Ascent sequences avoiding a set of length-3 patterns

Toufik Mansour
University of Haifa, Israel

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Introduction

• An ascent, short for ascent index, in an integer sequence $s_1 s_2 \cdots s_m$ is an index $1 \le j \le m-1$ such that $s_j < s_{j+1}$.

• An ascent sequence $a_1 a_2 \cdots a_n$ is one consisting of non-negative integers satisfying

$$a_1 = 0, \ a_i \le \mathrm{asc}(a_1 a_2 \cdots a_{i-1}) + 1, \quad i = 2, 3, \ldots, n,$$

where $\mathrm{asc}(a_1 a_2 \cdots a_k)$ is the number of ascents in the sequence $a_1 a_2 \cdots a_k$.

• For example, the sequence 0102321401 is an ascent sequence, whereas 0104 is not.

• Bousquet-Mèlou, Claesson, Dukes, and Kitaev connected ascent sequences to $(2+2)$-free posets. Since then, ascent sequences have been considered in a series of papers where connections to many other combinatorial structures have been found.

• Let $a = a_1a_2\cdots a_n$ be any sequence and $\tau = \tau_1\cdots\tau_m$ be any pattern, that is, a word in $\{0,\ldots,\ell\}^m$ which contains each letter $0, 1, \ldots, \ell$ for some $m \geq 1$, $\ell \geq 0$.

• We say the sequence a contains τ if a has a subsequence that is order isomorphic to τ, that is, there is a subsequence

$$a_{f(1)}, a_{f(2)}, \ldots, a_{f(m)} \text{ with } 1 \leq f(1) < f(2) < \cdots < f(m) \leq n,$$

such that $a_{f(i)} X a_{f(j)}$ if and only if $\tau_i X \tau_j$, for all $X \in \{<, >, =\}$ and $1 \leq i, j \leq m$. Otherwise, a is said to avoid τ.

• For instance, the ascent sequence 0101304351 has two occurrences of the pattern 110, namely, the subsequences 110 (0101304351) and 331 (0101304351), but avoids the pattern 3120.

• We denote the set of all ascent sequences that avoid a list of patterns $\tau^{(1)}, \ldots, \tau^{(s)}$ by $A_n(\tau^{(1)}, \ldots, \tau^{(s)})$ or $A_n(\{\tau^{(1)}, \ldots, \tau^{(s)}\})$.

• We say that two sets of patterns P and Q are A-Wilf-equivalent, denoted $P \overset{a}{\sim} Q$, if $|A_n(P)| = |A_n(Q)|$ for every $n \geq 0$.

• There are 13 patterns of length 3: 000, 001, 010, 100, 011, 101, 110, 012, 021, 102, 120, 201, and 210.

• The number of A-Wilf-equivalence classes among single patterns of length 3 is 9 (Duncan and Steingrímsson).

• The number of A-Wilf-equivalence classes among pairs of patterns of length 3 is 35 (Baxter and Pudwell).

• Let aw_k be the number of A-Wilf-equivalence classes of k length-3 patterns.

Theorem We have $aw_3 = 62$, $aw_4 = 74$, $aw_5 = 61$, $aw_6 = 47$, $aw_7 = 35$, $aw_8 = 25$, $aw_9 = 18$, $aw_{10} = 12$, $aw_{11} = 7$, $aw_{12} = 3$, and $aw_{13} = 1$.

• Note that there are $2^{13} = 8192$ subsets of length-3 patterns. So, we need a general method to deal with all these subsets.

Generating trees

• Let P be any set of patterns such that the length of each pattern is at least two. Define $\mathcal{A}(P) = \cup_{n=0}^{\infty} A_n(P)$.

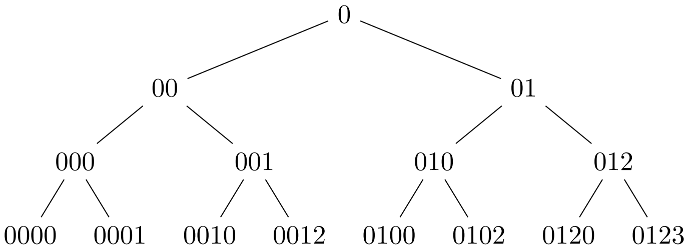
• We will construct a pattern-avoidance generating tree $\mathcal{T}(P)$ for the class of pattern-avoiding ascent sequences $\mathcal{A}(P)$.

• Starting with the root 0 which stays at level 1, we construct in a recursive manner the non-root nodes of the tree $\mathcal{T}(P)$ such that the nth level of the tree consists of exactly the elements of $A_n(P)$ arranged so that the parent of an ascent sequence $a_1 \cdots a_n \in A_n(p)$ is the unique ascent sequence $a_1 \cdots a_{n-1} \in A_{n-1}(P)$.

- The children of $a_1 \cdots a_{n-1} \in A_{n-1}(P)$ are obtained from the set $\{a_1 \cdots a_{n-1} a_n \mid a_n = 0, 1, \ldots, \mathrm{asc}(a_1 \cdots a_{n-1}) + 1\}$ by applying the pattern-avoiding restrictions of the patterns in P.

- We arrange the nodes from the left to the right so that if $a = a_1 \cdots a_{n-1} i$ and $a' = a_1 \cdots a_{n-1} i'$ are children of the same parent $a_1 \cdots a_{n-1}$, then a appears on the left of a' if $i < i'$.

