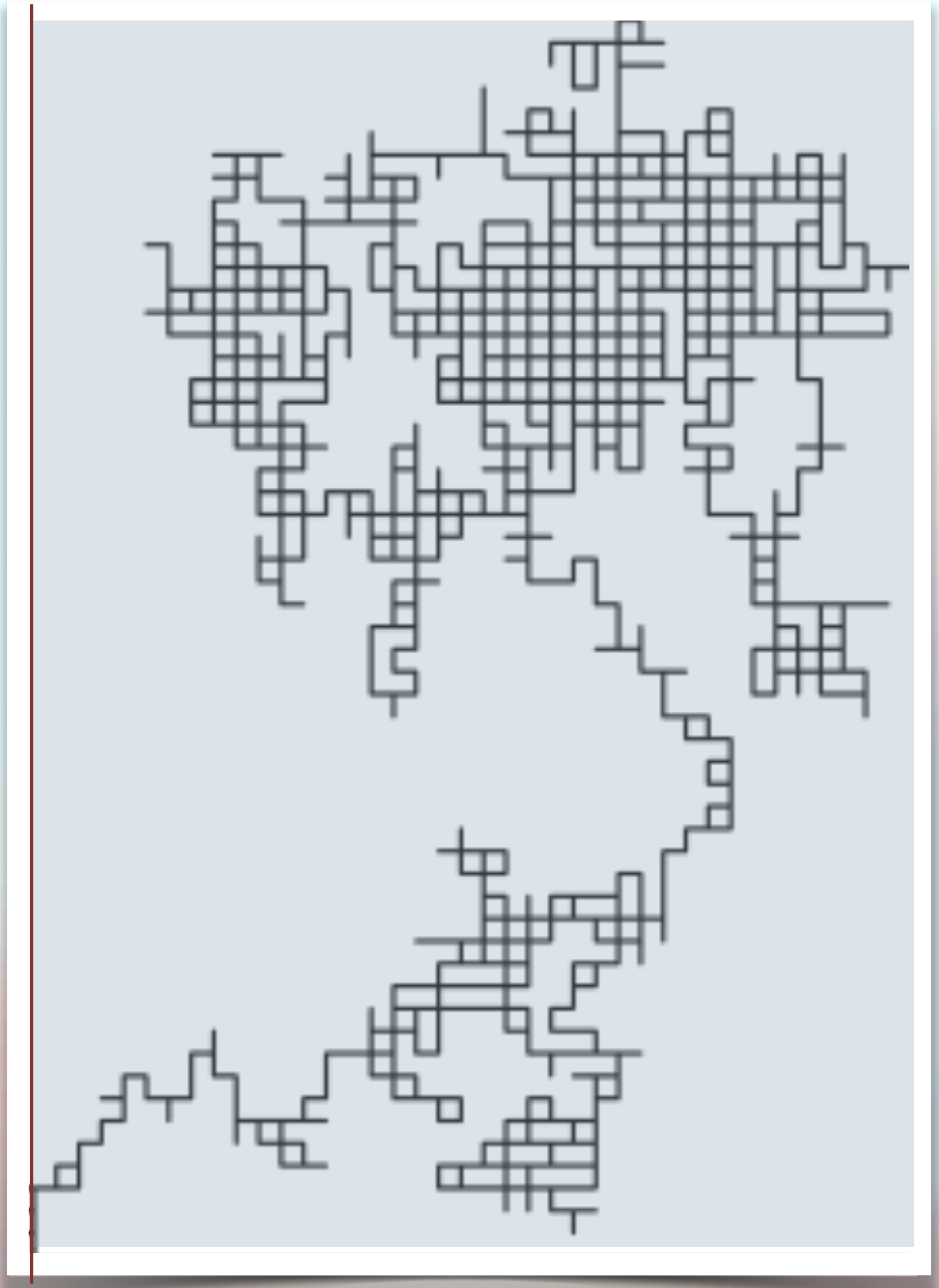
A formal power series $\sum f(n)x^n$ is said to be differentiably finite if it satisfies a linear differential equation with polynomial coefficients. Such power series arise in a wide variety of problems in enumerative combinatorics. The basic properties of such series of significance to combinatorics are surveyed. Some reciprocity theorems are proved which link two such series together. A number of examples, applications and open problems are discussed.