- The next figure presents the first few levels of $\mathcal{T}(\{011\})$.



- Clearly, $|A_n(P)|$ =the number of nodes in the nth level of $\mathcal{T}(P)$.

- Let $\mathcal{T}(P;a)$ denote the subtree consisting of the ascent sequence a as the root and its descendants in $\mathcal{T}(P)$.

- For any $a, a' \in \mathcal{T}(P)$, we say that the subtrees $\mathcal{T}(P;a)$ and $\mathcal{T}(P;a')$ are isomorphic, and write $\mathcal{T}(P;a) \cong \mathcal{T}(P;a')$, if these subtrees are isomorphic in the sense of plane (ordered) tree.

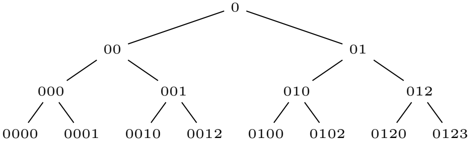
- For any two nodes $a, a' \in \mathcal{T}(P)$, we say that a is equivalent to a', denoted by $a \sim a'$, if and only if $\mathcal{T}(P;a) \cong \mathcal{T}(P;a')$.

- Define $V[P]$ to be the set of all equivalence classes in the quotient set $\mathcal{T}(P)/\sim$.

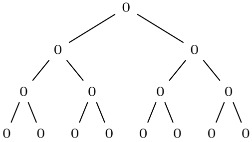
- We will represent each equivalence class $[v]$ by the label of the unique node v which appears on the tree $\mathcal{T}(P)$ as the left-most node at the lowest level among all other nodes in the same equivalence class.

• Let $\mathcal{T}[P]$ be the same tree $\mathcal{T}(P)$ where we replace each node a by its equivalence class label.

• So in our previous example, we see that the generating tree is given by $\mathcal{T}(\{011\})$ is



and the generating tree $\mathcal{T}[\{011\}]$ is



This because $01 \cong 00 \cong 0$.

Example: Let $P = \{000, 001, 021\}$. The generating tree $\mathcal{T}[P]$ has a root a_0 and satisfies the following rules

$$a_m \rightsquigarrow b_m, c_m, a_{m+1}, \quad c_m \rightsquigarrow b_m,$$

where $a_m = 01 \cdots m$, $b_m = a_m 0$, and $c_m = a_m m$.

• To show that, we need to verify the succession rules of the generating tree:

• The children of a_m are of type $a_m j$, where $j = 0, 1, \ldots, m+1$, so a_m has only three children b_m, c_m, a_{m+1} in $\mathcal{T}[P]$.

• Note that any child of b_m contains a pattern in P, so there are no children for b_m in $\mathcal{T}[P]$.

• Also, any child of c_m that avoids P is $c_m 0$. But it is not hard to see $a = c_m 0 v \in A_n(P)$ if and only if $a' = a_m 0 v \in A_{n-1}(P)$ by removing the second occurrence of the letter m in a.

• Thus, $a_m \rightsquigarrow b_m, c_m, a_{m+1}$ and $c_m \rightsquigarrow b_m$.

General procedure

Step 1:

- Let P be any set of patterns and let D be any positive number (here we use $D = 8$).

- We find the first D levels of the generating tree $\mathcal{T}(P)$.

- By (2), we guess all the succession rules of $\mathcal{T}(P)$.

- Based on (3), we try to prove these succession rules. If we fail, then we increase D by 1 and go back to Step (2). Otherwise, the succession rules of the generating tree $\mathcal{T}[P]$ are found.

Step 2:
After we guessed and proved (if possible) the rules of the generating tree $\mathcal{T}[B]$, we translate these rules into a system of equations and we solve for

$$F_P(x) = \sum_{n\geq 0} |A_n(P)|x^n.$$

Note that the rule $e \rightsquigarrow v^{(1)}, \ldots, v^{(s)}$ can be translated to

$$A_e(x) = x + x\sum_{j=1}^{s} A_{v^{(s)}}(x),$$

where

$$A_w(x) = \sum_{n\geq 0} (\#\text{the nodes at level } n \text{ in } \mathcal{T}(B; w))x^n$$

is the generating function for the number of nodes at level $n \geq 1$ in the subtree of $\mathcal{T}(B; w)$, where its root stays at level 1.

Triples

- Let L be the set of all triples of patterns of length-3. A candidate class is a maximal subset C of L such that for any $P, P' \in C$, $|A_n(P)| = |A_n(P')|$, for all $n = 1, 2, \ldots, 11$.

- Table for all triples of patterns of length-3 shows all the 62 candidate classes of L.

- A candidate class is called trivial if it contains exactly one triple, otherwise, it is called nontrivial. Clearly, any A-Wilf equivalence class is contained in a candidate class.

- There are 35 trivial candidate classes and 27 nontrivial candidate classes of triples of length-3 patterns.

• To prove $aw_3 = 62$, we show that the 27 nontrivial candidate classes of triples of length-3 patterns are indeed 27 A-Wilf equivalences.

• To establish this, we use the generating tree method as described above:

Table 1: Rules of generating trees for ascent sequences avoiding a triple of length-3 patterns.

Beginning of Table 1			
Class	B triple	Rules of $\mathcal{T}(B)$	$G_B(x)$/Reference
2	{000,001,012}	$0 \rightsquigarrow 00, 01$; $01 \rightsquigarrow 010, 011$; $011 \rightsquigarrow 010$	
	{000,010,012}	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 01$; $01 \rightsquigarrow 011$	$x + 2x^2 + 2x^3 + x^4$
3	{000,001,010}	$a_m \rightsquigarrow b_m, a_{m+1}$, where $a_m = 01 \cdots m$, $b_m = a_m m$	
	{000,001,011}	$a_m \rightsquigarrow b_m, a_{m+1}$, where $a_m = 01 \cdots m$, $b_m = a_m 0$	
	{000,010,011}	$a_0 \rightsquigarrow b_0, a_1$; $a_m \rightsquigarrow a_{m+1}$; $b_m \rightsquigarrow b_{m+1}$, where $a_m = 01 \cdots m$, $b_m = 0a_m$	
	{001,010,011}	$a_0 \rightsquigarrow 00, a_1$; $a_m \rightsquigarrow a_{m+1}$; $00 \rightsquigarrow 00$, where $a_m = 01 \cdots m$	
	{001,010,012} {001,011,012}	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 00$; $01 \rightsquigarrow 01$	
	{010,011,012}	$a_m \rightsquigarrow a_{m+1}, 01$, where $a_m = 0^m$	$\frac{x(1+x)}{1-x}$
4	{000,012,101}	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 001$; $01 \rightsquigarrow 010, 001$; $001 \rightsquigarrow 010$	
	{000,012,110}	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 001$; $01 \rightsquigarrow 001, 011$; $001 \rightsquigarrow 011$	$x + 2x^2 + 3x^3 + 2x^4$
5	{000,012,021} {000,012,100} {000,012,102} {000,012,120} {000,012,201}		

Continuation of Table 1			
Class	B triple	Rules of $\mathcal{T}(B)$	$G_B(x)$/Reference
	{000,012,210}	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 001$; $01 \rightsquigarrow 001, 001$; $001 \rightsquigarrow 0011$	$x + 2x^2 + 3x^3 + 3x^4$
6	{000,011,102}	$a_m \rightsquigarrow b_m, a_{m+1}$; $c_m \rightsquigarrow c_{m+1}$, where $a_m = 01 \cdots m$, $b_m = a_m 0$, $c_m = 0a_m$ $(b_0 = c_0)$	
	{000,011,120}	$a_0 \rightsquigarrow b_0, a_1$; $a_1 \rightsquigarrow b_1, a_2$; $a_m \rightsquigarrow a_{m+1}$; $b_m \rightsquigarrow b_{m+1}$, where $a_m = 01 \cdots m$, $b_m = 0a_m$	
	{001,011,100}	$a_m \rightsquigarrow b_m, a_{m+1}$; $b_0 \rightsquigarrow b_0$, where $a_m = 01 \cdots m$, $b_m = 0a_m$	
	{001,011,120}	$a_m \rightsquigarrow b_m, a_{m+1}$; $b_m \rightsquigarrow b_m$, where $a_m = 01 \cdots m$, $b_m = 0a_m$	
	{001,012,100}	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 00$; $01 \rightsquigarrow 010, 01$	
	{001,012,110}	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 00$; $01 \rightsquigarrow 010, 010$; $010 \rightsquigarrow 010$	
	{011,012,100}	$a_m \rightsquigarrow a_{m+1}, 01$; $01 \rightsquigarrow 010$, where $a_m = 0^m$	$x + 2x^2 + \frac{3x^3}{1-x}$
7	{000,001,021}	$a_0 \rightsquigarrow c_0, a_1$; $a_m \rightsquigarrow c_m, b_m, a_{m+1}$; $b_m \rightsquigarrow c_m$, where $a_m = 01 \cdots m$, $b_m = a_m m$, $c_m = a_m 0$	
	{000,001,120}	$a_0 \rightsquigarrow c_0, a_1$; $a_m \rightsquigarrow b_m, c_m, a_{m+1}$; $b_m \rightsquigarrow c_m$, where $a_m = 01 \cdots m$, $b_m = a_m m$, $c_m = a_m(m-1)$	$x + 2x^2 + 3x^3 + \frac{4x^4}{1-x}$
8	{000,001,110}	$a_m \rightsquigarrow (b_m)^{m+1}, a_{m+1}$, where $a_m = 01 \cdots m$, $b_m = a_m 0$	
	{000,011,021} {000,011,100} {000,011,101} {000,011,110} {000,011,201} {000,011,210}	$a_m \rightsquigarrow b_m, a_{m+1}$; $b_m \rightsquigarrow b_{m+1}$, where $a_m = 01 \cdots m$, $b_m = 0a_m$	
	{001,010,021} {001,010,100} {001,010,101} {001,010,102} {001,010,110} {001,010,120} {001,010,201} {001,010,210}	$a_m \rightsquigarrow b_m, a_{m+1}$; $b_m \rightsquigarrow b_m$, where $a_m = 01 \cdots m$, $b_m = a_m m$	