1	3	5
2	4	8
6	9	10
7		



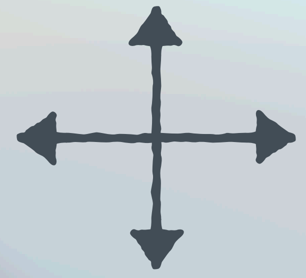
Lattice Paths

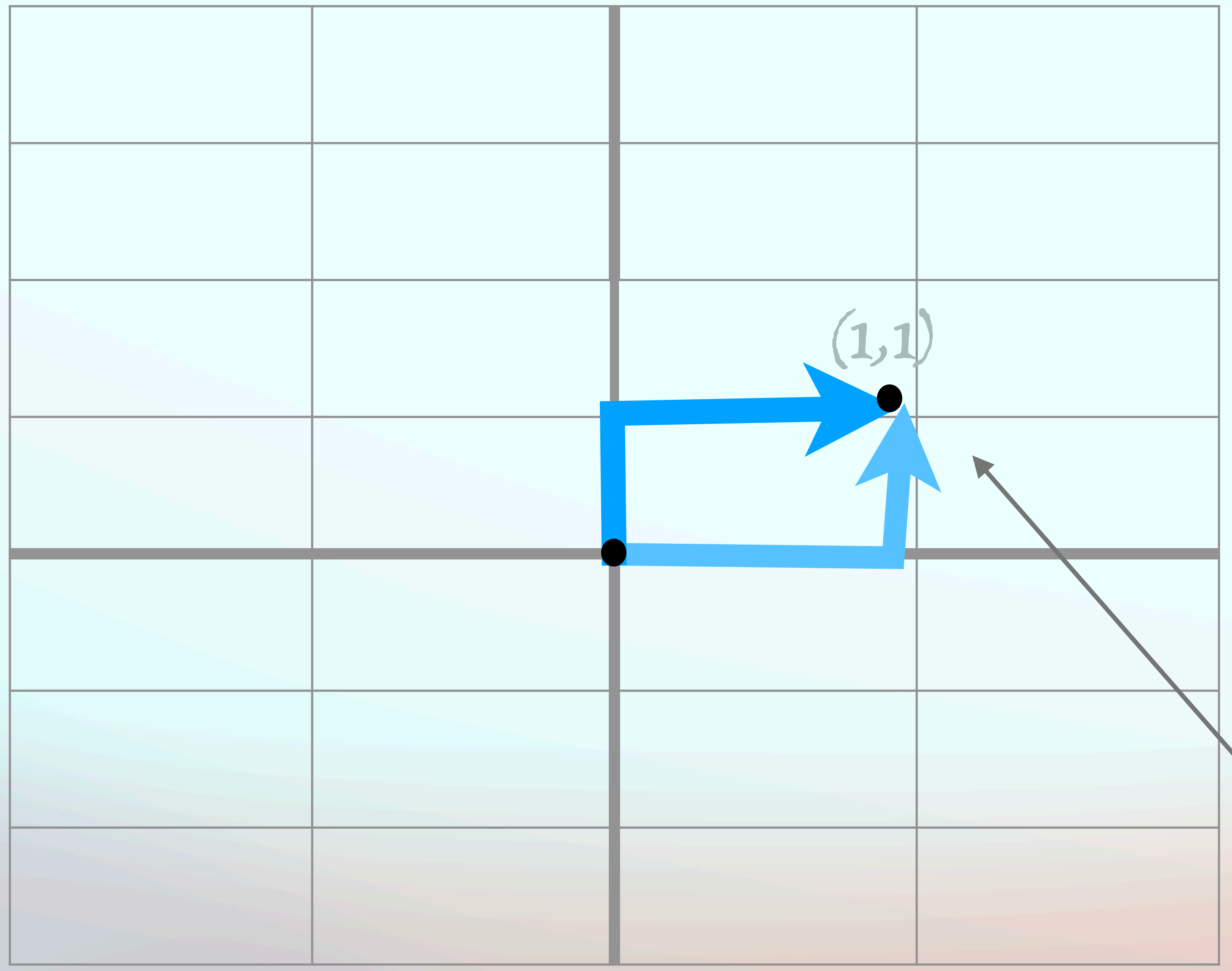


A walk is a sequence of steps

Consider (fixed) finite sets of possible steps (set of vectors). Enumerate the class of walks encoded by sequences of steps.

Strategy : Encode each walk with a monomial marking its endpoint.





Multiplying monomials captures the action of steps in sequence.

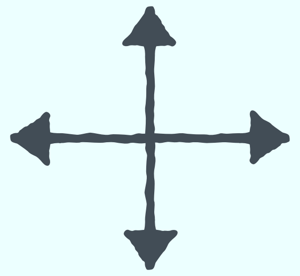
$$\frac{1}{1 - t\left(x + 1/x + y + 1/y\right)}$$