Continuation of Table 1			
Class	B triple	Rules of $\mathcal{T}(B)$	$G_B(x)$/Reference
	{001,011,021} {001,011,101} {001,011,102} {001,011,110} {001,011,201} {001,011,210}	$a_m \rightsquigarrow b_m, a_{m+1}$; $b_m \rightsquigarrow b_m$, where $a_m = 01\cdots m$, $b_m = a_m0$	
	{001,012,021} {001,012,101} {001,012,102} {001,012,120} {001,012,201} {001,012,210}	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 00$; $01 \rightsquigarrow 010, 01$; $010 \rightsquigarrow 010$	
	{010,011,021} {010,011,100} {010,011,101} {010,011,102} {010,011,110} {010,011,120} {010,011,201} {010,011,210}	$a_m \rightsquigarrow a_{m+1}, b_{m,1}$; $b_{m,j} \rightsquigarrow b_{m,j+1}$, where $a_m = 0^m$ and $b_{m,j} = 0^m12\cdots j$	
	{010,012,021} {010,012,100} {010,012,101} {010,012,102} {010,012,110} {010,012,120} {010,012,201} {010,012,210} {011,012,021} {011,012,101} {011,012,102} {011,012,110} {011,012,120} {011,012,201} {011,012,210}	$a_m \rightsquigarrow a_{m+1}, 01$; $01 \rightsquigarrow 01$, where $a_m = 0^m$	$\frac{x}{(1-x)^2}$
10	{000,001,100} {000,001,101} {000,001,102}		

Continuation of Table 1			
Class	B triple	Rules of $\mathcal{T}(B)$	$G_B(x)$/Reference
	{000,001,201}	$a_m \rightsquigarrow b_{m,0}, \ldots, b_{m,m}, a_{m+1}$; $b_{m,j} \rightsquigarrow b_{m,0}, \ldots, b_{m,k-1}$, where $a_m = 01\cdots m$, $b_{m,j} = a_m j$	
	{000,010,021} {000,010,100} {000,010,101} {000,010,102} {000,010,110} {000,010,120} {000,010,201} {000,010,210}		See [?]
14	{001,021,100}	$a_0 \rightsquigarrow c_0, a_1$; $c_0 \rightsquigarrow c_0$; $a_m \rightsquigarrow b_m, c_m, a_{m+1}$; $c_m \rightsquigarrow b_m, c_m$, where $a_m = 01\cdots m$, $b_m = a_m 0$, $c_m = a_m m$	
	{001,021,110}	$a_0 \rightsquigarrow b_0, a_1$; $a_m \rightsquigarrow (b_m)^2, a_{m+1}$; $b_m \rightsquigarrow b_m$, where $a_m = 01\cdots m$, $b_m = a_m 0$	
	{001,021,120}	$a_0 \rightsquigarrow b_0, a_1$; $b_0 \rightsquigarrow b_0$; $a_1 \rightsquigarrow 010, b_1, a_2$; $b_1 \rightsquigarrow 010, b_1$; $a_m \rightsquigarrow b_m, a_{m+1}$; $b_m rub_m$, where $a_m = 01\cdots m$, $b_m = a_m m$	
	{001,100,110}	$a_m \rightsquigarrow (b_m)^m, c_m, a_{m+1}$; $c_m \rightsquigarrow c_m$, where $a_m = 01\cdots m$, $b_m = a_m 0$, $c_m = a_m m$	
	{001,100,120}	$a_0 \rightsquigarrow b_0, a_1$; $b_0 \rightsquigarrow b_0$; $a_m \rightsquigarrow c_m, b_m, a_{m+1}$; $b_m \rightsquigarrow c_m, b_m$, where $a_m = 01\cdots m$, $b_m = a_m(m-1)$, $c_m = a_m m$	
	{001,110,120}	$a_0 \rightsquigarrow 00, a_1$; $00 \rightsquigarrow 00$; $a_m \rightsquigarrow (b_m)^2, a_{m+1}$; $b_m \rightsquigarrow b_m$, where $a_m = 01\cdots m$, $b_m = a_m(m-1)$	
	{011,100,102}	$a_m \rightsquigarrow a_{m+1}, b_{m,1}$; $b_{m,j} \rightsquigarrow c_m, b_{m,j+1}$, where $a_m = 0^m$, $b_{m,j} = a_m 1\cdots j$, $c_m = 01\cdots m0$	