→ ← ↑ ↓

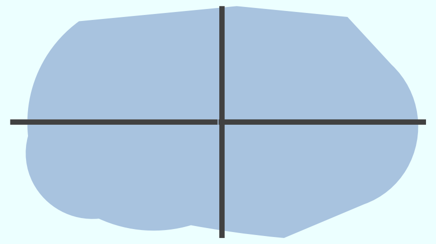
$(x + 1/x + y + 1/y) \times (x + 1/x + y + 1/y) = x^2 + 1/x^2 + y^2 + 1/y^2 + 4 + \boxed{2xy} + 2x/y + 2y/x + 2/xy$

$$\frac{1}{1 - 4t}$$

NESW-walks in various regions



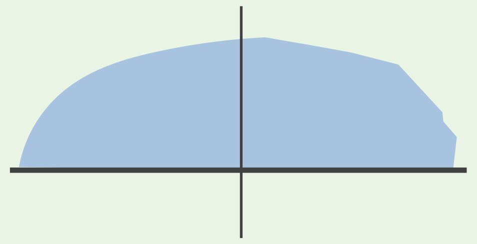
$$W(t) = \sum w(n)t^n = \sum_{n\geq 0} (\text{\#walks of length } n \text{ that stay in the blue region})\ t^n$$



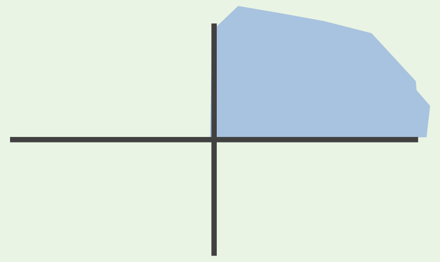
$$\frac{1}{1-4t} = \sum 4^n t^n$$

Rational

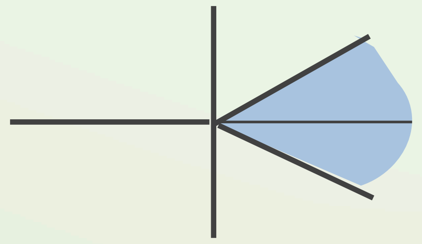
We can *classify* the nature of generating functions of NESW walks in some regions



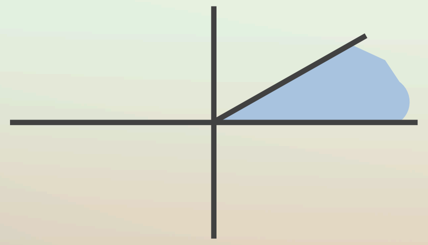
Algebraic



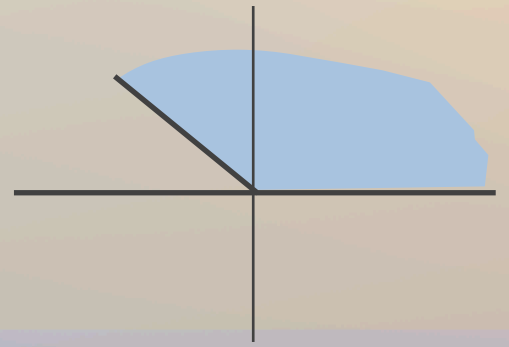
D-finite



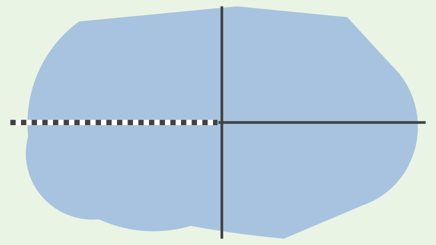
D-finite



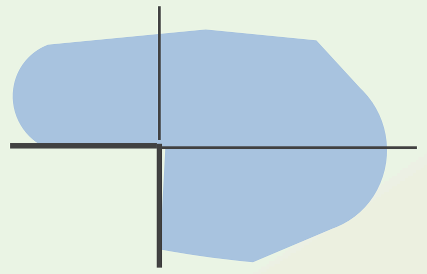
D-finite
(Gouyou-Beauchamps)



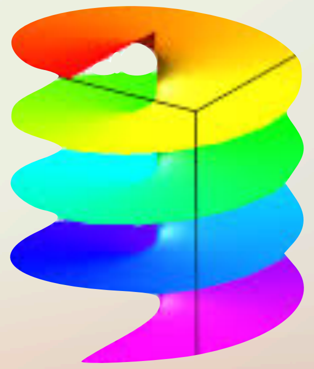
Nasty algebraic
(Bostan & Kauers…)



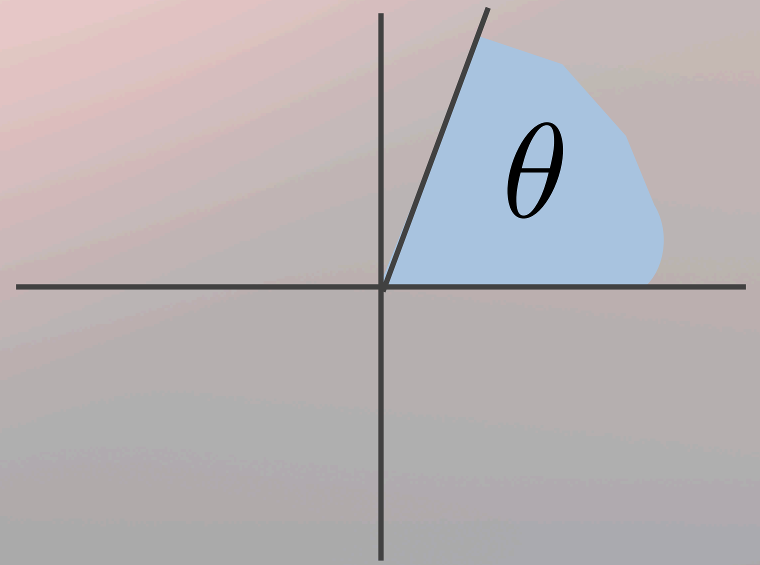
Slit plane model
Algebraic
(Bousquet-Mélou & Schaeffer)