Class	B triple	Rules of $\mathcal{T}(B)$	$G_B(x)$/Reference
	{011,100,120}	$a_m \rightsquigarrow a_{m+1}, b_{m,1}$; $b_{m,1} \rightsquigarrow c_m, b_{m,2}$; $c_m \rightsquigarrow b_{m+1,2}$; $b_{m,j} \rightsquigarrow b_{m,j+1}$, where $a_m = 0^m$, $b_{m,j} = a_m 1 \cdots j$, $c_m = a_m 10$	
	{011,102,120}	$a_m \rightsquigarrow a_{m+1}, b_{m,1}$; $b_{m,1} \rightsquigarrow 010, b_{m,2}$; $010 \rightsquigarrow 010$; $b_{m,j} \rightsquigarrow b_{m,j+1}$, where $a_m = 0^m$, $b_{m,j} = a_m 1 \cdots j$	
	{012,100,101}	$a_m \rightsquigarrow a_{m+1}, 01$; $01 \rightsquigarrow 010, 01$, where $a_m = 0^m$	
	{012,100,110}	$a_m \rightsquigarrow a_{m+1}, 01$; $01 \rightsquigarrow 010, 010$; $010 \rightsquigarrow 0101$; $0101 \rightsquigarrow 0101$, where $a_m = 0^m$	
	{012,101,110}	$a_m \rightsquigarrow a_{m+1}, 01$; $01 \rightsquigarrow 010, 010$; $010 \rightsquigarrow 010$, where $a_m = 0^m$	$\frac{x(1+x^2)}{(1-x)^2}$
15	{001,021,101} {001,021,102} {001,021,201} {001,021,210}	$a_0 \rightsquigarrow b_0, a_1$; $a_m \rightsquigarrow b_m, c_m, a_{m+1}$; $b_m \rightsquigarrow b_m$; $c_m \rightsquigarrow b_m, c_m$, where $a_m = 01 \cdots m$, $b_m = a_m 0$, $c_m = a_m m$	
	{001,100,210}	$a_m \rightsquigarrow (b_m)^m, c_m, a_{m+1}$; $c_m \rightsquigarrow (b_m)^m, c_m$, where $a_m = 01 \cdots m$, $b_m = a_m 0$, $c_m = a_m m$	
	{001,101,110} {001,102,110} {001,110,201} {001,110,210}	$a_m \rightsquigarrow (b_m)^{m+1}, a_{m+1}$; $b_m \rightsquigarrow b_m$, where $a_m = 01 \cdots m$, $b_m = a_m 0$	
	{001,101,120} {001,102,120} {001,120,201} {001,120,210}	$a_0 \rightsquigarrow b_0, a_1$; $b_0 \rightsquigarrow b_0$; $a_m \rightsquigarrow c_m, b_m, a_{m+1}$; $c_m \rightsquigarrow c_m$; $b_m \rightsquigarrow c_m, b_m$, where $a_m = 01 \cdots m$, $b_m = a_m(m-1)$, $c_m = a_m m$	

Continuation of Table 1

Continuation of Table 1			
Class	B triple	Rules of $\mathcal{T}(B)$	$G_B(x)$/Reference
	{011,021,100} {011,100,101} {011,100,110} {011,100,201} {011,100,210}	$a_m \rightsquigarrow a_{m+1}, b_{m,1}$; $b_{m,j} \rightsquigarrow c_{m,j}, b_{m,j+1}$; $c_{m,j} \rightsquigarrow c_{m,j+1}$, where $a_m = 0^m$, $b_{m,j} = 0^m 12 \cdots j$, $c_{m,j} = 0^m 102 \cdots j$	
	{011,021,102} {011,101,102} {011,102,110} {011,102,201} {011,102,210}	$a_m \rightsquigarrow a_{m+1}, b_{m,1}$; $b_{m,j} \rightsquigarrow c_j, b_{m,j+1}$; $c_m \rightsquigarrow c_m$, where $a_m = 0^m$, $b_{m,j} = 0^m 12 \cdots j$, $c_m = 01 \cdots m0$	
	{011,021,120} {011,101,120} {011,110,120} {011,120,201} {011,120,210}	$a_m \rightsquigarrow a_{m+1}, b_{m,1}$; $b_{m,1} \rightsquigarrow c_{m+1,1}, b_{m,2}$; $b_{m,j} \rightsquigarrow b_{m,j+1}$, where $a_m = 0^m$, $b_{m,j} = 0^m 12 \cdots j$	
	{012,021,100} {012,100,102} {012,100,120} {012,100,201} {012,100,210}	$a_m \rightsquigarrow a_{m+1}, 01$; $01 \rightsquigarrow 010, 01$; $010 \rightsquigarrow 0101$; $0101 \rightsquigarrow 0101$, where $a_m = 0^m$	
	{012,021,101} {012,101,102} {012,101,120} {012,101,201} {012,101,210}	$a_m \rightsquigarrow a_{m+1}, 01$; $01 \rightsquigarrow 010, 01$; $010 \rightsquigarrow 010$, where $a_m = 0^m$	
	{012,021,110}		

Continuation of Table 1			
Class	B triple	Rules of $\mathcal{T}(B)$	$G_B(x)$/Reference
	{012,102,110} {012,110,120} {012,110,201} {012,110,210}	$a_m \rightsquigarrow a_{m+1}, 01$; $01 \rightsquigarrow 01, 011$; $011 \rightsquigarrow 011$, where $a_m = 0^m$	$\frac{x(1-x+x^2)}{(1-x)^3}$
16	{001,100,101} {001,100,102} {001,100,201}		See [**?**]
21	{001,101,210} {001,102,210} {001,201,210}	$a_m \rightsquigarrow (b_m)^m, c_m, a_{m+1}$; $b_m \rightsquigarrow b_m$; $c_m \rightsquigarrow (b_m)^m, c_m$, where $a_m = 01\cdots m$, $b_m = a_m 0$, $c_m = a_m m$	$\frac{x(1-2x+2x^2)}{(1-x)^4}$
22	{000,021,101} {000,021,110}		See [**?**]
24	{000,101,102} {000,101,110}		See Subsection **??** See Subsection **??**
	{001,101,102} {001,101,201} {001,102,201}	$a_m \rightsquigarrow b_{m,0}, \ldots, b_{m,m}, a_{m+1}$; $b_{m,j} \rightsquigarrow b_{m,0}, \ldots, b_{m,j}$, where $a_m = 01\cdots m$, $b_{m,j} = a_m j$	
	{010,021,100} {010,021,101} {010,021,102} {010,021,110} {010,021,120} {010,021,201} {010,021,210} {010,100,101} {010,100,102} {010,100,110} {010,100,120} {010,100,201} {010,100,210} {010,101,102} {010,101,110} {010,101,120} {010,101,201} {010,101,210} {010,102,110}		