D-finite
(Bousquet-Mélou…)

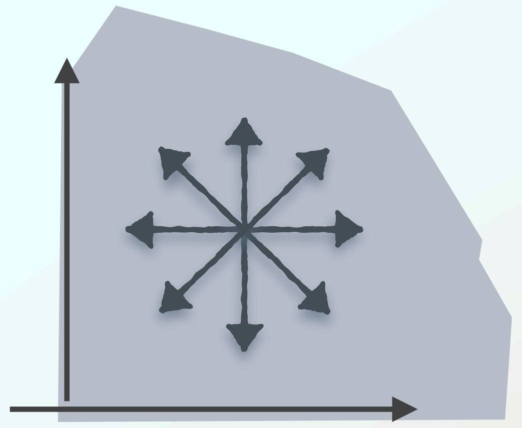


Classification via winding angle
(Excursions: Budd 2017; Elvey-Price)



Varied nature, including not D-finite
(Denisov-Wachtel, Bostan, Raschel & Salvy)

Small step walks in the first quadrant



Fix set of steps (i.e. vectors) $\mathcal{S} \subseteq \{(i,j) \mid i,j \in \{0,1,-1\}\} \backslash \{(0,0)\}$.

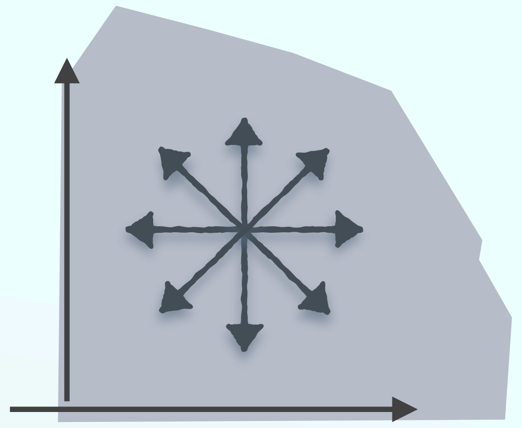
$$Q_{\mathcal{S}}(t) := \sum_{n \geq 0} \sum_{(i,j) \in \mathbb{N}^2} \#\text{walks}_{\mathcal{S}}\, (0,0) \xrightarrow{n} (i,j)\, t^n$$

Theorem *(M. + Rechnitzer 2009)*
There exist models with NON D-finite generating functions

Conjecture *(M. 2007; Bousquet-Mélou + M. 2010)*
Conditions for D-finiteness for the 79 nontrivial, distinct models.

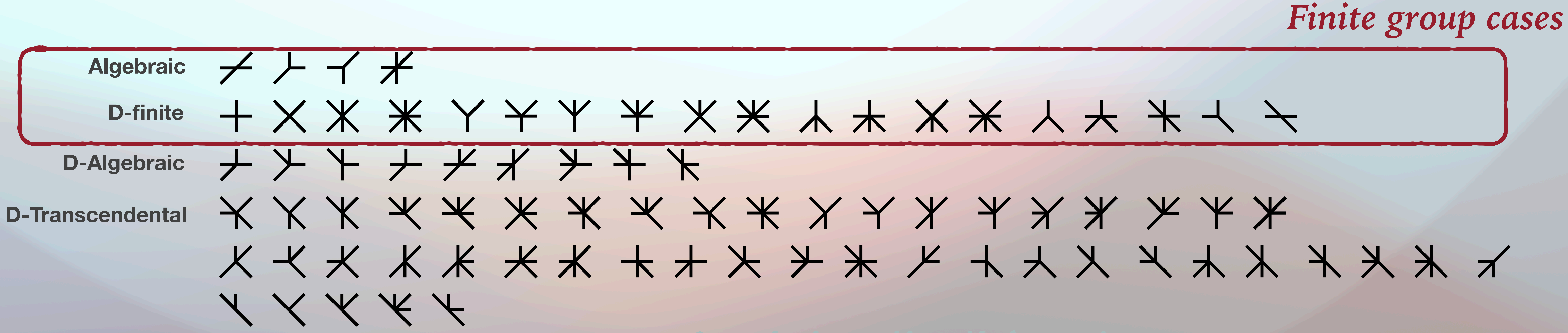
Small step walks in the quarter plane

Conjecture *(Bousquet-Mélou & M. 2010)*
$Q_{\mathcal{S}}(x, y; t)$ D-finite iff a certain group is finite.