Continuation of Table 1			
Class	B triple	Rules of $\mathcal{T}(B)$	$G_B(x)$/Reference
	{010,102,120} {010,102,201} {010,102,210} {010,110,120} {010,110,201} {010,110,210} {010,120,201} {010,120,210} {010,201,210} {011,021,101} {011,021,110} {011,021,201} {011,021,210} {011,101,110} {011,101,201} {011,101,210} {011,110,201} {011,110,210} {011,201,210}	$a_m \rightsquigarrow a_{m+1}, b_{m,1}$; $b_{m,j} \rightsquigarrow b_{m,j+1}, b_{m,j+1}$, where $a_m = 0^m$, $b_{m,j} = 0^m 12 \cdots j$	
	{012,021,102} {012,021,120} {012,021,201} {012,021,210} {012,102,120} {012,102,201} {012,102,210} {012,120,201} {012,120,210} {012,201,210}	$a_m \rightsquigarrow a_{m+1}, 01$; $01 \rightsquigarrow 01, 01$, where $a_m = 0^m$	$\frac{x}{1-2x}$
28	{000,021,100} {000,021,201} {000,021,210}		See Section **??**
34	{000,100,101} {000,101,201}		See Section **??**
41	{021,101,102}	$a_m \rightsquigarrow a_{m+1}, b_{m,1}$; $b_{m,j} \rightsquigarrow c_j, b_{m+1,j}, b_{m,j+1}$; $c_m \rightsquigarrow c_m$, where $a_m = 0^m$, $b_{m,j} = 0^m 12 \cdots j$, $c_m = 01 \cdots m0$	

Continuation of Table 1			
Class	B triple	Rules of $\mathcal{T}(B)$	$G_B(x)$/Reference
	{021,101,120}	$a_m \rightsquigarrow a_{m+1}, b_{m,1}$; $b_{m,1} \rightsquigarrow c_m, b_{m+1,1}, b_{m,2}$; $b_{m,j} \rightsquigarrow b_{m+1,j}, b_{m,j+1}$; $c_m \rightsquigarrow c_{m+1}, b_{m,2}$, where $a_m = 0^m$, $b_{m,j} = 0^m 12 \cdots j$, $c_m = a_m 10$	
	{021,102,120}	$a_m \rightsquigarrow a_{m+1}, b_{m,1}$; $b_{m,1} \rightsquigarrow 010, b_{m+1,1}, b_{m,2}$; $b_{m,j} \rightsquigarrow b_{m+1,j}, b_{m,j+1}$; $010 \rightsquigarrow 010, 0101$; $0101 \rightsquigarrow 0101, 0101$, where $a_m = 0^m$, $b_{m,j} = 0^m 12 \cdots j$	
	{100,102,120}	$a_m \rightsquigarrow a_{m+1}, b_{m,1}$; $b_{m,j} \rightsquigarrow c_j, b_{m+1,j}, b_{m,j+1}$; $c_m \rightsquigarrow d_m$; $d_m \rightsquigarrow d_m$, where $a_m = 0^m$, $b_{m,j} = 0^m 12 \cdots j$, $c_m = 01 \cdots m(m-1)$, $d_m = c_m m$	
	{101,102,110}	$a_m \rightsquigarrow 010, a_0, a_{m+1}$; $010 \rightsquigarrow 010$, where $a_m = 01 \cdots m$	
	{101,102,120}	$0 \rightsquigarrow 0, 01$; $01 \rightsquigarrow 010, (01)^2$; $010 \rightsquigarrow 010$	
	{102,110,120}	$0 \rightsquigarrow 0, 01$; $01 \rightsquigarrow 010, 0, 01$; $010 \rightsquigarrow 010, 0101$; $0101 \rightsquigarrow 0101$	$\frac{x(1-2x+2x^2)}{(1-2x)(1-x)^2}$
42	{021,100,101} {021,100,110} {021,100,120} {021,101,110} {021,110,120}		See Section **??**
43	{100,101,110}	$a_m \rightsquigarrow \epsilon^m, a_0, a_{m+1}$; $\epsilon \rightsquigarrow a_0$, where $a_m = 01 \cdots m$	
	{100,101,120}	$0 \rightsquigarrow 0, 01$; $01 \rightsquigarrow \epsilon, (01)^2$; $\epsilon \rightsquigarrow 0$	
	{101,110,120}	$0 \rightsquigarrow 0, 01$; $01 \rightsquigarrow 010, 0, 01$; $010 \rightsquigarrow 010, 0$	$\frac{x(1-x+x^2)}{1-3x+2x^2-x^3}$
47	{021,102,201} {021,102,210}	={021,102} ={021,102}	See [**?**]
	{102,110,210}	$a_m \rightsquigarrow (010)^m, a_0, a_{m+1}$; $010 \rightsquigarrow 010, 0101$; $0101 \rightsquigarrow 0101$, where $a_m = 01 \cdots m$	$\frac{x(1-3x+4x^2-x^3)}{(1-2x)(1-x)^3}$
49	{101,102,210}		