$$Q_{\mathcal{S}}(x, y; t) \equiv Q_{\mathcal{S}}(x, y) = \sum_{n \geq 0} t^n \sum_{(i,j) \in \mathbb{N}^2} \left(\#\text{walks}_{\mathcal{S}}\, (0,0) \xrightarrow{n} (i, j) \right) x^i y^j$$

*A decade-long, international collaboration determined the **classification of $Q_{\mathcal{S}}(x, y)$***



Histogram: Quarter plane lattice models sorted by the nature of $Q_{\mathcal{S}}(x, y)$

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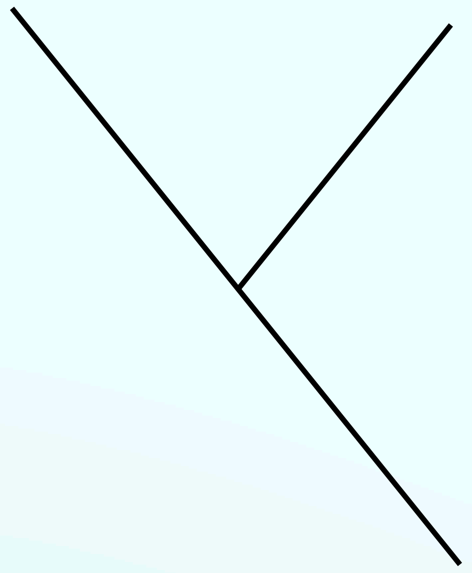
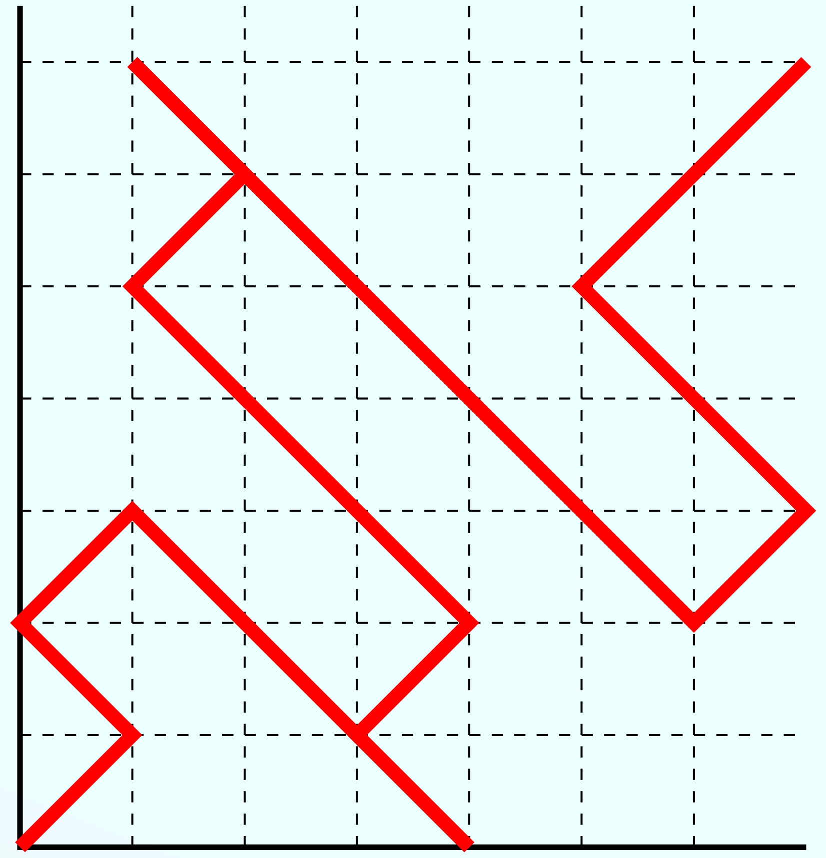
Differential Transcendence

A non-D-finite lattice model





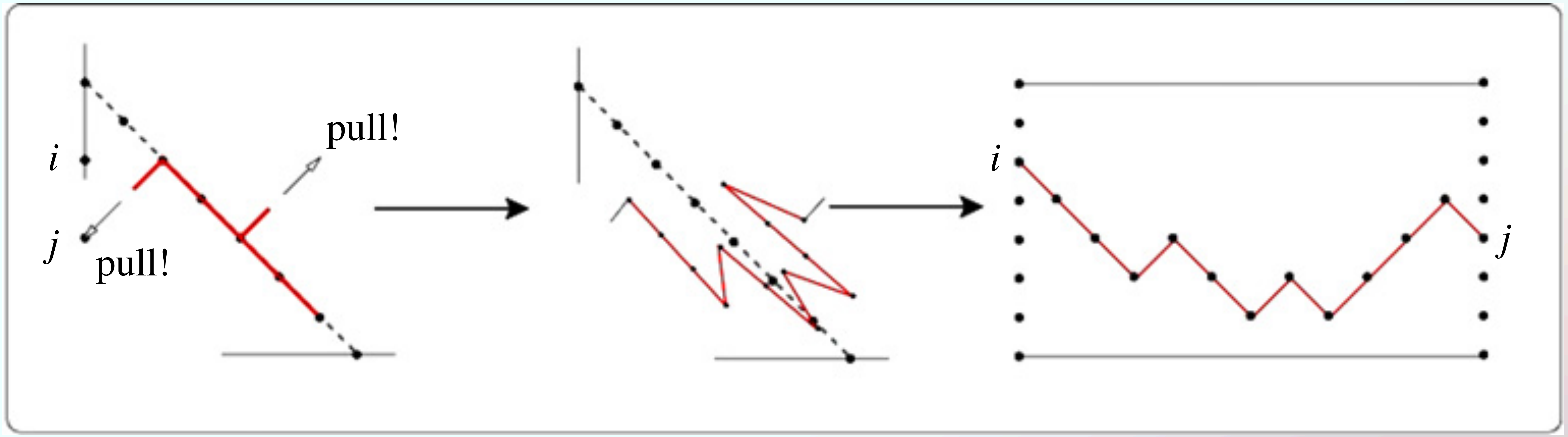
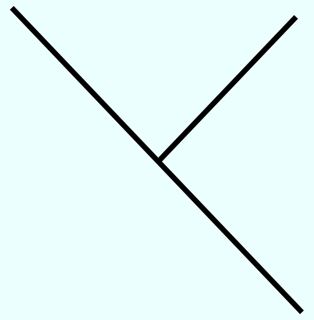


Fig. 5. Stretching the walk to find a directed path in a strip.

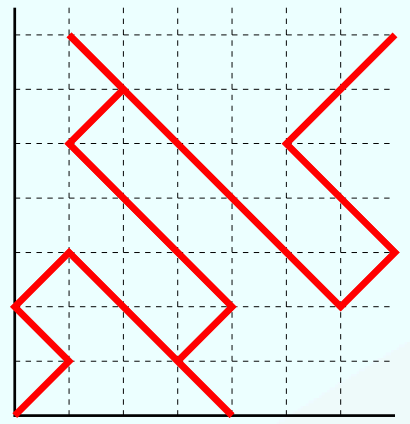
- **Theorem** *(M., Rechnitzer, 2009)*
  The univariate OGF has an *infinite number of singularities* and is not D-finite.

- A possible combinatorial explanation: A sequence of directed paths in strips of increasing height

- Similar models proved in an *ad hoc* manner.



In fact.. D-transcendental



Dreyfus+Hardouin+Roques+Singer 17 • Bostan 19

Step	
Combinatorial recurrence	*A walk is either the* **empty walk**, *or it is* **a shorter walk with a step appended**, *but you must* **exclude** *those walks that then* **step out** *of the quarter plane*
Functional equation for $Q_{\mathcal{S}}(x,y)$	$Q(x,y) = 1 + z(x/y + y/x + xy)Q(x,y) - z(x/y)Q(x,0) - z(y/x)Q(0,y)$
Rewrite so LHS is $K_{\mathcal{S}}(x,y)Q_{\mathcal{S}}(x,y)$	$K(x,y)Q(x,y) = xy - R(x) - R(y)$
Find rational parametrization for $E_{\mathcal{S}}$	$x(s) = \frac{v(1-v^2)s}{(s^2+1)}, y(s) = \frac{(1-v^2)s}{v^2s^2+1}, z = \frac{v}{v^2+1} \implies 0 = x(s)y(s) - R(x(s)) - R(y(s))$
Deduce an Ishizaki/Ogawara style equation for R(x(s))	$f(qt) = a(t)f(t) + b(t)$
Conclude D-transcendance	

Solution dichotomy

Lemma *(Ishizaki 1998; Ogawara 2015)*

Given a Laurent series $f(t)$, $a(t), b(t) \in \mathbb{C}(t)$, and q, a complex number that is not a root of unity such that

$$f(qt) = a(t)f(t) + b(t)$$

then $f(t)$ is EITHER rational or D-transcendental.

Strategy

Rough principle: *(ref. Adamczewski, Dreyfus, Hardouin 2021)*

A Laurent series solution *f(t)* of a linear [shift | Mahler | q-shift] equation is EITHER *rational*, or D-*transcendental*

- q-shift: $f(t) \mapsto f(qt)$ (q not a root of unity)
  Example: Genus 0 quarter plane walks
- Shift operator: $f(t) \mapsto f(t+h)$
  Example: $\Gamma(t+1) = t\Gamma(t)$
- Mahler operator: $f(t) \mapsto f(t^k)$
  Example:
  $f(t) = \sum t^{2^n}$ satisfies $f(t) = t + f(t^2)$
- $f(t) \mapsto f\left(\frac{t}{1+t}\right)$
  Example: Bell numbers
  $B(t) = \sum B_n t^n \implies B\left(\frac{t}{t+1}\right) = tB(t) + 1$
  (Klazar 2003; Bostan, DiVizio, Raschel 2020+)

...not rational, hence it must be D-transcendental.