Continuation of Table 1			
Class	B triple	Rules of $\mathcal{T}(B)$	$G_B(x)$/Reference
	{102,120,201}		
	{102,120,210}		See Section ??
51	{101,120,201}		
	{101,120,210}		See [?]
52	{021,100,201}	={021,100}	
	{021,100,210}	={021,100}	
	{021,110,201}	={021,110}	
	{021,110,210}	={021,110}	See [?]
53	{021,101,201}	={021,101}	
	{021,101,210}	={021,101}	
	{021,120,201}	={021,120}	
	{021,120,210}	={021,120}	
	{100,101,210}		
	{101,102,201}		
	{101,110,201}		
	{101,110,210}		See [?, ?]
55	{100,120,201}		
	{110,120,201}		See Theorem ??
56	{100,120,210}		
	{110,120,210}		See Theorem ??
61	{021,201,210}	={021}	See [?]
	{100,201,210}		
	{110,201,210}		See Section ??
End of Table 1			

Classes not covered by Table 1

Class 24

- From the results listed in Table 1, it remains to show first that

$$|A_n(000, 101, 102)| = |A_n(000, 101, 110)| = 2^{n-1}.$$

- For any word $w = w_1w_2 \cdots w_n$ and integer k, we define $k + w$ as $(w_1 + k)(w_2 + k) \cdots (w_n + k)$.

- Let $a_n = |A_n(000, 101, 102)|$. Clearly, $a_1 = 1$ and $a_2 = 2$. So, from now on, we assume that $n \geq 3$.

- Note that any ascent sequence π in $A_n(000, 101, 102)$ can be decomposed as either

$$\pi = 0(1 + \pi'), \pi = 0(1 + \pi'')0, \pi = 00(1 + \pi'')$$

such that $\pi' \in A_{n-1}(000, 101, 102)$ and $\pi'' \in A_{n-2}(000, 101, 102)$.

- Hence, $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 1$ and $a_1 = 2$. By induction on n, we have $a_n = 2^{n-1}$.

- For the case $A_n(000, 101, 110)$, based on a small modification of Proposition 15 of work of (Baxter and Pudwell), the ascent sequences of $A_n(000, 101, 110)$ can be characterized as a generating tree with a root (2) and the rules

$$(k) \rightsquigarrow (1)^{k-1}, (k+1); \quad (1) \rightsquigarrow (2).$$

- Let $A_k(x)$ be the generating function for the number of nodes at level n in the subtree with a root (k), where the root stays at level 1.
- Hence,

$$A_1(x) = x + xA_2(x),\ A_k(x) = x + (k-1)xA_1(x) + xA_{k+1}(x).$$

Define $A(x; v) = \sum_{k\geq 2} A_k(x)v^{k-2}$. Then

$$A(x; v) = \frac{x}{1-v} + \frac{x}{(1-v)^2}A_1(x) + \frac{x}{v}(A(x; v) - A_2(x)).$$

- By taking $v = x$, we have

$$A_2(x) = \frac{x}{1-x} + \frac{x}{(1-x)^2} A_1(x).$$

Thus, from $A_1(x) = x + xA_2(x)$, we obtain that $A_2(x) = \frac{x}{1-2x}$, as required.

Class 28: • Since an ascent sequence begins with a 0, if it contains either 201 or 210, then it also contains 021.

• Also, if an ascent sequence contains 100, then it also contains either 000 or 021.

• Hence,

$$\{000, 021, 100\} \overset{a}{\sim} \{000, 021, 201\} \overset{a}{\sim} \{000, 021, 210\}.$$

Class 61: Formula for $|A_n(100, 201, 210)|$

• Based on our algorithm, we find that the generating tree

$$\mathcal{T}[100, 201, 210]$$

has a root $\alpha_{0,0}$ and satisfies the following rules:

$$\begin{aligned}\alpha_{a,m} &\rightsquigarrow \beta_{a+1}^{m-a}, \alpha_{a,m}, \alpha_{a,m+1}, \ldots, \alpha_{0,m+1},\\ \beta_a &\rightsquigarrow \alpha_{a,a}, \ldots, \alpha_{0,a},\end{aligned}$$

where

$$\alpha_{a,m} = 0101212 \cdots a(a-1)a\,(a+1)(a+2) \cdots m$$

and

$$\beta_a = 0101212 \cdots (a-1)(a-2)(a-1)a(a-1).$$

• To see the rules, we note that the children of $\alpha_{a,m}$ (respectively, β_a) are exactly $\alpha_{a,m}j$ with $j = a, a+1, \ldots, a+m+1$ (respectively, $\beta_a j$ with $j = a, a+1, \ldots, 2a$).

- Now, we show the following equivalences:
  - $\mathcal{T}(\{100, 201, 210\}; \alpha_{a,m}j) \cong \mathcal{T}(\{100, 201, 210\}; \beta_{a+1})$, for all $j = a, a+1, \ldots, m-1$: Let $\pi = \alpha_{a,m}j\pi'$ be any ascent sequence that avoids $\{100, 201, 210\}$. So π' does not contain any letter from the set $\{0, 1, \ldots, m-1\}$. Thus, π avoids $\{100, 201, 210\}$ if and only if $\beta_{a+1}(1 + a - m + \pi')$ avoids $\{100, 201, 210\}$, which proves the equivalence.
  - $\mathcal{T}(\{100, 201, 210\}; \alpha_{a,m}m) \cong \mathcal{T}(\alpha_{a,m})$: Note that the ascent sequence $\pi = \alpha_{a,m}m\pi'$ avoids $\{100, 201, 210\}$ if and only if the ascent sequence $\alpha_{a,m}\pi'$ avoids $\{100, 201, 210\}$ (just remove the letter m).

- $\mathcal{T}(\{100, 201, 210\}; \alpha_{a,m} j) \cong \mathcal{T}(\{100, 201, 210\}; \alpha_{a+m+1-j,m+1})$, for all $j = m+1, m+2, \ldots, a+m+1$: Let $j = m + 1 + j'$ and $\pi = \alpha_{a,m} j \pi'$ be any ascent sequence that avoids $\{100, 201, 210\}$. Note that π' does not contain any letter from the set $\{0, 1, \ldots, j'\}$. So by removing the letters $0, 1, \ldots, j'$ from π' we obtain that π avoids $\{100, 201, 210\}$ if and only if the ascent sequence $\alpha_{a+m+1-j,m+1}(-j' + \pi') = \alpha_{a-j',m+1}(-j' + \pi')$ avoids $\{100, 201, 210\}$, which proved the equivalence.

- Hence, the first rule is holding:

$$\alpha_{a,m} \rightsquigarrow \beta_{a+1}^{m-a}, \alpha_{a,m}, \alpha_{a,m+1}, \ldots, \alpha_{0,m+1}.$$

Similarly, the second rule is holding.

- Define $A_{a,m}(x)$ and $B_a(x)$ to be the generating function for the number of nodes at level n in the tree

$$\mathcal{T}(\{100, 201, 210\}; \alpha_{a,a})$$

and

$$\mathcal{T}(\{100, 201, 210\}; \beta_a),$$

where the root stay at level 1.

- Hence, the above rules can be translated to

$$A_{a,m}(x) = x + (m-a)xB_{a+1}(x) + xA_{a,m}(x) + x\sum_{j=0}^{a} A_{j,m+1}(x),$$

$$\text{where } m \geq a \geq 0$$

$$B_a(x) = x + x\sum_{j=0}^{a} A_{j,a}(x), \quad a \geq 1.$$

- Now, we define

$$A(x;v,u)=\sum_{a\geq 0}\sum_{m\geq a}A_{a,m}(x)v^{a}u^{m}$$

and

$$B(x;v)=\sum_{a\geq 1}B_{a}(x)v^{a-1}.$$

- Then the last two recurrences can be written as

$$\begin{aligned}A(x;v,u)&=\frac{x}{(1-u)(1-vu)}+xA(x;v,u)\\&\quad+\frac{x}{u(1-v)}(A(x;v,u)-A(x;1,vu))\\&\quad+\frac{xu}{(1-u)^{2}}B(x;vu),\\B(x;v)&=\frac{x}{1-v}+\frac{x}{v}(A(x;1,v)-A(x;0,0)).\end{aligned}$$

- This implies

$$
\begin{aligned}
A(x;v/u,u) = {} & \frac{x}{(1-u)(1-v)} + xA(x;v/u,u) \\
& + \frac{x}{u-v}(A(x;v/u,u) - A(x;1,v)) \\
& + \frac{xu}{(1-u)^2}\left(\frac{x}{1-v} + \frac{x}{v}(A(x;1,v) - A(x;0,0))\right).
\end{aligned}
$$

- In order to solve this equation, we assume that $A(x;0,0) = C(x) - 1 = \frac{1-\sqrt{1-4x}}{2x} - 1$.
- By substituting $u = \frac{vx-v-x}{x-1}$, we obtain that

$$
A(x;1,v) = \frac{x(xC(x)(vx-v-x) + v - xv + x^2)}{(v-1)(v^2(x-1)^2 - v(3x-1)(x-1) + x^3)}.
$$

- Hence, by substituting expression of $A(x;1,v)$ into the equation and replacing v with vu, we obtain

$$\begin{aligned}
&A(x;v,u)\\
&= \frac{x((x-1)u^3v^2+(2-3x)u^2v+(3x-1)u-x)\sqrt{1-4x}}{2((x-1)^2u^2v^2-(3x-1)(x-1)uv+x^3)(1-uv)(1-u)^2}\\
&+ \frac{x((x-1)u^3v^2+2(1-x)u^2v^2-x(2x-1)u^2v)}{2((x-1)^2u^2v^2-(3x-1)(x-1)uv+x^3)(1-uv)(1-u)^2}\\
&+ \frac{x(2(2x-1)uv+(4x-1)(x-1)u-2x^2+x)}{2((x-1)^2u^2v^2-(3x-1)(x-1)uv+x^3)(1-uv)(1-u)^2}.
\end{aligned}$$

• Note that the expression of $A(x;v,u)$ satisfies the equation and the assumption $A(x;0,0)=C(x)-1$. Hence, $A(x;v,u)$ is the solution of the equation, which implies the following result.

Theorem The number of ascent sequences in $A_n(100,201,210)$ is given by $\frac{1}{n+1}\binom{2n}{n}$, the nth Catalan number.

Further results

By using our algorithm as in the previous sections, one can show that the number aw_k of A-Wilf-equivalence classes of k length-3 patterns is given by

$$aw_4 = 74, \quad aw_5 = 61, \quad aw_6 = 47, \quad aw_7 = 35, \quad aw_8 = 25,$$
$$aw_9 = 18, \quad aw_{10} = 12, \quad aw_{11} = 7, \quad aw_{12} = 3, \quad aw_{13} = 1.$$

Thanks for attention

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