NEW!

Extending the strategy

Theorem I. *(Di Vizio, Fernandes, M. 2023+)*

If $f(t) \in \mathbb{C}[[t]]$ satisfies $f(R(t)) = f(t) + b(t)$
with $R(t), b(t)$ rational, and furthermore $R(0) = 0,\ R'(0) \in \{0, 1, \text{roots of unity}\}$
no iterate of R is the identity , then $f(t)$ is either rational or D-transcendental.

Theorem II. *(Di Vizio, Fernandes, M. 2023+)*

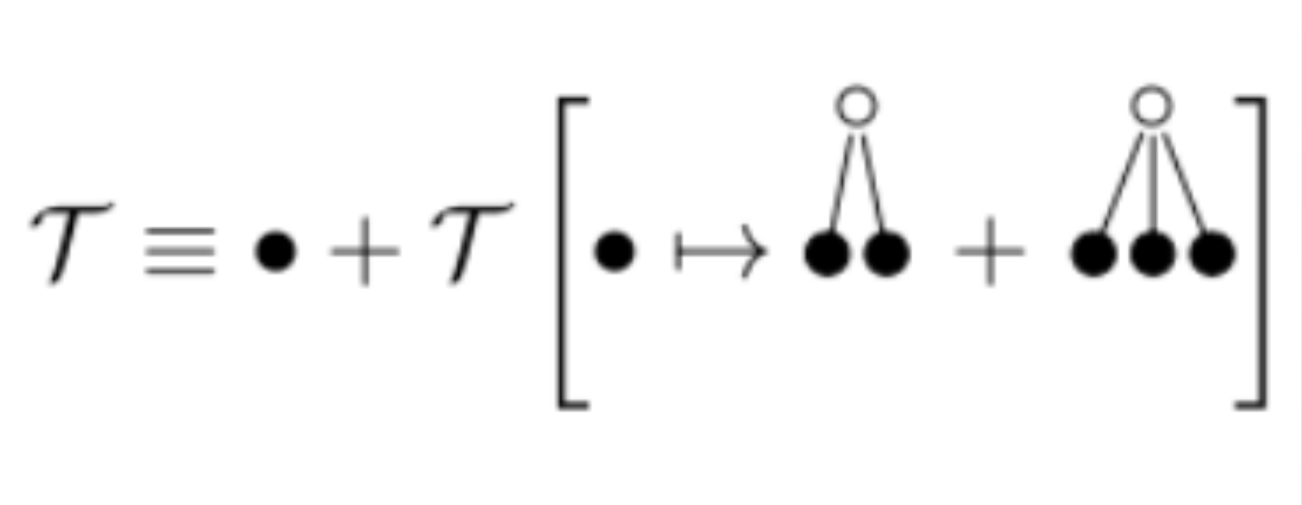
If $f(t) \in \mathbb{C}[[t]]$ satisfies $f(R(t)) = a(t)f(t)$
with $R(t), a(t), b(t)$ rational, and furthermore $R(0) = 0,\ R'(0) \in \{0, 1, \text{roots of unity}\}$
no iterate of R is the identity ,then $f(t)$ is either algebraic or D- transcendental.

New Examples: $R(t) = t^2 + t^3, R(t) = \dfrac{t}{1+t^2}$

Complete 2-3 Trees

$$T(z) = z + z^2 + z^3 + z^4 + 2z^5 + 2z^6 + O(z^7)$$

$$T(z) = z + T(z^2 + z^3).$$



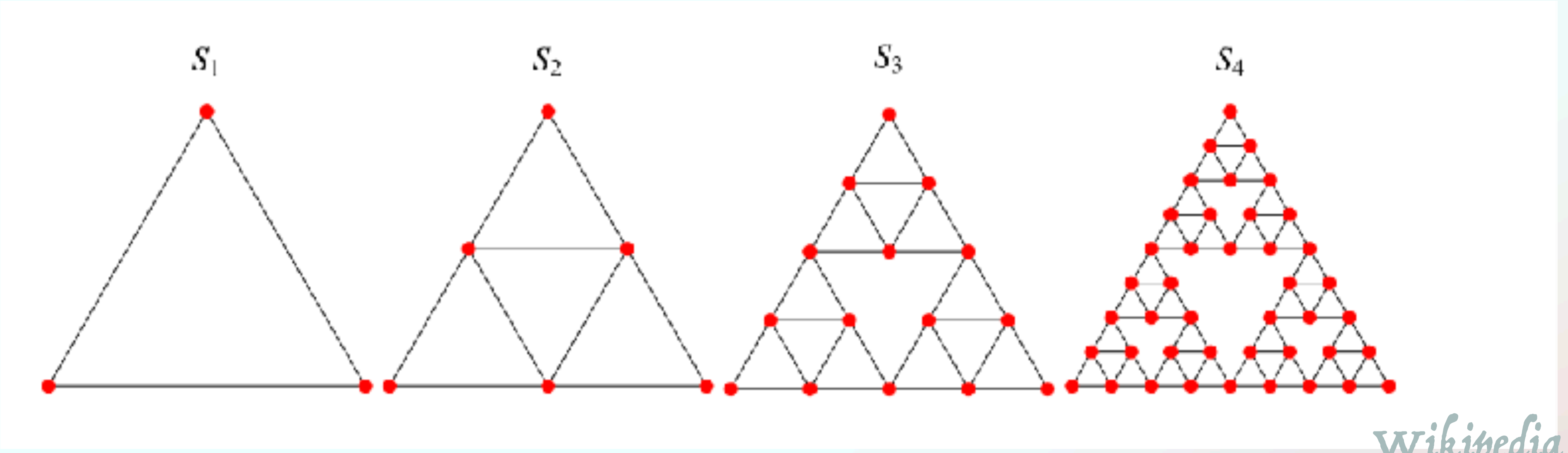
When *R(t)* is a polynomial we have a stronger result that we can apply here.

Corollary 1.2. *In the notation and under the assumptions of Theorem 1.1, we suppose moreover that* $R \in t^2\mathbb{C}[t]$, *and that* $b \in t\mathbb{C}[t]$, *with* $b \neq 0$ *and* $\deg_t b \leq \deg_t R$. *Then* f *is differentially transcendental over* $\mathbb{C}(t)$.

T(t) is D-transcendental (Indeed, most complete tree classes are similar)

Walks on self-similar graphs

The **Green function** of a graph is a probability generating function which describes the *n*-step displacement starting and returning to a certain origin vertex.



Wikipedia

(Grabner + Woess) The Green function associated to the Sierpinski Graph satisfies:

$$G\left(\frac{t^2}{4-3t}\right) = \frac{(2+t)(4-3t)}{(4+t)(2-t)}\,G(t)\,.$$

The asymptotics of the coefficients *(Teufl)* are incompatible with algebraicity. By Theorem II:

G(t) is D-transcendental

NEW!

Extending the strategy

Theorem III. *(Di Vizio, Fernandes, M. 2023+)*

If $f(t) \in \mathbb{C}[[t]]$ satisfies $f(R(t)) = a(t)f(t)+b(t)$
with $R(t), a(t), b(t)$ algebraic, and furthermore $R(0) = 0,\ R'(0) \in \{0,1,\text{roots of unity}\}$,
no iterate of R is the identity
then $f(t)$ is either D-finite or D- transcendental.

Permutations avoiding consecutive patterns

- A permutation σ of n avoids the pattern 1423 if there is no $0 \leq i \leq n-4$ so that $\sigma(i+1) < \sigma(i+4) < \sigma(i+2) < \sigma(i+3)$. The EGF of 1423-avoiding permutations can be written using $S(t)$ satisfying the following: *(Elizalde and Noy 2012)*

$$S(t) = 1 + \frac{1}{1+t} S\left(\frac{1}{1+t^2}\right)$$

- $S(t)$ has an infinite number of singularities. *(Beaton, Conway and Guttmann 2018)*

- Since $S(t)$ is not D-finite, by Theorem III, $S(t)$ is D-transcendental.

- Similar situation for $1m23\ldots(m-1)$ avoiding permutations

Concluding remarks

Rational

$$\frac{1}{1-t} = 1 + t + t^2 + \dots$$

Unconstrained simple walks

Regular languages

Fibonacci numbers

Algebraic

$$\frac{-1+\sqrt{1-4t^2}}{2t} = 1 + t + 2t^2 + \dots$$

Walks in half plane

Excursions on Cayley graphs of free products of finite groups

Context free languages

132- avoiding permutations

2-3 Trees

Catalan numbers

Differentiably Finite

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots$$

Simple walks in quarter plane

Constrained regular languages

Baxter permutations

K-regular graphs

Differentially Algebraic

Tree decorated maps

Bell numbers (EGF)

Complex

Simple walks in “transcendental” region

Excursions on Sierpinski gasket

*Consecutive 1432-avoiding permutations ****

Complete 2-3 Trees

Bell numbers (OGF)

Open questions & future work

- Identify combinatorial contexts that result in such functional equations.
- Simplify proofs of non-D-finiteness by proving D-transcendence.
- Higher order equations.
- Automated "guessing" tools for other kinds of functional equations.

Thank you for your attention!